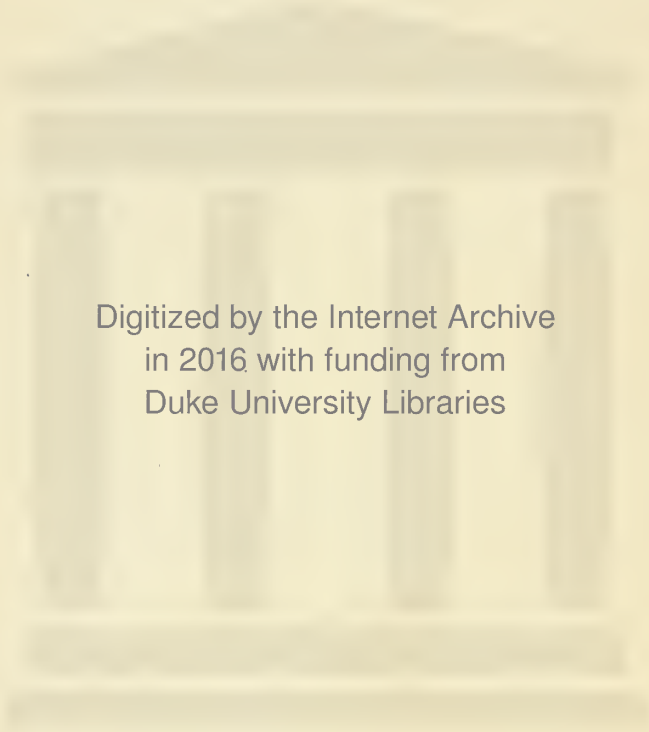
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Engineering Science Series

ALTERNATING CURRENTS AND ALTERNATING CURRENT MACHINERY

ENGINEERING SCIENCE SERIES

EDITED BY

DUGALD C. JACKSON, C.E.

Professor of Electrical Engineering
Massachusetts Institute of Technology
Fellow and Past President A.I.E.E.

EARLE R. HEDRICK, Ph.D.

Professor of Mathematics, University of Missouri
Member A.S.M.E.

ALTERNATING CURRENTS AND ALTERNATING CURRENT MACHINERY

NEW EDITION, REWRITTEN AND ENLARGED

BY

DUGALD C. JACKSON, C.E.

PROFESSOR OF ELECTRICAL ENGINEERING, MASSACHUSETTS INSTITUTE OF TECHNOLOGY
PAST-PRESIDENT AND FELLOW OF THE AMERICAN INSTITUTE OF ELECTRICAL ENGINEERS, PAST-PRESIDENT OF THE SOCIETY FOR THE PROMOTION OF ENGINEERING EDUCATION, ETC.

AND

JOHN PRICE JACKSON, M.E., Sc.D.

COMMISSIONER OF LABOR AND INDUSTRY, COMMONWEALTH OF PENNSYLVANIA, DEAN OF THE SCHOOL OF ENGINEERING, PENNSYLVANIA STATE COLLEGE, FELLOW OF THE AMERICAN INSTITUTE OF ELECTRICAL ENGINEERS, MEMBER OF THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS, ETC.

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PREFACE

This is a new edition of the authors' book on Alternating Currents and Alternating Current Machinery which was first published in 1896, and which has now been rewritten and greatly extended. The former editions met so favorable a response that the authors cannot help but believe that it served an important place in connection with the development of the methods of teaching the subject of alternating currents and their applications; and they earnestly hope that the new edition will be equally useful.

In this edition are maintained the well-known features of the earlier book in which were worked out the characteristics of electric circuits, their self-induction, electrostatic capacity, reactance and impedance, and the solutions of alternating current flow in electric circuits in series and parallel. More attention is paid to the transient state in electric circuits than was the case in the original edition. A considerable amount of related matter has been introduced in respect to vectors, complex quantities, and Fourier's series which the authors believe will be useful to students and engineers. The treatment of power and power factor has been given great attention, and a full chapter is now allotted to the hysteresis and eddy current losses which are developed in the iron cores of electrical machinery. More space and more complete treatment have been assigned to synchronous machines and to asynchronous motors and generators. The treatment of the self-inductance and mutual inductance of line circuits and skin effect in electric conductors which was found in the old book has been extended, and it has been supplemented by a treatment of the electrostatic capacity of lines and the influences of distributed resistance, inductance, and capacity.

In all these features as well as in others the book has been

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brought up to the requirements of present day teaching. The book covers the ground that is needed to give a fairly complete course in the essential elements of alternating currents and their applications to machinery. It is longer than will be needed in the courses in many of the engineering schools, but chapters may be selected so as to meet the requirements of each school. Thus, if an abbreviated course is required, Chapters V and XIII may be omitted, and if the course does not go into electrical machinery, the first eight chapters only need be considered. In those colleges where the students have a course in alternating current measurements previous to their entering upon the subject of alternating current machinery the second chapter may be omitted during the study of this text, but the chapter will be useful in connection with the course in alternating current measurements. The book is also serviceable in connection with instruction in the electrical transmission of power. For this purpose the authors suggest more particularly the use of the portions of Chapter I which deal with complex quantities and Fourier's series and the whole of Chapters IV, V, VI, VII, VIII, XIII. These are all particularly related to matters which are fundamental not only to electrical machinery but also to the transmission of power and the other branches in which alternating currents are usefully applied.

The authors have endeavored to make the phraseology simple and to illustrate the applications of the principles by examples drawn from the best practice in the art. As in the first edition, original methods have been introduced in various instances to gain simple paths to results, every effort being made to present a full physical conception of phenomena to the reader's mind. The mathematics used are merely logical means for accomplishing the end, and are by no means to be considered from any other standpoint. It has been sought by the authors to avoid either the error of presenting unnecessary formulas or, on the other hand, of giving results without supporting them on reasons, both of which are fatal to a student's effective progress, since they leave him without a true physical conception of the phenomena studied. It has been the authors' aim to produce a text which avoids the fault of being cursorily descriptive and at the same time avoids the reasonable objections to a treatise which is unrelieved mathematics.

The original edition of the book was noticeable for utilizing the word "active" for representing the true power component of voltage or current, and that practice, which has now been approved by the Standards Committee of the American Institute of Electrical Engineers, is followed in this edition, while the component at right angles is in this edition called either the quadrature component or the reactive component, the latter phraseology being also now in accordance with the recommendations of the Standards Committee of the American Institute of Electrical Engineers. In various respects the book contains original treatments which will be obvious, but the authors have made an effort to set forth the best treatment of each of the problems of alternating currents and therefore have brought under contribution most of the literature and practice in the art in the course of their selections of methods of presentation.

The examples which are introduced at intervals throughout the text for the purpose of illustrating the way in which the principles may be utilized will be useful to teachers and students. Footnotes referring mostly to correlated articles in the book itself are numerous. The many historical footnotes which were in the former editions have been mostly omitted from this edition, since the early literature has been now far outstripped and is no longer an essential source of knowledge for the undergraduate student.

The book is placed in the hands of electrical engineers in the belief that it will prove valuable as a textbook in the engineering schools, and also prove of service as a reference book with respect to the principles used in the numerous applications of alternating currents to practical purposes which now make so large a part of electrical engineering.

The authors wish to express their sincere thanks to Mr. H. P. Wood, Professor of Electrical Engineering at the Georgia School of Technology, Mr. Charles L. Kinsloe, Professor of Electrical Engineering at the Pennsylvania State College, and Mr. Charles W. Green, Instructor in Electrical Engineering at the Massachusetts Institute of Technology, who have read the proof and made many suggestions. It is not to be expected that a book of the extent and importance of this, the manuscript of which is written and the proof read in the midst of the exacting employments of men in important duty in engi-

neering schools, can be entirely free from ambiguities and errors; but the authors venture the hope that any errors or ambiguities which have escaped their vigilance are few. They will greatly appreciate having brought to their attention any which readers may find.

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ALTERNATING CURRENTS

CHAPTER I

THE VOLTAGE DEVELOPED BY ALTERNATORS. THE USE OF VECTORS AND THE COMPLEX QUANTITY. FOURIER'S SERIES

1. Direct and Alternating Currents obey the Same Laws. — A deeply rooted belief seems to have been cultivated in the minds of many that phenomena connected with the flow of direct electric currents and of alternating electric currents are almost entirely unrelated. This popular idea, however, is erroneous; the principles which relate to the flow of electric currents, whether direct or alternating, and which are applied to the design and construction of machines and circuits, are one and the same.

When Oersted, in 1820, made known his signal discovery that an electric current exerts a magnetic influence in the space around it, the foundation was begun for our knowledge of the laws of the flow of alternating currents. Within a dozen or fifteen years thereafter much knowledge of the electric current had been thrashed out experimentally by men like Ampère, Arago, Faraday, Henry, and others. The last two named laid the finishing stone on the foundation by searching out and making known the laws of electro-magnetic induction. These basic laws, developed early in the nineteenth century, apply with equal force to continuous, pulsating, and alternating currents.

In dealing with alternating currents, several variables enter into the problem which make it impracticable to use, without modification, the results gained in the study of direct currents. But, as already said, the same fundamental laws control the phenomena of both, and if care is taken to apply these with

due regard to the limiting conditions, it will be found that the subject of alternating currents may be clearly grasped and be reduced almost to the simplicity of direct-current work.

The design and operation of alternating-current machinery may differ considerably from those of direct-current apparatus in certain particulars because the commercial requirements are materially different, but the same principles fundamentally apply in both, and in the matter of good workmanship and substantial construction there should be no difference in the two classes of machines.

2. Alternating Voltage and Current. — An alternating voltage or electric pressure, as the term indicates, is an uninterrupted, rapid succession of electric pressure impulses which are alternately in opposite directions. These voltage impulses produced by ordinary commercial alternating-current machines reverse direction many times per second * and may be very irregular in form, though the typical and simplest form is that of a sinusoid. Figure 1 shows the plot of an alternating voltage curve of sinusoidal form. The ordinates of the curve represent voltages, while the abscissas represent time. The abscissas of such a curve are ordinarily marked in degrees, or in fractions of π; the base of the two loops which constitute a complete cycle of a sine wave being given the assigned value of 360°, or 2π radians. The ordinates may be scaled in volts. The alternate loops are drawn above and below the zero or datum line to indicate reversal of voltage.

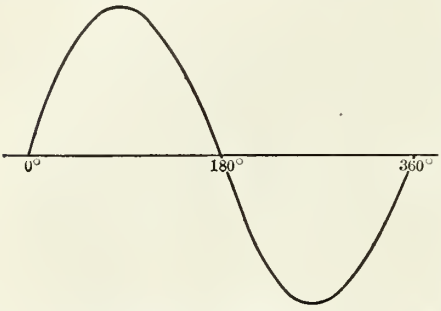


FIG. 1. — Two Loops of a Sinusoidal Alternating Voltage or Current Curve.

An alternating electric current flows whenever an alternating voltage is applied in a closed electric circuit of fixed constants. This current is not necessarily proportional at each instant to the voltage ordinate at the same instant, as will be shown later.† A curve representing the form of an alternating current wave may be drawn exactly as was indicated for an alternating volt-

* Art. 10. † Chap. IV.

age, but the ordinates represent amperes instead of volts. The simplest typical form of current curve is also sinusoidal, but this is very seldom attained in commercial practice; while on the other hand highly irregular current curves are quite common.

The successive loops in either current or voltage curves are usually assumed to be of identical form and dimensions as long as the conditions of the circuit are constant.*

3. Period and Frequency of an Alternating Current. — The time in seconds, T, required to pass through a complete cycle (that is, two loops of a curve) is called the **Period** of an alternating electric current or voltage. The number of periods in a second is called the **Frequency** of the current or voltage (this term was adopted by the Paris Electrical Congress). Usually an alternating current or voltage is designated by its *effective† value and frequency.* It is common, however, to use the number of half-periods, or the number of **Alternations**, in a minute, instead of the frequency. In this case, the number of alternations is equal to $2 \times 60 \times$ the frequency. Example : a current with a frequency of 60 periods per second makes 7200 alternations per minute. The term **Periodicity** is sometimes used for frequency.

4. Effective Alternating Voltages and Currents. — The heating effect or power activity of a current flowing through electrical resistance varies directly as the square of the current. This was proved by Joule in 1841, and the statement is often called Joule's law. Putting the statement of the law in symbols gives

$$H = .24\ I^2RT, \text{ and the power is } P = I^2R,$$

where H is heat measured in calories ; I, current measured in amperes ; T, the time measured in seconds ; R, resistance measured in ohms ; and P is the power expended in the circuit, measured in watts.

Alternating currents and voltages are constantly varying in intensity, and the heating effect of an alternating current is found by summing up instantaneous values through a complete cycle, as follows :

$$H = .24\ R \int_0^T i^2 dt,$$

in which i represents instantaneous values of current ; and the power is

* Art. 12. † Art. 4.

$$P = 1/T\int_0^T p\,dt = R/T\int_0^T i^2 dt.$$

The expression $1/T\int_0^T i^2 dt$ obviously represents the *average of the squared instantaneous currents*, and is equal to the average ordinate of a curve plotted on the same base as the original alternating-current curve, but with each ordinate of a length equal to the square of the corresponding ordinate of the original curve.

If the average value or ordinate of an alternating current is squared (that is, $[1/T\int_0^T i\,dt]^2$), the result is not the same as taking the average of squared ordinates or instantaneous values; and the amount of the difference depends on the form of the current curve. It is evident that this must be true, since squaring the larger instantaneous values produces more than a proportional effect upon the resulting average. It is well known that the average of the squares of a series of numerical values is always larger than the square of the average of the values, unless the numerical values are all equal to each other.

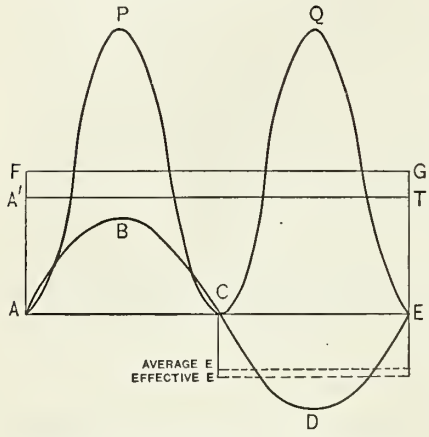


FIG. 2. — Curve of Squared Instantaneous Ordinates.

Figure 2 shows a sine curve and its curve of squared instantaneous ordinates designated respectively A, B, C, D, E, and A, P, C, Q, E. The height of the line FG represents the average value of the squared current ordinates, while the height of the line $A'T$ represents the square of the average of the current ordinates.

When a sinusoidal current flows through electrical resistance, the average power expended in producing heat is proportional to the average of the squares of instantaneous values of current, that is, to the height of the line FG; and the square root of this ordinate is equal in amperes to a direct current which (flowing

through the same resistance) will have the same heating effect as the alternating current under consideration. This square root of the average of the squared instantaneous values, that is, $\sqrt{\text{Av. } i^2}$ or $\sqrt{\text{Av. } e^2}$, is called the **Effective** value of the current or voltage.

Prob. 1. The resistance of a circuit is 20 ohms and 500 watts are expended in heating it. What is the effective current in the circuit?

Prob. 2. Four circuits in parallel and having 1, 2, 4, and 5 ohms resistance respectively, absorb 400, 200, 100, and 80 watts. What is the effective current in each branch?

Prob. 3. What are the effective voltages across the terminals of the circuits in problems 1 and 2, supposing that there is no live counter-voltage in the circuits?

Prob. 4. A circuit of 10 ohms resistance is in series with the four parallel circuits of problem 2, and 2000 watts are expended in the system. What are the effective currents and voltages in the series portion and in the parallel portion of the whole circuit, supposing no live counter-voltages are present?

5. Value of Effective Current and Voltage in Terms of the Maximum when the Curve is Sinusoidal. — If the current curve is of the sine form, it may be expressed as follows:

$$i = i_m \sin \alpha,$$

where i is the *instantaneous* value of the current for any angle α and i_m is the maximum instantaneous value. To obtain the average heating effect of this current when flowing through a resistance, square both sides of the equation, multiply through by R, and find by integration the area of one loop or one-half period of the squared curve. The area thus found divided by the base π gives the average of the squared ordinates. The expression may conveniently take the form,

$$(\text{Av. } i^2)R = \frac{R}{\pi} i_m^2 \int_0^{\pi} \sin^2 \alpha d\alpha,$$

from which, $$(\text{Av. } i^2)R = \frac{Ri_m^2}{2}.$$

Dividing by R, $\quad \text{Av. } i^2 = \frac{i_m^2}{2}$,

or $$\sqrt{\text{Av. } i^2} = \frac{i_m}{\sqrt{2}} = .707 i_m.$$

The value of $\sqrt{\text{Av. } i^2}$ is called the **Effective** value of the alternating current, as has already been pointed out.

The effective value of an alternating current is the square root of the average of the squared instantaneous values of the current. This is correct whether or not the current is of sinusoidal form. The foregoing equations show that, when the current is sinusoidal in form, there is a fixed relation between the effective and maximum values of the current and that the effective value is $.707\left(=\frac{1}{\sqrt{2}}\right)$ times the maximum value.

In the case of a sinusoidal pressure (voltage) curve the formula may be written,

$$e = e_m \sin \alpha,$$

where e is the instantaneous value of the voltage for any angle α and e_m is the maximum instantaneous value. By writing the integral for the heating effect as before, a similar result is obtained, or

$$\frac{\text{Av. } e^2}{R} = \frac{e_m^2}{2R}$$

and $$\sqrt{\text{Av. } e^2} = \frac{e_m}{\sqrt{2}} = .707\ e_m.$$

Effective voltage is the square root of the average of the squared instantaneous values of the voltage. When the form of the voltage wave is sinusoidal, the effective voltage is equal to $.707\left(=\frac{1}{\sqrt{2}}\right)$ times the maximum value of the voltage.

The average value of the ordinates of a sine curve may be found from the expression

$$\text{Av. } y = \frac{y_m}{\pi}\int_0^{\pi} \sin \alpha d\alpha,$$

where y is the instantaneous ordinate and y_m the maximum ordinate of the curve; and therefore

$$\text{Av. } y = \frac{2\, y_m}{\pi} = .637\ y_m,$$

or the average value of the instantaneous ordinates is $\frac{2}{\pi}$ (=.637) times the maximum ordinate. From this it is seen that

$$\sqrt{\text{Av. } y^2} = \frac{\pi}{2\sqrt{2}} (\text{Av. } y),$$

or

$$\sqrt{\text{Av. } y^2} = 1.11 \text{ (Av. } y).$$

Therefore the ratio of effective current or voltage to the average ordinate of the curve, in the case of sinusoidal curves, has the fixed value of 1.11.

The same considerations follow for alternating current and voltage waves of any form, but the ratios of the average and the effective values to the maximum are different, and depend upon the form of each curve considered. Alternating current and voltage waves are always single-valued; that is, a single value of the ordinates corresponds to each single value of the abscissas. They may be represented by the expression

$$x = af(\alpha).$$

Manifestly, the ratio which the average bears to the maximum ordinate takes a value nearer unity for a curve that is flatter than a sinusoid, and recedes farther from unity for a curve more peaked than a sinusoid.

For a curve made up of alternating rectangular loops, such as is shown in Fig. 17, the maximum, average, and effective values are equal to each other; but for a curve made up of alternating triangular (isosceles) loops, the average value is one half the maximum value, and the effective value exceeds the average value in a ratio of 1.155 to 1. The ratio of the effective to the average value may be quite large in the case of a curve which is very peaked. This ratio is sometimes called the **Form factor** of the curve.

Prob. 1. What are the maximum ordinates of the sinusoidal voltages having effective values of 100 and 200 volts, and of sinusoidal currents having the effective values of 50 and 75 amperes?

Prob. 2. What are the average ordinates of voltages and currents in problem 1?

Prob. 3. What are the effective and average values of a

voltage wave which is of the form of an equilateral triangle, and has an altitude of 110 volts?

Prob. 4. What are the effective and average values of a voltage wave in the form of a right triangle in which the altitude is 10 and the base 10?

Prob. 5. What are the average and effective values of a current wave having the form of a semicircle with the base equal to 100? That is, compute the average ordinate and the square root of the average of the squared ordinates, in terms of the radius.

6. Vectors representing Sinusoidal or Harmonic Voltages and Currents; Lead and Lag.— A physical quantity which may be represented by the length, position, and direction of a line is called a **Vector quantity** and the line so representing the quantity is called a **Vector.*** Thus, two forces may be represented (1) in magnitude, sometimes termed **Scalar** value, by two lines having lengths proportional to the intensities of the forces; (2) in relative angular position, or **Phase**, by the angle at which the lines, extended if necessary, intersect; and (3) in direction by arrow heads placed upon the lines. In the following discussions, if one or more vectors radiate from or turn upon a common center, and no arrow heads are shown, it is assumed that their directions are from the center outward. Vectors may be combined or resolved into components by the well-known laws of mechanics relating to the composition or resolution of forces or velocities. Thus, the centrifugal force acting upon a uniformly rotating crank may be represented by a vector *rotating uniformly around one end as a center of revolution.* and its component value at any instant in the direction of some fixed line (assume for convenience a vertical line passing through the axis of rotation) is found by projecting the vector from its position at the instant, upon that line. If the center is moved (for convenience, horizontally to the right) a given distance for each degree of angular rotation of the vector, and the instantaneous vertical projections for each degree of advance are plotted at the proper points along the path of the center, the points thus obtained when joined form a sine curve. In Fig. 3, OA is a rotating vector. OX is the initial line or

* For a discussion of the use of vectors in graphical methods of solving electrical problems, see Chapter VI.

horizontal axis from which the angle α is measured, OY is the vertical axis upon which the point A is projected as the vector rotates, and SS is a sine curve produced as explained above. In the figure, OA has advanced 30°, and the ordinate a' of the sine curve, which is placed over the 30° division of its base, is made equal to OA sin 30° or Oy. Likewise, when the vector has advanced to 60° the ordinate a'' of the curve is equal to OA sin 60°. When the vector is at 90°, the ordinate a''' is equal to OA sin 90°, which equals OA; hence *the length of a rotating vector required to produce a given sine curve must be equal to the maximum ordinate of the curve.*

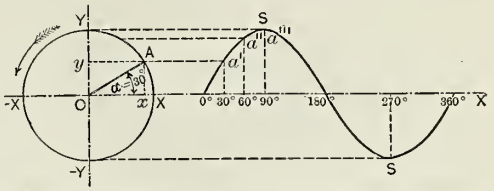


FIG. 3.—Rotating Vector and Harmonic Curve Produced Thereby.

By geometry it is seen that in the figure

$$A = \sqrt{y^2 + x^2} = A\ (\sin^2\alpha + \cos^2\alpha)^{\frac{1}{2}},$$

where $A = OA$, $x = Ox = A \cos \alpha$, $y = Oy = A \sin \alpha$; and

$$\tan \alpha = \frac{y}{x} \text{ and } \tan^{-1}\frac{y}{x} = \alpha.$$

These relations express the angular position and value of a **Rotating vector** in terms of its instantaneous rectangular components with great simplicity, and are much used in the computations of electrical phenomena.

Rotating vectors may be combined, by the ordinary methods of mechanics, with other vectors having the same speed of rotation, since their relative angular positions will remain always the same. Thus, in Fig. 4, OA and OB are two rotating vectors having the same speed or frequency, and therefore at a fixed **Phase difference** or angular position with respect to each other. Their resultant is OC, the diagonal of the parallelogram $OACB$. It may be seen by inspection that the vertical projection of OC, which is Oc, is equal at any instant to $A \sin \alpha + B \sin (\alpha - \theta) = Oa + Ob$, since BC is equal and parallel to OA and ac is therefore equal to Ob. These vectors A, B, and C are the generators of the sine curves or harmonics

A', B', and C'. If the ordinates of A' and B' corresponding to any abscissa are added together, they give the corresponding

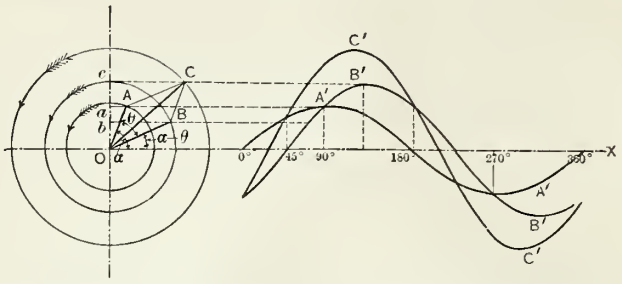


FIG. 4.—Composition of Rotating Vectors OA and OB by Parallelogram of Forces.

ordinate of the curve C'. This follows from the construction of the curves. If the vector OC is given, it can be resolved into the component vectors OA and OB with their particular phase relations to OC, or it may be resolved into components at any other fixed angular relations, by reversing the process of construction. Or, if OC and one component, OA, are given, the other component may be found by completing the parallelogram of which OC is the diagonal and OA one side. Three or more vectors may be combined by similar processes.

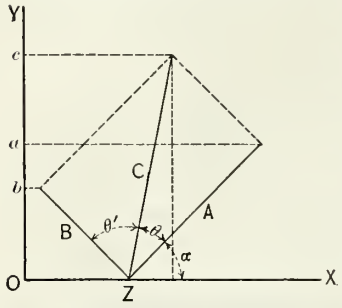


FIG. 5.—Illustration of the Relations of Two Vectors 90° apart.

In the case of two vectors 90° apart the relations are very simple. Thus in Fig. 5, if A and B are two such vectors, rotating about Z, which form the resultant C,

$$C = \sqrt{A^2 + B^2},\ \tan\theta = \frac{B}{A},\ \tan\theta' = \frac{A}{B},$$

$$a = A\sin\alpha,\ \text{and}\ b = B\sin(\alpha + 90°) = B\cos\alpha.$$

where A, B, and C are the values of the vectors, a and b are the vertical components of A and B, α is the angle of advance of A, and θ and θ' are the fixed angles AZC and CZB. Further,

$$c = C\sin(\alpha + \theta),$$

and

$$c = a + b,$$

where a, b, and c are the instantaneous vertical components of A, B, and C; from which, remembering that $\cos \alpha = \sin (\alpha + 90°)$,

$$c = A \sin \alpha + B \cos \alpha;$$

hence, $$A \sin \alpha + B \cos \alpha = C \sin (\alpha + \theta).$$

Also $$A \sin \alpha + B \cos \alpha = C \cos (\alpha - \theta').$$

In using these expressions it is desirable to always measure α in the positive (counter-clockwise) direction in order to prevent confusion in the algebraic signs of the angles.

Since $B \cos \alpha = B \sin (\alpha + 90°)$, it is evident that a sine curve generated by a rotating vector 90° in advance of the vector OA of Fig. 3 would be a curve with its positive maximum at $\alpha = 0°$, its negative maximum at $\alpha = 180°$, and its zero values at $\alpha = 90°$, 270°, etc. That is, this cosine curve would be ahead of or in the **Lead** of the curve drawn in Fig. 3 by 90°, and *vice versa*, the sine function **Lags** 90° behind the cosine function. Also, the phase or relative angular position of a resultant vector is between the phases of its components, and the angles between a resultant and its components depend upon the phase difference and the ratio of scalar values of the components.

Vectors having different speeds of rotation cannot be combined by the parallelogram of forces, as the resultant vector varies in length with the time.*

Harmonic (sinusoidal) voltages and currents can evidently be represented by rotating vectors and can be combined with as much facility as mechanical forces or velocities. In doing this the maximum ordinate, or **Amplitude**, must be taken for the scalar value of the vector if it is desired to determine the instantaneous components; but if it is desired to find the effective resultant of two or more voltages or currents, the effective values of the components may be used. This is because $E = \frac{e_m}{\sqrt{2}}$ and therefore the parallelograms using maximum and effective values are similar, the former being convertible into the latter by dividing its sides and diagonals by $\sqrt{2}$. Effective voltages and currents may possibly not be represented as vectors in the broadest mathematical sense, but for our purposes such representation is eminently satisfactory.

* See Art. 71.

If a number of vectors are to be combined, the resultant of a pair may be found and this then combined with another vector, and so on until the final resultant is obtained. In Fig. 6 the resultant of the vectors A, A', A'', and A''' is found by the parallelograms $OAa'A'$, $Oa'a''A''$, and $Oa''a'''A'''$. This is cumbersome and can be simplified by drawing OA, Aa', $a'a''$, and $a''a'''$, respectively, parallel and equal in length to the vectors A, A', A'', and A'''. By joining the free points O and a''' the resultant OR is obtained. The figure $OAa'a''a'''O$ is called a **Vector polygon.** If only two vectors are combined, giving, say, OAa', the result is called a **Vector triangle.** When the figure shows all the vectors radiating from a center as do OA. OA', OA'', and OA''', it is called a **Phase diagram,** as the relative angular positions of the vectors are directly indicated.

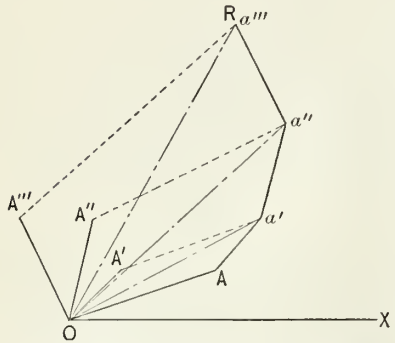


FIG. 6. Vector Addition by Parallelograms of Forces and Vector Polygon.

Prob. 1. Two current vectors differing 30° in phase have values respectively of 20 and 40 amperes. What is their combined value?

Prob. 2. Four voltages of 25, 50, 75, and 100 are represented by vectors having 0°, 30°, 60°, and 90° absolute phase positions. What is the combined value of all the voltages in series, and what is its angle with reference to the component vector having 0° angle?

Prob. 3. A current of 100 amperes divides into two branches in such a manner that one branch current leads the main current by 30°, and the second branch current lags behind the main current by an angle of 60°. What are the values of the two currents?

Prob. 4. A current vector of 50 amperes is divided into two rectangular components, one of which lags behind it by an angle of 45°. What is the value of the second component and its angular position?

Prob. 5. A voltage vector of 100 volts is composed of two components, one of which lags behind it by an angle of 10°, and the other leads by an angle of 20°. What are the scalar values of the components?

Prob. 6. There are four voltages, A, B, C, D, in series in a circuit. Assuming the first to have an angle of 0° and the others to be measured in angular positions with reference to it, the scalar and angular values of the vectors are $10, \angle 0°$; $20, \angle 15°$; $50, \angle -15°$; $30, \angle -45°$. What is the resultant voltage and its angular position with reference to the component having $\angle 0°$?

Prob. 7. The resultant in problem 6 is to be divided into two rectangular components, one on the horizontal axis ($\angle 0°$), and the other on the vertical axis. What are the scalar values of the components?

7. Polar Coördinates and the Method of obtaining Effective Voltages from the Polar Curve. — It is sometimes convenient to plot alternating voltage and current curves to polar coördinates; *i.e.* the instantaneous values of the voltage or current are laid off on *radius vectors* occupying the corresponding instantaneous positions of the rotating vector thought of as generating the alternating function.

The effective value may be directly derived from the primary polar curve, as originally shown by Steinmetz,* if it is plotted on polar coördinates taking 360° to a complete period. This gives a symmetrical curve which crosses the origin at 0°, 180°, 360°, etc. For an exact sinusoid the curve is of the form shown in Fig. 7, and has its maximum values, positive and negative, at 90° and 270°; *i.e.* each loop is a circle with the pole on its circumference and the initial line, OX, tangent to the circumference, the maximum

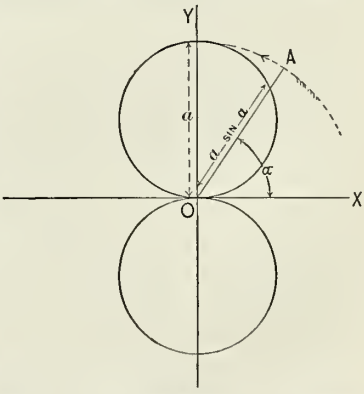


Fig. 7.—Polar Diagram of a Harmonic Function.

* *Trans. Amer. Inst. E. E.*, Vol. 10, p. 527; *Elektrotechnische Zeitschrift*, June 20, 1890.

ordinate, a, being equal to the diameter. The area of the curve in this form may be shown to be directly proportional to the square of the effective value of the ordinates as follows: In the case of a sinusoidal curve, the polar curve has the equation $e = a \sin \alpha$, where e is the instantaneous voltage corresponding to an angular advance α. In plotting the curve, values of e are laid off on the radius vectors having vectorial angles equal to the corresponding values of α, and a line is drawn through the points thus located (Fig. 7). Each loop of this curve, that is, the part of the curve taken between $\alpha = 0°$ and $\alpha = 180°$, or $\alpha = 180°$ and $\alpha = 360°$, is a circle, and its area is $A = \frac{1}{4}\pi d^2$, where d is the diameter of the circle. By the construction, d is equal to a of the formula $e = a \sin \alpha$, and the area of a loop of the curve is therefore $A = \frac{1}{4}\pi a^2$. The effective ordinate of a sinusoid has already been shown to be *

$$E = \frac{1}{\sqrt{2}} e_m = \frac{a}{\sqrt{2}}.$$

Consequently $$E = \sqrt{\frac{2A}{\pi}} = .798\ \sqrt{A}.$$

This may be taken for most purposes as $E = .8\sqrt{A}$.

The same expression applies to any single-valued periodic curve of equal positive and negative loops. The area of one loop of any such polar curve is

$$A = \tfrac{1}{2}\int_0^{\pi} e^2 d\alpha;$$

for if the curve is divided up into elementary triangles having their apices at O and bases equal to $e\,\Delta\alpha$, each triangle has an area approximately equal to $\frac{1}{2}\Delta\alpha \times e^2$, where e is the average altitude of the triangle; and the total area is

$$A = \sum_0^{\pi} \tfrac{1}{2}\, e^2 \Delta\alpha.$$

This, when $\Delta\alpha$ is reduced to the limit, becomes the integral given above. The mean of the squared ordinates of the function is Av. $e^2 = \frac{1}{\pi}\int_0^{\pi} e^2 d\alpha$. The effective ordinate is therefore

$$E = \sqrt{\text{Av.}\, e^2} = \sqrt{\frac{1}{\pi}\int_0^{\pi} e^2 d\alpha} = \sqrt{\frac{2A}{\pi}} = .798\sqrt{A}.$$

As before, this may be taken as $E = .8\sqrt{A}$.

* Art. 5.

Figure 8 shows the voltage curve plotted to rectangular coördinates, the curve of squared ordinates, and the polar curve for the voltage wave of an alternator which by special design gave a very distorted wave. The mean ordinates of the rectangular curves are readily determined by measuring the area by planimeter and dividing the area by the length of the base.

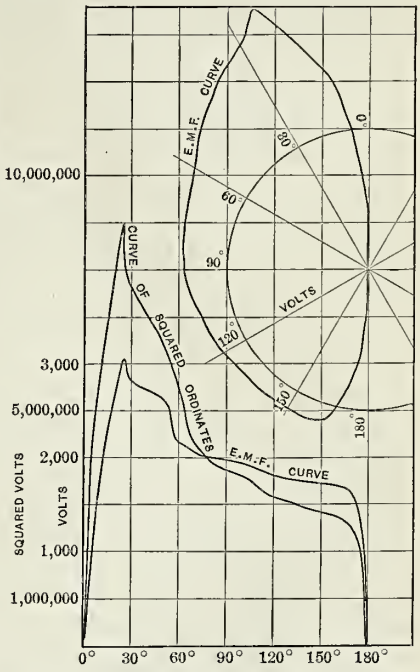


FIG. 8. — Irregular Alternator Voltage Loop plotted to Rectangular and Polar Coördinates, and the Curve of Squared Ordinates.

Prob. 1. Construct the polar loops of a wave of voltage which, laid out to rectangular coördinates, has rectangular loops with a base of 10 and an altitude of 5.

Prob. 2. Make the same construction as in problem 1, when the wave plotted on rectangular axes is a semicircle having a diameter of 8.

Prob. 3. Find the effective values of the voltages in problems 1 and 2 by the method explained in this article.

8. A Fundamental Conception of the Vector Quantity. — In Art. 6 it has been pointed out that currents and voltages may be shown in magnitude and relative phase by means of a phase diagram. Thus, in Fig. 9 suppose OX to be the initial line and OA', OA'', and OA''' to be voltages in series, or currents entering or leaving a junction, which are represented in relative phase by the angular positions and in magnitude by the lengths of the lines. It has been pointed out that the resultant of two or more similar electrical quantities may be found by treating their representative lines as vectors; such vectors may be combined by algebraically adding the vertical and horizontal components of the individual lines, by which

means the vertical and horizontal components of the resultant are determined. If a', b'; a'', b''; and a''', b''' are the horizontal and vertical components of OA', OA'', and OA''', and A, B, the components of the resultant, then

$$A + B = (a' + a'' + a''') + (b' + b'' + b'''),$$

or

$$A + B = (a' + b') + (a'' + b'') + (a''' + b''').$$

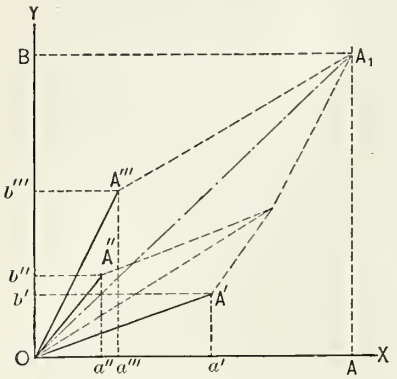


Fig. 9. — Illustration of Relation of Vectors and their Components.

In this expression there is nothing to distinguish the horizontal from the vertical components except the difference between the letters A or a and the letters B or b; or, in general, there is nothing to indicate the angular positions of the components, or of the lines represented by them, with reference to the initial line; and the ordinary algebraic processes afford no convenient means for giving these indications. To fully indicate the magnitude and position of a line by its rectangular components, we must abandon the methods of algebra for geometric processes. Therefore we may consider, for the moment, that the *components t and u of the vector $\bar{A}$* (Fig. 10) *both lie on the initial line OX*, but in order that t and u may determine the vector $\bar{A}$, u must be vertical, that is, rotated 90°.* To indicate such a rotation, a prefix such as j may be used. Then $\bar{A}$ will be represented in magnitude and angular position by the expression

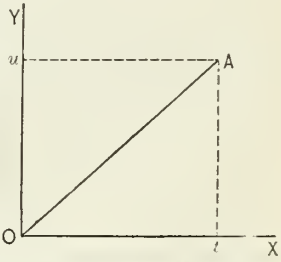


Fig. 10. — Single Vector and Components.

$$\bar{A} = t + ju,$$

where the sole function of the letter j is to indicate that the component u stands 90° from the initial line and *the addition is*

* The letter denoting a vector will be written with a vinculum, thus, $\bar{A}$, to distinguish it from its scalar value. When its components are on the horizontal or vertical they will have this designation omitted.

geometric; that is, *the joint effect of two vectors in the direction of their resultant is equal to the effect of their resultant.* The effect of t and ju is equal to the effect of $\bar{A}$. The expression $t + ju$ does not indicate arithmetical addition, but represents the combination of the effects of t and u. It is sometimes written t, ju. As ju is positive, it is said to have rotated u ahead 90°; $-ju$ would indicate that u had been rotated 90° in a negative direction, or that u is measured downwards (the negative direction) from the origin. If t and ju are both negative, they are both measured in the negative direction; hence, if $t + ju$ be multiplied by -1, there results $-t - ju$, t and u are both reversed in direction, and the vector line OA is rotated 180° (Fig. 11); $jt - u$ means that the line has been rotated forward 90°, since t is positive but stands at 90° from the initial line, and u is negative; $-jt + u$ means that the line has been rotated back 90°. Finally, multiplying by j means advancing the vector line 90°; for, $j(t + ju) = jt + (+j)(ju)$, and as j^2 indicates rotation twice forward, j^2u becomes $-u$, and therefore $j(t + ju)$ is geometrically equal to $jt - u$; j is, therefore, seen to be similar to the imaginary term $\sqrt{-1}$ since $(\sqrt{-1})^2 = -1$. Also multiplying by $-j$ means turning the vector back 90°, for $-j(t + ju) = -jt + (-j)(ju)$, and as $-j^2$ (namely, $+j \times -j$) indicates rotation forward 90° and back 90°, $-j^2u = u$, and there results $-jt + u$.

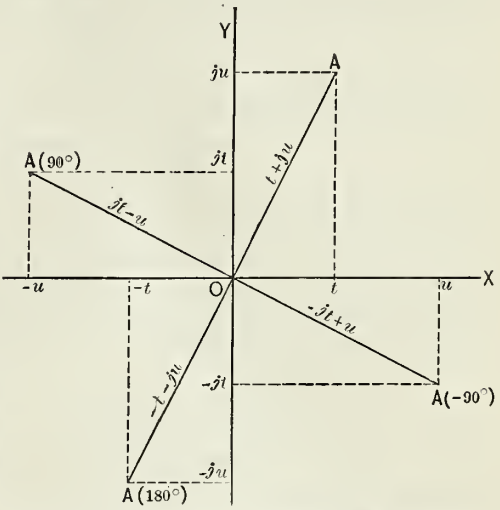


FIG. 11.—Illustration of the Effect of the Operator j.

Quantities comprising associations of real and imaginary quantities, such as the quantities described in the preceding paragraph, are called **Complex quantities.**

The vector expressing a sine wave may now be represented in magnitude or scalar value, as heretofore shown, by

$$A = \sqrt{t^2 + u^2},$$

where A is the length or scalar value of the vector $\bar{A}$ and is a purely arithmetical quantity; in phase by

$$\tan \theta = \frac{u}{t};$$

in phase and magnitude by the complex quantity

$$A(\cos \theta + j \sin \theta),$$

since $A \cos \theta = t$ and $A \sin \theta = u$; and also, as just indicated, by the equivalent complex quantity

$$t + ju.$$

The addition of the vectors given in the first illustration of this article now becomes

$$A + jB = (a' + a'' + a''') + j(b' + b'' + b''').$$

The assumptions made with respect to the symbol j leave it subject to the fundamental laws of algebra, even though it does not represent an ordinary numerical quantity and may be considered purely as a sign of operation.

For a further discussion of the complex quantity and its application to various types of electric circuits, refer to Chapter VI.

Prob. 1. The complex quantity expressing a vector of voltage is $10 + j\,8$. What is the angular position of this vector with reference to the horizontal axis, $\angle 0°$, and what is its scalar value in volts?

Prob. 2. A vector of current is $\bar{I} = 15 - j\,6$. What is its angular position and scalar value?

Prob. 3. Three vectors of voltage $\bar{E}_1 = 5 + j\,6$, $\bar{E}_2 = 8 + j\,10$, $\bar{E}_3 = 20 + j\,2$ are to be added together. What is the angular position and scalar value of the resultant voltage?

Prob. 4. Complex quantities expressing three voltages which are in series are $\bar{E}_1 = 80 - j\,10$, $E_2 = 100 + j\,0$ and $\bar{E}_3 = -j\,50$. What is the scalar value and angular position of the resultant voltage?

Prob. 5. A current having a scalar value of 100 amperes and an angular position of $\angle 30°$ may be expressed as a complex quantity in what terms?

Prob. 6. Two vectors of voltage having respectively scalar values of 20 and 50 and angles of $\angle 30°$ and $\angle -30°$ are to be combined. Form their complex expressions and from these obtain the complex expression of the resultant, after which find the scalar value and angular position of the resultant.

Prob. 7. Find the scalar value and angular position of the resultant of the four following voltages:

$$\bar{E}_1 = 8 + j\,4, \quad \bar{E}_2 = 10 - j\,6, \quad \bar{E}_3 = 15, \quad \bar{E}_4 = j\,20.$$

9. The Principle of an Alternating-current Generator. — The simplest form of **Alternating-current generator**, or **Alternator**, is a single coil revolving in a constant magnetic field and having its terminals connected to two rings fastened upon the shaft. Such a simple arrangement is shown diagrammatically in Fig. 12. The rings to which the ends of the coil are attached are called **Collector rings** or **Slip rings**, and upon these bear copper or carbon brushes for the purpose of connecting the coil with the external circuit. Revolving the coil in the magnetic field causes a voltage to be generated in one direction during the time it is under the influence of one pole piece, and the direction of the voltage reverses as soon as the coil begins cutting lines in the opposite direction under the influence of the other pole. If the instantaneous voltages set up in the revolving coil are plotted as explained in Art. 2, a curve of two loops, or one complete period, will represent the results in each complete revolution (Fig. 1). As an alternating current will not serve to magnetize the fields of alternating-current generators of this type, which are usually called **Synchronous generators**, some special arrangement for obtaining a direct current for the excitation must be made. This may be done either by commutating or rectifying all or a part of the alternating current produced by the machine, or

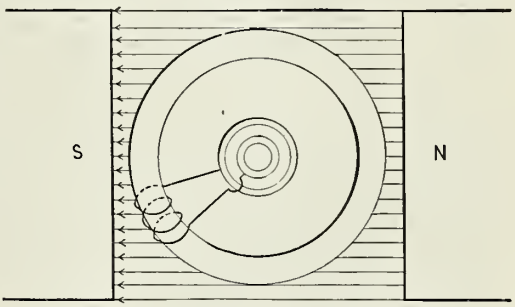


FIG. 12. — Simple Alternator Diagram.

a small auxiliary direct-current dynamo called an **Exciter** may be supplied for the purpose. Sometimes the exciter is mounted on the bed plate of the alternator.

10. Commercial Frequencies and Multipolar Generators. — The frequencies of alternating currents used for the general commercial purposes of the present day vary widely, but in nearly all cases fall within the limits of 15 and 135 periods or **Cycles** per second (1800 and 16,200 alternations per minute). The majority of American alternating-current dynamos, or **Alternators**, give a frequency of 25, 40, and 60 periods per second.

The rotation of a coil in a two-pole field gives one complete period, or two alternations, for each revolution; and the frequency is therefore equal to the number of revolutions per second. Since the armatures of two-pole machines, unless driven by direct-connected steam turbines or the like, would be required to run at impracticable speeds in order to give the ordinary commercial frequencies, alternators are usually made with a considerable number of poles. The number of poles depends upon the size of the alternator and other conditions which may control the speed of the armature, but, in general, it may be said to vary from eight upwards. An exception to this statement must be made in the case of generators which are designed for direct connection with steam turbines. The unusually high speed of these turbines renders a small number of field poles, even as few as two or four, requisite in the direct-connected alternator.

Figure 13 is a diagram of a six-pole generator with revolving armature. In the figure N, S, N, S, N, S, are the pole pieces, Y is the yoke, A the armature, B the brushes, and W, W, the field windings. Two of the magnetic circuits of the machine are shown by the dotted lines marked n, n. Instead of using one coil on the armature as was shown in Fig. 12, it is usual in this type of machine to use as many as there are poles for the purpose of utilizing the working space to the best advantage. These coils may be conveniently connected in series and their free ends connected to the collector rings, as is shown at D, Fig. 13, but they may be connected in series parallel or parallel relations if desired. In the series connection, the alternate coils, which move under pole pieces of opposite polarities, must

be connected in circuit in relatively opposite directions so that their voltages may add together.

The frequency which is produced by an alternator of this type is equal to one sixtieth of the product of the number of its pairs of poles by the number of revolutions per minute made by the armature. It is evident that this must be the case, since an armature coil will have completed one cycle or period when its center has passed from a point under one magnetic pole to a similar position with respect to the next pole of the same sign. The number of alternations per minute is equal to the frequency, as shown above, multiplied by 120.

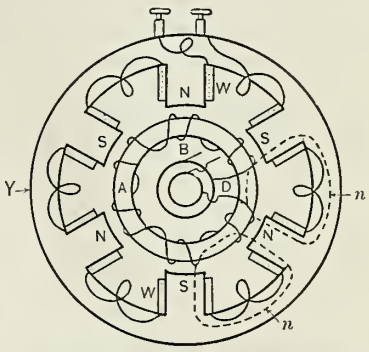


FIG. 13. — Diagram of Multipolar Generator.

In any alternator, one complete cycle of the alternating voltage is generated *while an armature conductor moves from a given point under a pole of one polarity to a like point under the next pole of the same polarity;* and the frequency is equal to the number of cycles per second. This definition applies to all alternators, while the commoner definition involving the number of poles, given in the preceding paragraph, only applies to alternators possessing field magnets with adjacent poles (in the direction of rotation) of opposite signs.

The angular distance of movement of the conductor required to generate one complete cycle is called 360 **Electrical degrees.** The electrical degrees correspond with the mechanical degrees of armature rotation only in a bipolar machine or its equivalent. Three hundred and sixty electrical degrees are embraced in the span from a given point in the magnetic field under one pole to the corresponding point in the magnetic field under the next pole of the same sign; so that there are as many times 360 electrical degrees measured around an armature as there are poles of like sign in the related field magnet. In speaking of the movement of the armature conductor in the magnetic field, the relative movement of the armature with respect to the field is here referred to, and it may be obtained

by the actual rotation of either the armature or the field magnet. In modern machines the latter is most frequently rotated.

Prob. 1. What is the frequency of an alternator having 10 pairs of poles and a speed of 600 revolutions per minute? How many alternations does it give per minute?

Prob. 2. An alternator gives a frequency of 25 periods per second and has 5 pairs of poles. At what speed (in revolutions per minute) does it run?

Prob. 3. An alternator is to give a frequency of 60 periods per second at a speed of 1200 revolutions per minute. How many poles must it have?

Prob. 4. Can an alternator be built to furnish a frequency of 47 periods per second when run at a speed of 600 revolutions per minute? Why?

Prob. 5. Can an alternator be built to furnish a frequency of 60 periods per second when running at 700 revolutions per minute? Why?

11. Form of the Voltage Curve of an Alternator. — The voltage set up by a conductor or coil moving in a magnetic field is equal to the rate of cutting lines of force, or to the rate of change of the number of linkages of lines of force with the coil, divided by 10^8; or (for a single conductor),

$$e = \frac{1}{10^8} \times \frac{d\phi}{dt},$$

where e is the instantaneous voltage and $\frac{d\phi}{dt}$ is the time rate of cutting lines of force. In the case of an armature revolving in a magnetic field as shown in Fig. 12 the voltage will be at any instant,

$$e = \frac{1}{10^8} \times \frac{S}{p'} \times \frac{d\phi}{dt},$$

where S is the number of active conductors on the armature, p' is the number of paths for the current through the armature, and $\frac{S}{p'}$ is the number of active conductors in series cutting the lines of force. If the armature revolves at a constant speed of V revolutions per minute in a magnetic field of p poles, with Φ

useful lines of force emanating from each north pole, the average voltage set up may be deduced from this formula to be,

$$E_{av.} = \frac{pS\Phi V}{p' \times 10^8 \times 60},$$

since $\frac{pV\Phi}{60}$ is the average rate of cutting lines of force by each conductor.

This is the formula used to state the continuous voltage of an ordinary direct-current dynamo. Since the voltage produced by direct-current dynamos is essentially constant, the effective and average values are equal to each other in this case.*

In the alternating-current dynamos the same averaging does not occur as in direct-current commutating machines with armature conductors uniformly distributed over the surface of the armature, and the voltage between alternator armature terminals varies from instant to instant.

Thus, suppose the coil of Fig. 12 has negligible width but comprises S conductors arranged to rotate in the uniform magnetic field as illustrated; then the voltage in the conductors is, at any instant, as before stated,

$$e = \frac{1}{10^8} \times \frac{S}{p'} \times \frac{d\phi}{dt}.$$

If Φ is the number of useful lines of force emanating from the north pole, then $\frac{d\phi}{dt}$, under the conditions described, becomes, for the bipolar machine,

$$\frac{d\phi}{dt} = \frac{2\pi r}{T''} \times \frac{\Phi}{2r} \times \sin \alpha = \frac{\pi}{T''} \Phi \sin \alpha,$$

where α is the angular displacement of the conductor from a diameter perpendicular to the lines of force, r is the radius of the armature, and T'' is the time in seconds of one revolution.

If there are more than two poles, each having Φ lines of force uniformly distributed over its face so as to make a multipolar magnetic field analogous to the bipolar magnetic field illustrated in Fig. 12, the voltage induced is obviously increased in proportion to the number of poles, and the last formula becomes, in general,

* Jackson's *Electromagnetism and the Construction of Dynamos*, Chap. 4.

$$\frac{d\phi}{dt} = \frac{p}{2} \times \frac{2\pi r}{T''} \times \frac{\Phi}{2r} \sin \alpha.$$

It must be remembered that α is an angle measured with respect to a normal to the lines of force, and it changes by the extent of 360° in passing from a given point in the magnetic field to the corresponding point under the next pole of like sign.

The voltage at any instant now becomes

$$e = \frac{\pi}{10^8} \times \frac{S}{p'} \times \frac{p}{2} \times \frac{\Phi}{T''} \sin \alpha.$$

But $\frac{1}{T''}$ is equal to $\frac{V}{60}$, where V stands for the speed of the armature in revolutions per minute. The formula, therefore, reduces to

$$e = \frac{\pi p S \Phi V}{2p' \, 10^8 \times 60} \sin \alpha,$$

which is the equation for a sinusoidal or harmonic curve * such as is shown in Fig. 1. The equation may be written

$$e = e_m \sin \alpha,$$

in which the maximum ordinate, e_m, is equal to

$$e_m = \frac{\pi p S \Phi V}{2 \times p' \times 10^8 \times 60}.$$

That is, under the conditions assumed for the magnetic field, and the arrangement of the armature conductors, the alternating voltage of the armature partakes of the form of a sine curve.

The symbol p in this formula has been defined as the number of poles comprised within the magnetic field. Since one complete cycle of the voltage is generated while an armature conductor moves from a given point under a pole of one polarity to a like point under the next pole of the same polarity, — that is, while the conductor moves through twice the polar pitch, — it is plain that p in the last formulas may be replaced by $2\frac{60}{V}f$, where f represents the frequency, and the formulas become

$$e = \frac{\pi S \Phi f}{p' \times 10^8} \sin \alpha,$$

and

$$e_m = \frac{\pi S \Phi f}{p' \times 10^8},$$

* Art. 5.

The effective voltage for this case is

$$E = \frac{\pi S \Phi f}{\sqrt{2}\, p' \times 10^8} = \frac{2.22\, S \Phi f}{p' \times 10^8}.$$

Now, if it is assumed that the field strength under the pole faces illustrated in Fig. 12 is altered so that it is weakest near the center, then the rate of cutting lines no longer varies as a sine curve, but the curve rises more rapidly in its lower portions, and does not reach so great a maximum. On the other hand, if the field is concentrated towards the center, the curve becomes higher at the center, but does not rise so rapidly in its lower portions. Figure 14 shows the curve of voltage developed in a coil which revolves in a field of the same total number of lines of force as that of Fig. 12, but in which the magnetic density at the centers of the poles is about 35 per cent less than the average. Figure 15 shows a similar curve when the magnetic density at the center is 50 per cent greater than the average. In each case, the polar density is assumed to change gradually and uniformly from the center to the edges. In all cases where a certain total number of lines are cut per revolution by a coil revolving at constant

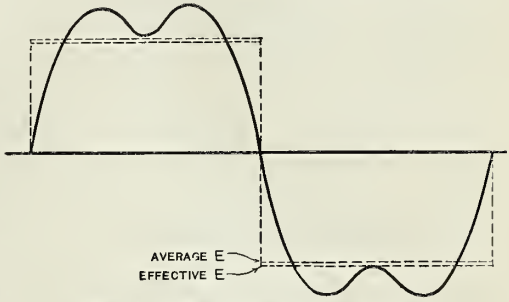


FIG. 14. — Flat-topped Voltage Curve.

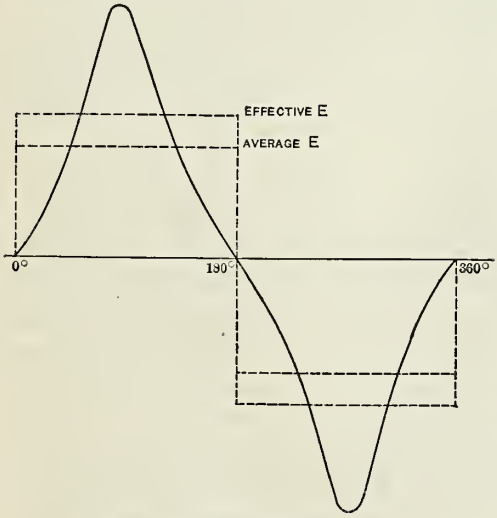


FIG. 15. — Peaked Voltage Curve.

speed, the average voltage remains constant regardless of the magnetic distribution, but the effective voltage ($\sqrt{\text{Av.}e^2}$) is by no means independent of the distribution. Taking the maximum voltage of the sine curve shown in Fig. 1 as the unit, the average voltage* in each figure is $\frac{2}{\pi} = .637$. The effective voltage of the sine curve shown in Fig. 1 is $\frac{1}{\sqrt{2}} = .707$. The maximum voltages shown in the curves of Figs. 14 and 15 are .84 and 1.5, and the effective voltages are .58 and .84, respectively. If the field were distributed as in Fig. 16 (the total magnetization remaining constant, and the coil of inappreciable width), the maximum, average, and effective voltages would be equal to each other (Fig. 17), and of a numerical value of .637. This form of curve gives the least possible relative effective voltage.

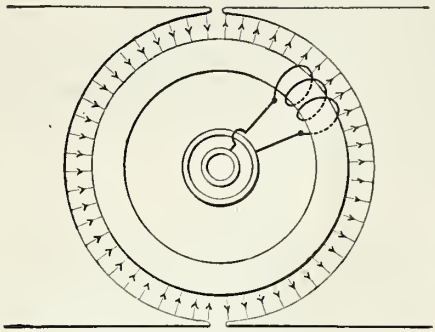


Fig. 16. — Imaginary Magnetic Field for obtaining Rectangular Voltage Curve.



Fig. 17. — Ideal Rectangular Voltage Curve.

On the other hand, if the field is greatly concentrated towards the center, as in Fig. 18, the maximum voltage is very great, and the effective voltage is considerably greater than the average, though it by no means approaches in value the maximum

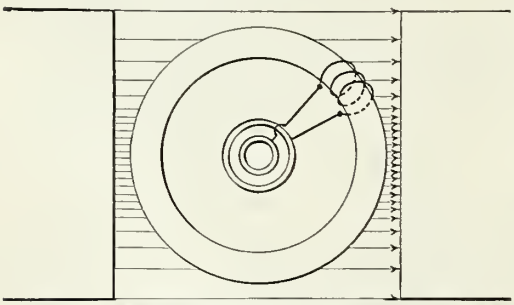


Fig. 18. — Magnetic Field for obtaining a Peaked Voltage Curve.

* Art. 5.

voltage. In the case of modern alternators the windings are usually **Distributed** * over a large portion of the polar area, so that the resulting curves of voltage tend to approximate the sine form.

12. Equations of Curves by Fourier's Series. — It has already been explained that the curves of current and voltage in commercial alternating-current circuits may deviate widely from the sine form, but are always single-valued. A general formula is needed, therefore, that will express such curves whatever may be their forms. The most usual and satisfactory method of obtaining such a formula is by means of Fourier's Series. This series is based upon the proposition that any single-valued recurrent function may be represented by a constant term, plus a sine term and a cosine term having the same period as the function in question, plus a sine term and a cosine term of one half the period, plus a sine term and a cosine term of one third the period, and so on up to a sine term and a cosine term of one nth the period †; where n is dependent upon the irregularity of the curve in question and the accuracy of representation desired.

If $f(t)$ is a single-valued recurrent function of time having a period T, it may be expressed in accord with the above statement as follows:

$$f(t) = A + A_1 \sin \frac{2\pi t}{T} + B_1 \cos \frac{2\pi t}{T} + A_2 \sin \frac{4\pi t}{T} + B_2 \cos \frac{4\pi t}{T}$$
$$+ A_3 \sin \frac{6\pi t}{T} + B_3 \cos \frac{6\pi t}{T} \cdot \cdot \cdot + A_n \sin \frac{2n\pi t}{T} + B_n \cos \frac{2n\pi t}{T},$$

where A, A_1, B_1, A_2, B_2, etc., are constants.

Since the expression $\frac{2\pi t}{T}$ represents a proportion of 2π (or 360° in angular measure), the equation may be written as follows:

$$f(\alpha) = A + A_1 \sin \alpha + B_1 \cos \alpha + A_2 \sin 2\alpha + B_2 \cos 2\alpha \cdot \cdot \cdot$$
$$+ A_n \sin n\alpha + B_n \cos n\alpha.$$

The measure of α at the end of a cycle is 2π.

Evidently $f(\alpha)$ is represented by a summation comprising a constant term plus a series of sine and cosine terms having

* Art. 20.

† Byerly, *Fourier's Series and Spherical Harmonics*, pp. 30 *et seq.*

maximum instantaneous values of A_1, B_1, A_2, B_2, etc., with frequencies of one, two, three, etc., times that of the original function, and with the zero ordinate of each cosine term occurring simultaneously with the maximum ordinate of the sine term of equal frequency. The value of the ordinate of the periodic curve in question at any time, t, or angle, α, is equal to the algebraic sum of the ordinates of all the terms at the particular instant. Any single-valued recurrent curve, no matter how irregular, may be thus resolved into component sinusoids.

a. Examples of Irregular Waves. — Figure 19 shows a curve, S, of alternating voltage, which is composed of two sine waves having respective maximum values A_1 and A_3. The sinusoidal component of the same frequency as the wave is called the **Primary** or **Fundamental harmonic**; remaining component waves may in general be called **Minor harmonics**, or specifically, the **Second**, **Third**, **Fourth**, etc., **harmonics**, depending upon their frequencies when compared with that of the primary wave. In Fig. 19 the primary and third harmonics only are present, so that the Fourier's Series for the curve is

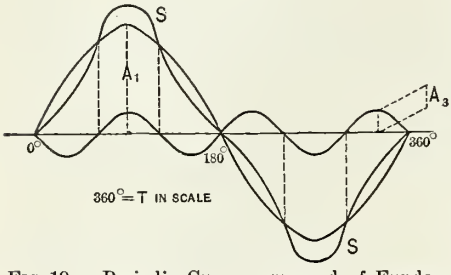


FIG. 19. — Periodic Curve composed of Fundamental and Third Harmonics.

$$f(\alpha) = y = A_1 \sin \alpha + (-A_3) \sin 3\alpha,$$

where y is the ordinate of the curve S for any time, t, or angle, α.

The constant A is zero in this as in all periodic curves having equivalent loops above and below the x axis.* All the other terms except those given are also zero as the sum of the first and third sine harmonics exactly satisfy the curve S. The constant term A_3 is negative because the *negative loop* of the third harmonic starts at zero of time. Figure 20 shows the curve that is formed by the same harmonics, but with the third harmonic moved forward 180°, so that A_3 is positive. Figure 21

* For proof see Art. 13.

shows a curve composed of a primary and a fifth harmonic, with A_1 and A_5 both positive; and Fig. 22 shows a curve derived from the following series :

$$x = A_1 \sin \alpha + A_3 \sin 3\,\alpha + A_5 \sin 5\,\alpha.$$

It is to be observed that the loops of the curves shown are symmetrical on each side of a vertical axis drawn through the

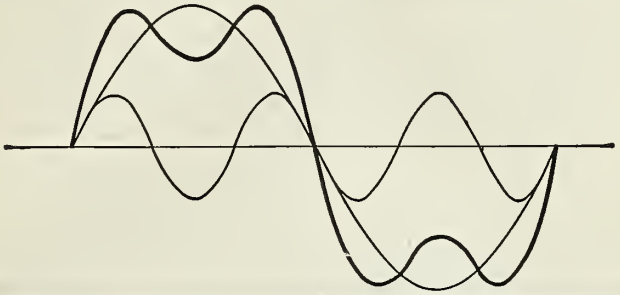


FIG. 20. — Periodic Curve containing Fundamental and Third Harmonics.

axis of abscissas at 90° or 270°. If, now, a cosine term is added to the components of Fig. 19, for instance, as in the expression

$$y = A_1 \sin \alpha - A_3 \sin 3\,\alpha - B_3 \cos 3\,\alpha,$$

which is shown by the curves in Fig. 23, this symmetry of the loops with respect to the vertical axis is destroyed. The loops

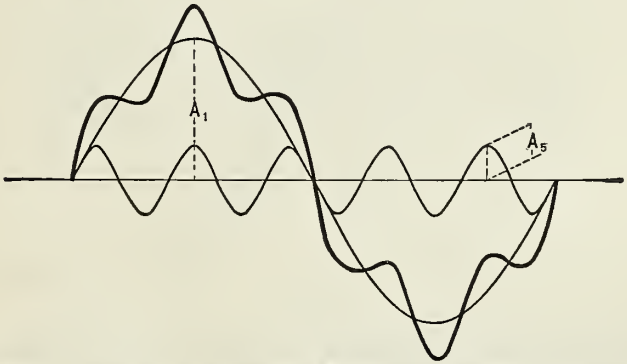


FIG. 21.— Periodic Curve containing Fundamental and Fifth Harmonics.

of this type, as of the preceding curves, are, however, symmetrical one with the other. That is, if the negative loops are revolved 180° on the axis of abscissas so as to stand between the positive loops, all of the loops look just alike, and if one

of the negative loops is then moved along the axis one half a period so as to lie upon one of the positive loops, it will exactly coincide with a positive loop. From the brief description already given of the arrangement of armature windings and of the magnetic field of the ordinary commercial alternator,* it is

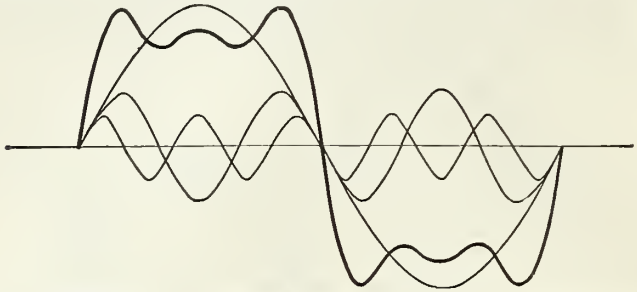


Fig. 22. — Periodic Curve containing Fundamental, Third, and Fifth Harmonics.

seen that the same magnetic lines are cut in the same order by exactly the same arrangement of conductors during the generation of both the positive and negative loops of voltage which are as a result alike in area and form, provided the machine is

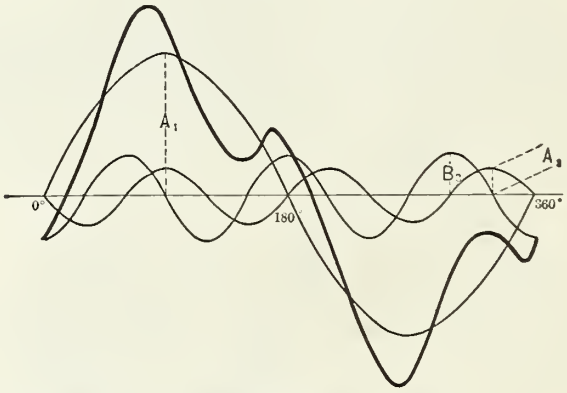


Fig. 23. — Periodic Curve containing Sine and Cosine Terms in the Third Harmonic.

mechanically and magnetically rigid. It will likewise be shown later that the negative and positive current loops also ordinarily have the same characteristic likeness. It is therefore evident that such waves can be expressed by such formulas as have preceded and which include only the odd harmonics.

* Arts. 10 and 11.

If the even harmonics appear, this characteristic symmetry no longer exists. Figure 24 shows a curve answering to the formula:

$$y = A_1 \sin \alpha + A_2 \sin 2\,\alpha.$$

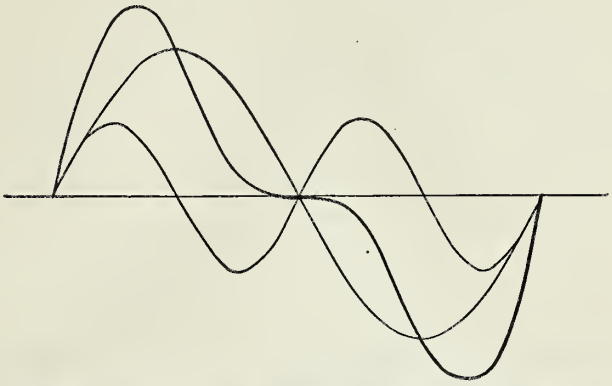


FIG. 24. — Periodic Curve Containing Fundamental and Second Harmonics.

Figure 25 shows a curve compounded of the first, second, and third sine harmonics; and Fig. 26 has first and second sine and second cosine harmonics — the latter two combined into

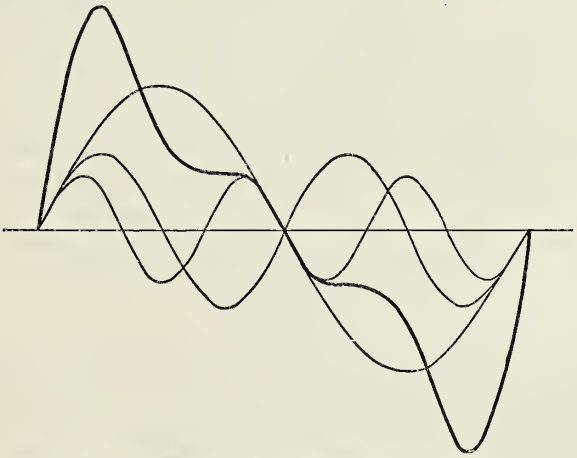


FIG. 25. — Periodic Curve Containing Fundamental, Second, and Third Harmonics.

one curve, which may be done by adding their ordinates algebraically. The resultant irregular curves in these figures do not have positive and negative loops which will superpose by revolution around and sliding along the axis of abscissas as

above described, because for each *half* period of the primary curve the even harmonics complete one or more entire cycles and have relatively opposite effects upon the positive and nega-

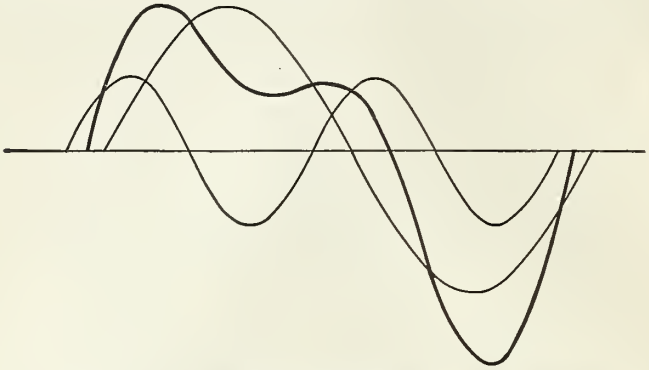


FIG. 26. — Periodic Curve having Second Harmonic which is composed of Sine and Cosine Terms.

tive loops of the primary curve. On the other hand, the odd harmonics keep at all times the same relation to the primary, *i.e.* when the primary reverses, the odd harmonics also reverse.

Prob. 1. Construct the voltage curve expressed by the formula $e = 100 \sin \alpha + 50 \cos \alpha + 30 \sin 3\,\alpha + 20 \cos 3\,\alpha$.

Prob. 2. Construct the curve of current expressed by the formula $i = 200 \sin \alpha - 25 \sin 5\,\alpha + 15 \cos 5\,\alpha$.

Prob. 3. Construct the voltage curve expressed by the formula $e = 200 \sin \alpha + 200 \cos \alpha + 50 \sin 3\,\alpha + 50 \cos 3\,\alpha - 25 \sin 7\,\alpha - 25 \cos 7\,\alpha$.

Prob. 4. Construct the curve represented by the formula $y = 50 + 50 \sin \alpha + 40 \cos \alpha + 30 \sin 2\,\alpha + 20 \cos 2\,\alpha$.

b. Fourier's Series with Sine and Cosine Terms Combined. — Sine and cosine harmonics of equal frequency may be combined into one curve, as explained with reference to Fig. 26, and a sine term is formed thereby having a maximum which may be designated C and which is out of phase with the original sine component harmonic by some angle β_n. As a result Fourier's Series may be expressed in a more general form, thus:

$$f(\alpha) = A + C_1 \sin(\alpha + \beta_1) + C_2 \sin(2\,\alpha + \beta_2) \cdots + C_n \sin(n\alpha + \beta_n);$$

or where β_n' is the angle between the new harmonic and the original cosine component,

$$f(\alpha) = A + C_1 \cos(\alpha - \beta_1') + C_2 \cos(2\alpha - \beta_2') \cdots + C_n \cos(n\alpha - \beta_n').$$

In these expressions $C_n = \sqrt{A_n^2 + B_n^2}$, $\tan \beta_n = \frac{B_n}{A_n}$, and $\tan \beta'_n = \frac{A_n}{B_n}$, since C_n is the resultant of A_n and B_n, which differ from each other in phase by 90°. It is usual in drawing the harmonics of a curve to plot the total harmonics as given by either of these formulas rather than to use their sine and cosine components, which add to the complication.*

13. Determination of the Value of the Constants in Fourier's Series. — The value of Fourier's Series to electrical engineering consists in the possibility of resolving irregular waves of current and voltage into their harmonics and by that means obtaining mathematical expressions to which the elementary laws of electricity can be applied. To do this it is evidently necessary to obtain the numerical values of the constants contained in the formulas.

Having given the formula,

$$f(\alpha) = A + A_1 \sin \alpha + B_1 \cos \alpha + A_2 \sin 2\alpha + B_2 \cos 2\alpha \cdots + A_n \sin n\alpha + B_n \cos n\alpha,$$

an expression for the value of A may be obtained by multiplying both sides of the equation by $d\alpha$, and integrating between 0 and 2π. Thus:

$$\int_0^{2\pi} f(\alpha)\, d\alpha = A\int_0^{2\pi} d\alpha + A_1\int_0^{2\pi} \sin \alpha d\alpha + B_1\int_0^{2\pi} \cos \alpha d\alpha \cdots + A_n\int_0^{2\pi} \sin n\alpha d\alpha + B_n\int_0^{2\pi} \cos n\alpha d\alpha.$$

* A theoretical discussion of Fourier's Series will be found in the first three chapters of Byerly, *Fourier's Series and Spherical Harmonics;* J. W. Mellor, *Higher Mathematics*, Chap. VIII; Merriman and Woodward, *Higher Mathematics;* and similar works. The following articles are of considerable interest in this connection: Periodic Functions Developed in Fourier Series; The Graphical Method, by Professor John Perry, *London Electrician*, Vol. 35, p. 285; Wave Form Synthesis, *London Electrician*, Vol. 35, p. 257; *Trans. Amer. Inst. Elect. Eng.*, Vol. 12, p. 476; Fleming's *Alternate Current Transformer in Theory and Practice*, Vol. II, p. 454; Graphical Analysis of Harmonic Curves, by Wedmore, *London Electrician*, Vol. 35, p. 512; Approximate Method, Houston and Kennelly, *Electrical World*, Vol. 16, p. 580; Graphical Computations in Alternating Current Circuits, *Éclairage Électrique*, Vol. 16, p. 397.

All the terms on the right, except the first, reduce to zero, as may be shown by integrating the general sine and cosine terms as follows:

$$A_n\int_0^{2\pi}\sin n\alpha d\alpha = A_n\left(-\frac{1}{n}\cos n\alpha\right)_0^{2\pi} = 0.$$

$$A_n\int_0^{2\pi}\cos n\alpha d\alpha = A_n\left(+\frac{1}{n}\sin n\alpha\right)_0^{2\pi} = 0.$$

Therefore $$A = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, d\alpha.$$

The integral $\int_0^{2\pi} f(\alpha)\, d\alpha$ is the net area of the curve for an entire period, and 2π is its base; so that A is the average ordinate for an entire period. Since, as has already been explained, the positive and negative loops in the ordinary alternating electric voltage and current curves are closely alike, $\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, d\alpha$ reduces to zero because the area of the positive loop in a period is equal to the area of the negative loop. The constant A, therefore, need not be included in writing the series for computations of such quantities; that is, the axis of abscissas is placed symmetrically with respect to the positive and negative areas.

To obtain A_1 multiply the series by $\sin\alpha d\alpha$ and integrate between the same limits as before. Thus $\int_0^{2\pi} f(\alpha)\sin\alpha d\alpha$

$$= A\int_0^{2\pi}\sin\alpha d\alpha + A_1\int_0^{2\pi}\sin^2\alpha d\alpha + B_1\int_0^{2\pi}\cos\alpha\sin\alpha d\alpha \cdots$$
$$+ A_n\int_0^{2\pi}\sin n\alpha\sin\alpha d\alpha + B_n\int_0^{2\pi}\cos n\alpha\sin\alpha d\alpha.$$

Each of the terms in the right-hand member of the equation except the second term is of the form of one of the following:

$$\int_0^{2\pi}\sin\alpha d\alpha,\quad \int_0^{2\pi}\sin\alpha\cos\alpha d\alpha,$$

$$\int_0^{2\pi}\sin p\alpha\sin q\alpha d\alpha,\quad \int_0^{2\pi}\sin p\alpha\cos q\alpha d\alpha,$$

$$\int_0^{2\pi}\sin p\alpha\cos p\alpha d\alpha,\ \text{or}\quad \int_0^{2\pi}\cos p\alpha\cos q\alpha d\alpha,$$

where p and q are unequal integers. As is well known, these expressions reduce to zero. This is equivalent to saying that

any curve which may be represented by one of these expressions incloses as much negative area below the axis of abscissas as positive area above the axis, in any length equal to one period measured along that axis. The rule also applies for full periods, when, of p and q, one is an integer and the other differs from an integral number by one half.

The second term in this equation is of the form

$$\int_0^{2\pi} \sin^2 p\alpha d\alpha,$$

where p is any integer. This reduces to π, as also does the analogous expression

$$\int_0^{2\pi} \cos^2 p\alpha d\alpha.$$

The original equation therefore reduces to

$$\int_0^{2\pi} f(\alpha) \sin \alpha d\alpha = \pi A_1,$$

and

$$A_1 = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \sin \alpha d\alpha.$$

A like process in which the original equation is multiplied by $\cos \alpha d\alpha$, and reduced to find the expression for B_1 by $\sin 2\, \alpha d\alpha$, and reduced to find the expression for A_2, etc., shows that the expressions representing the various constants are as follows:

$$B_1 = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \cos \alpha d\alpha,$$

$$A_2 = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \sin 2\, \alpha d\alpha,$$

$$B_2 = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \cos 2\, \alpha d\alpha,$$

.

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \sin n\alpha d\alpha,$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \cos n\alpha d\alpha.$$

14. Method of Finding the Harmonics of a Periodic Curve by Fourier's Series. — If $f(\alpha)$ (that is, each ordinate of the recurrent curve) is dependent upon α by some known law, its value can usually be incorporated into the formulas for the constants given above and the integration performed. For

instance, such would be the case if the loops of the curve were rectangular. In the case of current or voltage loops such a relation between the ordinates and abscissas of the curve is not apt to exist, so that it is necessary to change the integral sign into a summation sign and obtain the result as accurately as time will permit or necessity require.

Thus erect m equally spaced ordinates on the axis of abscissas of an experimentally determined curve, beginning at the inception of a period. The ordinates divide the base line into $m-1$ equal divisions in the period. The lengths of the ordinates, from the axis of the abscissas to the curve, may be called e_0, e_1, $\cdots e_m$. The base of one complete period being 2π radians in

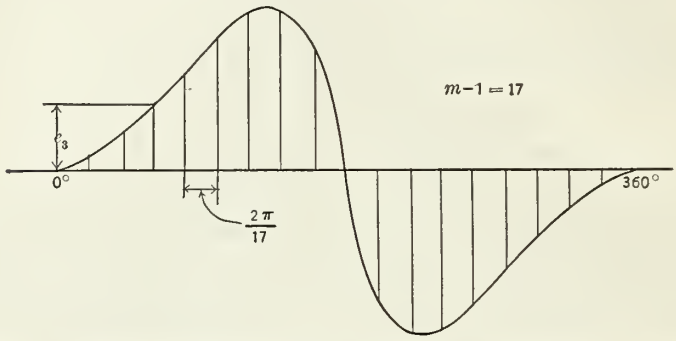


Fig. 27. — Showing Periodic Wave divided into Elementary Areas for obtaining Harmonics.

length, the distance between any adjacent two of the equally spaced ordinates is $\frac{2\pi}{m-1}$ radians (see Fig. 27). Then the value $A_1 = \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\sin\alpha d\alpha$ can be written approximately

$$A_1 = \frac{2}{m-1}\sum_1^m e_k \sin\left(k\frac{2\pi}{m-1}\right),$$

where e_k represents the ordinate of the curve or $f(\alpha)$ at any particular division point, k the designating number of the corresponding ordinate, and $\frac{2\pi}{m-1}$ distance between adjacent ordinates.

Hence,

$$A_1 = \frac{2}{m-1}\left[e_1\sin\left(\frac{2\pi}{m-1}\right) + e_2\sin\left(2\frac{2\pi}{m-1}\right)\cdots + e_m\sin\left(m\frac{2\pi}{m-1}\right)\right].$$

where e_1, e_2, etc., are the ordinates of the curve at the division points, 1, 2, 3, $\cdots$ m, and in general

$$A_n = \frac{2}{m-1}\left[e_1 \sin\left(n\frac{2\pi}{m-1}\right) + e_2 \sin\left(2n\frac{2\pi}{m-1}\right) \cdots + e_m \sin\left(nm\frac{2\pi}{m-1}\right)\right],$$

$$B_n = \frac{2}{m-1}\left[e_1 \cos\left(n\frac{2\pi}{m-1}\right) + e_2 \cos\left(2n\frac{2\pi}{m-1}\right) \cdots + e_m \cos\left(nm\frac{2\pi}{m-1}\right)\right].$$

The positive and negative loops of ordinary alternating currents and voltages are similar, and for them it is only necessary to divide up one loop; and for the same reason even harmonics of the series are commonly unimportant. The larger m is made, the more accurate will be the result; likewise the accuracy is increased by making n large, though it is seldom necessary to obtain more than the first, third, and fifth harmonic.

Having found A_1, A_3, etc., and B_1, B_3, etc., the values of C_1, C_3, and β_1, β_3 may be at once obtained for substitution into the general formula.* Having obtained these values, the harmonics represented by each term of the series can, if desired, be plotted with the periodic curve to which it belongs. It must be remembered in doing this that $\alpha = 0$ when the irregular curve crosses the X axis in an upward or positive direction.

The values of the constants may be obtained approximately by the use of a planimeter instead of by the laborious computation involved in solving the foregoing equations. Measure the ordinates e_1, e_2, e_3, $\cdots$ e_m as before and multiply each by the sine of the proper angle as shown in the formula of the desired constant (A_1 or A_2, or B_1 or B_2, etc.). Plot a curve over the points 1, 2, 3, 4, $\cdots$ m with ordinates equal to these products, that is, place the ordinate $e_1 \sin\left(n\frac{2\pi}{m-1}\right)$, or $e_1 \cos\left(n\frac{2\pi}{m-1}\right)$, over the point 1, $e_2 \sin\left(2n\frac{2\pi}{m-1}\right)$, or $e_2 \cos\left(2n\frac{2\pi}{m-1}\right)$, over the point 2, and so on (see curve z in Fig. 28). Measure the areas of the resultant loops by means of a planimeter and add them algebraically, considering areas above the axis of abscissas as positive and those below the axis as negative. If m is sufficiently large, this area, U, will be with sufficient accuracy

* Art. 12 *b*.

$$U = \int_0^{2\pi} f(\alpha) \sin n\alpha d\alpha \text{ or } \int_0^{2\pi} f(\alpha) \cos n\alpha d\alpha.$$

Hence, $$A_n = \frac{1}{\pi} \int_0^{2\pi} f(\alpha) \sin n\alpha d\alpha = \frac{U}{\pi}.$$

In Fig. 28 is shown a recurrent curve, X, and the process of obtaining A_3 is indicated. To do this, 37 values of e and the corresponding 37 values of $\sin 3\alpha$ were used, α being respectively 0, $\frac{2\pi}{36}$, $2\frac{2\pi}{36}$, etc., at the successive points. These values multiplied into values of e, e_1, e_2, etc., and plotted on the corresponding ordinates give the points on the curve Z. If the areas of the positive and negative loops of curve Z are meas-

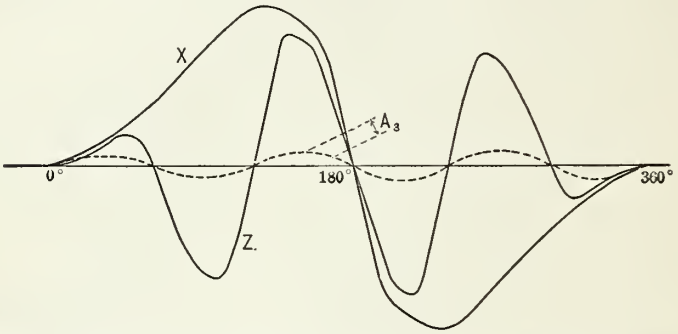


Fig. 28. — Product of an Alternating-current Curve with Sine Third Harmonic Analyzer. x is the Irregular Curve, and z the Curve of Products.

ured by a planimeter, it will be found that there is a small positive area in excess. The constant A_3 is therefore positive; *i.e.* the sine component of the third harmonic starts at zero with a positive loop, and has a maximum height equal to twice the length obtained by dividing the algebraic sum of the areas of the loops of curve Z by the length of the base for the period. This is readily converted into volts by applying the scale used in plotting the values of e. The dotted line in Fig. 28 shows the harmonic plotted in position and to scale.

The other constants may be obtained by following a similar process. While the foregoing example is applied to a complete period of the curve, it is sufficient to carry out the analysis on a single loop or half period of ordinary alternating current or voltage waves which have half periods alike, as already pointed out.

14 a. Components of Alternating Curves do not always fall within the Fourier Series. — The foregoing articles deal with the harmonic analysis of alternating voltage and current curves on the hypothesis that the components of such curves are all of frequencies represented by integral numbers of times the primary frequency, and this is an accurate hypothesis with respect to curves produced under perfectly stable recurrent conditions, such as voltage curves induced by the uniform rotation of the armature of a synchronous generator constructed with perfect mechanical rigidity and possessing a perfectly rigid magnetic field. But these conditions do not always exist and sometimes are departed from widely. The rotation velocity may lack uniformity, the shaft of the armature may spring and alter the magnetic conditions, and other occurrences may likewise introduce influences into the voltage curve that are independent of the primary frequency. Also, even though the voltage wave should be free from such effects, variations in the circuit through which the current flows may introduce distortions into the current wave that are independent of the primary frequency.

Variations of the kind here referred to are generally not strictly recurrent, but this may not be indicated by the analysis of a single period of the curve and is not likely to be indicated by the analysis of a half period. As such variations are likely to cause the successive loops to differ in form and area, the analysis of several periods is likely to indicate their existence and magnitude. They are generally too small a factor in most of the existing commercial conditions to make them of serious import, but in case of necessity they may be treated like interference phenomena which introduce recurrent effects embracing several or many of the periods of the alternating curve in question and have a frequency which is some fractional multiple of the primary frequency.

15. Concrete Examples of the Resolution of an Alternating-current Curve into its Harmonic Components. — The positive loop of a voltage wave of an alternator is represented by the heavy line of Fig. 29, and the constants of Fourier's Series for this curve have been determined up to the seventh harmonic, giving the values,

$A_1 = +98.6,$ $B_1 = -14.7,$

$A_3 = -13.3,$ $B_3 = +18.2,$

$$A_5 = -\ 1.6, \qquad B_5 = -\ 4.8,$$
$$A_7 = +\ 0.56, \qquad B_7 = +\ 1.2.$$

The equation for the curve as determined by this means is

$$e = 98.6 \sin \alpha - 13.3 \sin 3\alpha - 1.6 \sin 5\alpha + 0.56 \sin 7\alpha$$
$$- 14.7 \cos \alpha + 18.2 \cos 3\alpha - 4.8 \cos 5\alpha + 1.2 \cos 7\alpha.$$

Substituting various values of α in this equation, the corresponding values of e are given, and the corresponding curve, which is dotted, has been plotted in the figure. It will be noticed that the calculated curve very closely approximates to

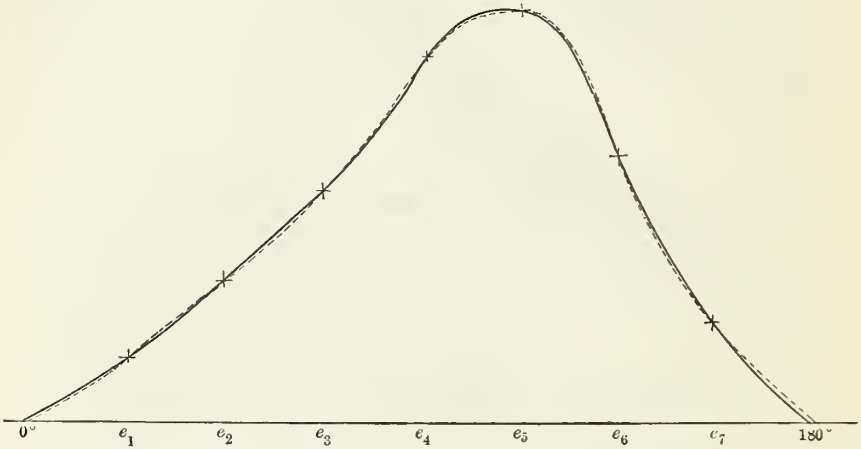


FIG. 29. Irregular Curve of Voltage, and Calculated Curve composed of the First, Third Fifth, and Seventh Harmonics.

the exact form of the original curve. If a larger number of constants had been determined, the calculated curve would have crossed the original curve a larger number of times, and the approximation would have been still closer. The calculated curve must coincide with the original curve at each ordinate which was used in the computation. The calculated curve cannot, except in special cases, exactly coincide with the original curve unless $m = \infty$. The series used is rapidly convergent, and in this particular curve the effect of the fifth and seventh harmonics is quite small, so that the curve is sufficiently well represented for practical purposes by the fundamental and third harmonics, in which case the equation is

$$e = 98.6 \sin \alpha - 13.3 \sin 3\alpha - 14.7 \cos \alpha + 18.2 \cos 3\alpha.$$

The corresponding sine and cosine terms of the series,

$$\left.\begin{matrix} A_1 \sin \alpha \\ + B_1 \cos \alpha \end{matrix}\right\} + \left\{\begin{matrix} A_3 \sin 3\alpha \\ + B_3 \cos 3\alpha \end{matrix}\right\} + \left\{\begin{matrix} A_5 \sin 5\alpha \\ + B_5 \cos 5\alpha \end{matrix}\right\} + \left\{\begin{matrix} + \text{etc.}, \\ + \text{etc.}, \end{matrix}\right.$$

may be conceived as representing the rectangular sine components of the terms of a single sine or cosine curve.* The equation when reduced to the more general form (using β) has the following constants:

$$\begin{aligned} C_1 &= 99.7, & \beta_1 &= -\ 8^\circ\ 29', \\ C_3 &= 22.5, & \beta_3 &= -\ 53^\circ\ 50', \\ C_5 &= 5.1, & \beta_5 &= +\ 71^\circ\ 34', \\ C_7 &= 1.3, & \beta_7 &= +\ 66^\circ, \end{aligned}$$

and the equation is

$$e = 99.7 \sin (\alpha - 8^\circ\ 29') - 22.5 \sin (3\alpha - 53^\circ\ 50') - 5.1 \sin (5\alpha + 71^\circ\ 34') + 1.3 \sin (7\alpha + 66^\circ),$$

and its value to a considerable degree of approximation is

$$e = 99.7 \sin (\alpha - 8^\circ\ 29') - 22.5 \sin (3\alpha - 53^\circ\ 50').$$

The example which has been taken fairly represents the complexity of the average distorted waves. Some alternating-current curves are so greatly distorted that larger numbers of terms of the series are required to closely represent them, but for practical purposes three or four terms are generally sufficient. In a large number of waves the forms are so simple that two terms of the series give sufficient approximation for practical purposes.

Following are examples of the calculation of the constants of these equations:

$$m = 9, \quad \left(\frac{180^\circ}{m-1}\right) = 22\tfrac{1}{2}^\circ.$$

Values of e from curve by measurement:

$$\begin{aligned} e_1 &= 18, & e_3 &= 70, & e_5 &= 125, & e_7 &= 30. \\ e_2 &= 42, & e_4 &= 110, & e_6 &= 80, & e_0 &= e_8 = 0. \end{aligned}$$

$$A_1 = \tfrac{1}{4}\left\{\begin{matrix} 18 \sin 22\tfrac{1}{2}^\circ + 42 \sin 45^\circ + 70 \sin 67\tfrac{1}{2}^\circ + \cdots \\ + 30 \sin 157\tfrac{1}{2}^\circ \end{matrix}\right\} = 98.6,$$

$$A_3 = \tfrac{1}{4}\left\{\begin{matrix} 18\sin 67\tfrac{1}{2}^\circ + 42\sin 135^\circ + 70\sin 202\tfrac{1}{2}^\circ \\ + \cdots + 30\sin 112\tfrac{1}{2}^\circ \end{matrix}\right\} = -13.3,$$

$$B_3 = \tfrac{1}{4}\left\{\begin{matrix} 18\cos 67\tfrac{1}{2}^\circ + 42\cos 135^\circ + 70\cos 202\tfrac{1}{2}^\circ + \cdots \\ + 30\cos 112\tfrac{1}{2}^\circ \end{matrix}\right\} = 18.2,$$

$$C_3 = \sqrt{A_3^2 + B_3^2} = 22.5,\ \beta_3 = \tan^{-1}\frac{B_3}{A_3} = -53^\circ\,50.'$$

As C is a second root, it is algebraically either positive or negative, but the conditions of the problem require it to enter the equation with the algebraic sign pertaining to the corresponding A.

Prob. 1. Find the primary and third harmonics of an equilateral triangular voltage wave having an altitude of 10 volts.

Prob. 2. Find the primary and third harmonics of a voltage wave which is in the form of a semicircle with a diameter of 10.

16. Effective Value of an Irregular Curve of Voltage or Current. — The effective ordinate of an alternating-current curve may be determined by integrating directly from its equation. The effective value squared is

$$E^2 = \text{Av.}e^2 = \frac{1}{\pi}\int_0^\pi e^2 d\alpha$$

$$= \frac{1}{\pi}\int_0^\pi (A_1\sin\alpha + A_3\sin 3\alpha + \text{etc.} + B_1\cos\alpha + B_3\cos 3\alpha + \text{etc.})^2\, d\alpha;$$

and since $\int_0^\pi \sin p\alpha \sin q\alpha d\alpha$, $\int_0^\pi \sin p\alpha \cos q\alpha d\alpha$, $\int_0^\pi \cos p\alpha \cos q\alpha d\alpha$. $\int_0^\pi \sin px \cos pxd\alpha$ are zero when p and q are unequal integers, there results

$$E^2 = \frac{1}{\pi}\int_0^\pi e^2 d\alpha = \frac{A_1^2}{\pi}\int_0^\pi \sin^2\alpha d\alpha + \frac{A_3^2}{\pi}\int_0^\pi \sin^2 3\alpha d\alpha$$
$$+ \frac{A_5^2}{\pi}\int_0^\pi \sin^2 5\alpha d\alpha + \text{etc.} + \frac{B_1^2}{\pi}\int_0^\pi \cos^2\alpha d\alpha$$
$$+ \frac{B_3^2}{\pi}\int_0^\pi \cos^2 3\alpha d\alpha + \frac{B_5^2}{\pi}\int_0^\pi \cos^2 5\alpha d\alpha + \text{etc.}$$

But $\int_0^\pi \sin^2 p\alpha d\alpha$ and $\int_0^\pi \cos^2 p\alpha d\alpha$ are always equal to $\frac{\pi}{2}$;

hence, $$E^2 = \frac{A_1^2}{2} + \frac{A_3^2}{2} + \frac{A_5^2}{2} + \text{etc.} + \frac{B_1^2}{2} + \frac{B_3^2}{2} + \frac{B_5^2}{2} + \text{etc.},$$

and $$E = \left\{\sum_{K=1}^{K=n}\frac{A^2}{2} + \sum_{K=1}^{K=n}\frac{B^2}{2}\right\}^{\frac{1}{2}} = \left\{\sum_{K=1}^{K=n}\frac{C^2}{2}\right\}^{\frac{1}{2}}.$$

The effective value of the curve is therefore independent of β. The area and the mean ordinate of the curve, on the other hand, are not independent of β, which, of course, depends on the relative magnitude and algebraic signs of the sine and cosine functions. That is, the effective value of the current depends only on the amplitudes of the harmonics, and is not influenced by their relative positions; but the average value depends jointly upon the amplitudes and relative positions of the harmonics.

The following table gives various ratios relating to periodic curves of various forms with known equations or which are illustrated in various figures in this book:

TABLE

CHARACTERISTIC FEATURES OF DIFFERENT FORMS OF ALTERNATING-CURRENT AND VOLTAGE CURVES

Name of Curve	Ratio of Average to Maximum Value	Ratio of Effective to Maximum Value	Ratio of Maximum to Effective Value	Ratio of Effective to Average Value	Area of Squared Curve corresponding to Unit Maximum Value
Triangle	.500	.577	1.732	1.155	.333
Sinusoid	.637	.707	1.414	1.112	.500
Parabola	.666	.730	1.369	1.096	.533
Semicircle	.785	.816	1.225	1.040	.667
Rectangle	1.000	1.000	1.000	1.000	1.000

Prob. 1. Find the effective value of a voltage having a wave expressed by the formula $e = 50 \sin(\alpha + 30°) + 30 \sin(3\alpha + 60°) + 20 \sin(5\alpha + 45°)$.

Prob. 2. Find the effective value of a current curve expressed by the formula $i = 200 \sin(\alpha + 90°) - 50 \sin(3\alpha + 20°)$.

17. Alternating Current Voltmeters and Amperemeters measure Effective Volts and Amperes. — Alternating voltages and currents cannot be measured by instruments depending upon permanent magnets, so that instruments for such purposes usually fall within three general classes:*

1. *Electromagnetic* instruments, depending on the magnetic attractions of two coils, one fixed and the other movable,

* Jackson's *Elementary Electricity and Magnetism*, Chaps. VIII and XIV.

carrying the current to be measured; or of a fixed coil or coils and a small movable vane-like bit of iron or disk of conducting metal. The coils may or may not be provided with soft iron cores, with the iron carefully subdivided to prevent excessive eddy currents.

2. *Hot wire* instruments, in which the expansion of a wire due to the heating effect of a current is used to obtain a deflection.

3. *Electrostatic* instruments, depending upon electrostatic forces for obtaining a deflection, as in an electrometer.

In the first two classes the force tending to deflect a needle, suitably attached, is proportional to the square of the current flowing in the coil or wire, and hence is also proportional to the square of the voltage at the terminals of the instrument when it is arranged for use as a voltmeter, except in the cases of instruments with iron cores or vanes, in which the forms of pole pieces or the saturation of the iron may be caused to modify the force. In the third class the tendency of the needle to deflect is proportional to the square of the voltage impressed upon the terminals; and it is proportional to the square of the current flowing through a shunt across the terminals of the instrument, if it is arranged with a shunt for the purpose of using it as an amperemeter.

When an instrument of either of these types is inserted in an alternating-current circuit, it tends at each instant to deflect its needle by a force proportional to the square of the instantaneous current flowing through it or instantaneous voltage impressed upon its terminals. The deflection of the needle is resisted by a spring or like device affording a resisting force proportional to the extent of movement. On account of the great frequency of commercial alternating-current circuits the resultant deflection of the needle is ordinarily constant and is proportional to the average of the squared instantaneous values of the current or voltage. The square root of this deflection is proportional to the effective amperes or volts impressed upon the instrument.

Such an instrument can therefore be calibrated to read effective amperes or volts. When the instrument is so calibrated, it will read direct or alternating currents or voltages

on the same scale, since the force causing the needle's deflection is in every case proportional to the average of the squares of the instantaneous values of the current or voltage. It is further evident, for the same reason, that instruments of these types will correctly measure alternating currents or voltages whatever may be the forms of their waves, provided that modifying influences from iron cores or similar extraneous influences are not allowed to creep in.

Direct currents and voltages and effective alternating currents and voltages are uniformly indicated throughout this book by the same symbols, I and E, since they are equivalent in their electro-dynamic or heating effects.

18. The Effective Voltage developed by an Alternator. — In Art. 11 it is shown that the instantaneous voltage of a multipolar armature generating a sine voltage curve is

$$e = \frac{\pi p S \Phi V}{2 \times p' \times 10^8 \times 60} \sin \alpha = \frac{\pi S \Phi f}{p' \times 10^8} \sin \alpha,$$

and that then the maximum voltage is

$$e_m = \frac{\pi p S \Phi V}{2 \times p' \times 10^8 \times 60} = \frac{\pi S \Phi f}{p' \times 10^8}.$$

To obtain the effective voltage of a machine which fulfills the conditions of this formula it is only necessary to divide by the square root of 2, since that is the ratio between the maximum and effective values of the sine voltage curve. Therefore,

$$E = \frac{e_m}{\sqrt{2}} = \frac{1.11\, p S \Phi V}{p' \times 10^8 \times 60} = \frac{2.22\, S \Phi f}{p' \times 10^8}.$$

The numerical coefficient in this formula is accurate only for a machine with very narrow coils and with such a distribution of magnetism that the voltage curve is of the sine form. It is, however, correct within a few per cent for a large proportion of the alternators of the present day, as will be seen later.*

Prob. 1. What is the voltage generated by an alternator having 500 active conductors in series, a magnetic flux from each pole of 2,000,000 lines of force, a speed of 1200 revolutions per minute, and five pairs of poles, supposing the formula above to be applicable?

* Art. 20.

Prob. 2. The magnetic field of an alternator has been designed with six pairs of poles so that 2,000,000 lines of force emanate from each pole. The armature is to run at 500 revolutions per minute and generates a sinusoidal voltage. What must be the number of conductors in series to give 1000 volts?

19. Comparison of the Voltages developed by an Alternator and a Direct-current Dynamo. — In multipolar direct current dynamos

$$E = \frac{p S \Phi V}{p' \times 10^8 \times 60}.$$

Assuming two machines in which S, p, p', Φ, and V are alike, one of which has the armature conductors uniformly spaced over the armature surface and produces direct currents, and the other of which has the conductors arranged in a narrow coil and produces alternating currents, the formulas show that the *average* voltage developed in the alternator armature is equal to that developed in the direct-current armature, but the *effective* voltage of the alternator should be greater than the average voltage. Calling the ratio of effective to average voltage k, it is seen that the effective voltage of the alternator is k times that of the direct-current machine under the conditions represented by the formulas. The ratio of effective to average voltage when the curve is sinusoidal is, as already shown,

$$\frac{1}{\sqrt{2}} \div \frac{2}{\pi} = 1.11.$$

Hence, the value of k is 1.11.

The foregoing value of k is computed on the hypothesis of a uniform magnetic field for the alternator, corresponding to that shown in Fig. 12, but the value of k in commercial alternators depends upon the ratio which the width of the poles bears to the polar pitch (distance between poles, center to center), the form of the **Polar surfaces** and the distribution of the magnetism thereover. It has been shown * to have a minimum limit of unity and a maximum limit which may be large, depending upon the form of the voltage wave set up in the individual armature conductors.

* Art. 11.

20. The Effect of Distributed Coils on Armature Voltage. — It is now well to examine the relative sizes of the two armatures compared above, and the effect of the arrangements of the windings upon their voltages. Thus far the alternator windings have been assumed to be in narrow coils, or arranged so that at each instant all of the conductors are equally effective. This is not the case in actual machines, as the coils must have appreciable width. If two collector rings, C and C_1, are placed upon the shaft of a direct-current armature, such as a Gramme ring mounted to rotate in a bipolar field, and are connected to armature conductors which are in opposite coils (Fig. 30), an alternating current may be taken from the rings. Such an arrangement is called a **Double-current machine.** The voltage between the rings has its maximum value when the conductors connected to the rings are under the points of commutation, and the voltage is zero when the armature has revolved 90°.*

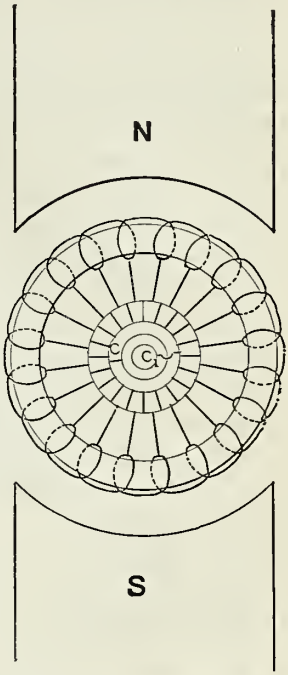


FIG. 30. — Gramme Ring in Two-pole Field, connected to Collector Rings.

The maximum of the alternating voltage is manifestly equal to the direct voltage for which the armature was designed, and the effective alternating voltage is .707 times the direct voltage if the field is uniform and the voltage wave sinusoidal. In commercial machines the field is not uniform, but it is not likely to be sufficiently irregular to materially disturb the ratio of the direct and effective alternating voltages when such an armature is carrying little current. When the armature carries considerable current, armature reactions may disturb the relations to a greater or less degree. An armature thus arranged, with the conductors uniformly distributed over its surface, therefore gives substantially a sine alternating voltage, and the value of k would be 1.11 except for the distribution of the conductors

* Jackson's *Electromagnetism and the Construction of Dynamos*, p. 90.

over the armature surface; but the effective alternating voltage is only .707 times that of the direct voltage developed by the same conductors. The question at once arises as to the cause of this loss of 40 per cent or more. A little consideration shows that the Gramme ring acts like two broad coils in parallel, which cover the whole armature and unite at the points where the collecting rings are connected. When in the position of zero voltage, the two halves of each coil are so located in the field as to cut lines of force in opposite directions, and hence the voltage is zero. Similar opposition but to less degree is found in the case of windings which cover only a portion of the armature core, the extent of the effect depending upon the ratio of the width and pitch of the poles to the width of the coils. It is largely avoided if the coils are not distributed over a greater width than the distance between the pole tips. Economy of material and the form of the voltage wave are also factors in determining the width the coils should have.

On account of the differential action of distributed coils which is here described, it is necessary to include another constant in the formula for the voltage developed by alternating-current armatures. Calling this constant k', and replacing the product $k'k$, by K, the formula for the voltage of an alternator armature becomes

$$E = \frac{KpS\Phi V}{p' \times 10^8 \times 60} = \frac{2KS\Phi f}{p' \times 10^8}.$$

Kapp states that K varies from .29 to 1.15,* but in the greater number of commercial cases it is between 1.00 and 1.11.

Instead of having the windings laid together upon the surface of the armature, it is usual to have them laid in slots as shown in Fig. 31, where the winding is distributed in four slots per pole. If the winding is progressive, as in the Gramme ring, the wires in different slots will set up voltage waves with their maxima separated by the angular distance between the centers of the slots, as illustrated in Fig. 32, where each slot is supposed to contain the same number of conductors, and the whole width covered by the winding is equal to one half the pitch. The resultant curve, due to connecting the conductors all in series, may be found by adding the instantaneous ordinates together

* Kapp's *Dynamos, Alternators, and Transformers*, p. 374.

at each abscissa. If the individual curves are sinusoids, the summation is of this form:

$$e = e_{sm}\{\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+(n-1)\beta)\},$$

where e is the instantaneous voltage of the resultant curve, e_{sm} is the maximum ordinate of any one of the individual curves, α is the angular distance from the point where the first

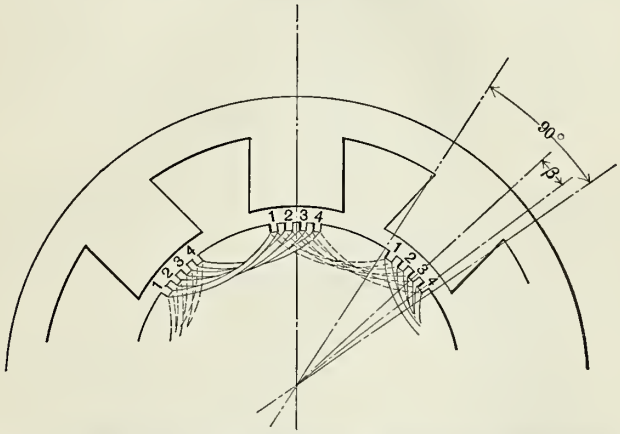


FIG. 31. — Multipolar Armature with Winding distributed in Four Slots. Dotted Lines show Cross Connections at the Rear.

curve cuts the x axis, β is the angular distance between slots, center to center, measured in electrical degrees, and n is the number of individual curves composing the resultant curve. By trigonometry this summation becomes*

$$e = e_{sm}\sin(\alpha + \frac{n-1}{2}\beta)\sin\frac{n\beta}{2}\operatorname{cosec}\frac{\beta}{2}.$$

For the purpose of finding the value of α at which the resultant curve has its maximum height, the first derivative of this function with respect to α may be equated to zero and reduced to

$$\cos\alpha\cos\frac{n-1}{2}\beta - \sin\alpha\sin\frac{n-1}{2}\beta = 0,$$

or,

$$\tan\alpha = \cot\frac{n-1}{2}\beta.$$

* Hobson's *Plane Trigonometry*, 2d ed., p. 89.

From this the value of α required to give the maximum ordinate of the resultant curve is found to be

$$\alpha_m = \frac{\pi}{2} - \frac{n-1}{2}\beta,$$

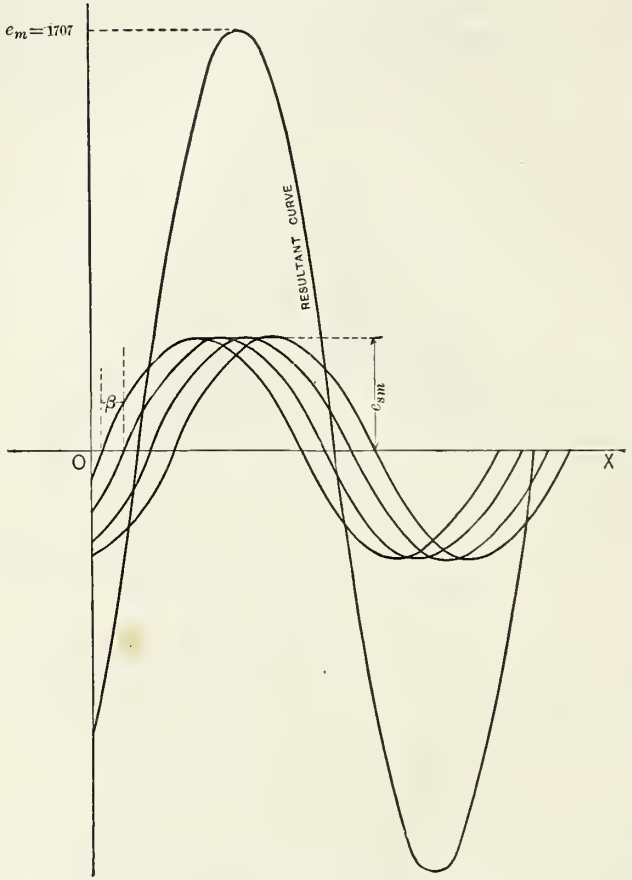


FIG. 32.—Curve of Voltage set up in the Separate Slots of Progressive Distributed Windings, and the Resultant Total Voltage.

and substituting this value of α in the formula for e gives

$$e_m = e_{sm} \sin\left(\frac{\pi}{2} - \frac{n-1}{2}\beta + \frac{n-1}{2}\beta\right) \sin\frac{n\beta}{2} \operatorname{cosec}\frac{\beta}{2}$$

$$= e_{sm} \frac{\sin(n\beta/2)}{\sin(\beta/2)}.$$

The effective voltage is therefore

$$E = \frac{e_{sm}}{\sqrt{2}} \frac{\sin (n\beta/2)}{\sin (\beta/2)}.$$

If $E' = \frac{ne_{sm}}{\sqrt{2}}$ is the voltage that would have been set up if all the wires had been in a single narrow slot, the formula becomes

$$E = \frac{E'}{n} \frac{\sin (n\beta/2)}{\sin (\beta/2)},$$

but E' is the voltage given by the formula in Art. 18; hence,

$$E = \frac{2.22 S\Phi f}{p' \times 10^8} \times \frac{\sin (n\beta/2)}{n \sin (\beta/2)}.$$

Evidently K of Art. 20, in this case, is equal to $\frac{1.11 \sin (n\beta/2)}{n \sin (\beta/2)}$, k and k' being respectively equal to 1.11 and $\frac{\sin (n\beta/2)}{n \sin (\beta/2)}$, in any instance in which the conductor voltages are of sinusoidal form.

Suppose there are four slots equally distributed through 90 electrical degrees in an alternator armature with the conductors of one coil equally distributed in those slots; then $n = 4$ and $\beta = 22\frac{1}{2}°$, whence

$$k' = \tfrac{1}{4} \sin 45° \operatorname{cosec} 11° 15' = .906 \text{ (approx.)}$$

and $K = 1.11 \times .906 = 1.006$ (approx.), provided the conductor voltages are of sinusoidal form. Suppose now that there are a total of 500 active conductors in series, and equally distributed in the slots, 2,000,000 lines of force per pole, and that the frequency is 60 periods per second, then

$$E = \frac{2 KS\Phi f}{p' \times 10^8} = \frac{2 \times 1.006 \times 500 \times 2{,}000{,}000 \times 60}{1 \times 10^8} = 1207 \text{ volts.}$$

Figure 31 shows the slots for a winding such as contemplated in the problem above, and Fig. 32 shows the voltage curves.

The following table gives the value of K for various proportions of the armature core covered with winding and various numbers of slots per pole up to four, assuming sinusoidal conductor voltages.

TABLE I

VALUES OF K FOR PROGRESSIVELY DISTRIBUTED WINDINGS

GROUP OF SLOTS LOCATED IN	NUMBER OF SLOTS IN A GROUP			
	1	2	3	4
45°	1.11	1.09	1.085	1.08
60°	1.11	1.075	1.07	1.065
90°	1.11	1.025	1.015	1.01
135°	1.11	.925	.895	.885
180°	1.11	.78	.735	.725

In a single phase winding, *i.e.* an armature having a single winding, any angular width for the group of slots may be used, though about 90° is common. Where there are two independent circuits on the armature, 90° is also used; while in three-phase circuits, where there are three independent circuits, 60° is common.*

The values given for K in the table can be used with sufficient accuracy for most commercial forms of alternators, as the deviations of the curves from sinusoidal form are usually not great enough to change the constant very much; but this condition cannot be relied on unchallenged.†

When desired, the curves of conductor voltage may be plotted, and the ordinates of the voltage curve of the machine may be obtained by adding corresponding ordinates of the conductor voltage curves.

The vector relations of sinusoidal conductor voltages and the machine voltage are illustrated in Fig. 33, where OA, AA', $A'A''$, and $A''S$ are the voltages developed by the conductors in the several slots of the foregoing example. These are laid off in a vector polygon. The effective voltage produced in the conductor (or conductors) of each slot is 333 volts, and the resultant of the four voltages is 1207 volts as already computed by the formula. These vectors are plotted in their proper relative positions. Their lengths are laid down equal to their respect-

* Art. 28.

† De la Tour's *Moteurs Asynchronous Polyphases*, Chap. 2.

ive effective values. The corresponding instantaneous values for the instant corresponding to Fig. 33 are proportional to the projections of the various vectors on the vertical axis. The reason that the coefficient k' is less than unity is plainly shown by this figure, in which it is seen that the component or conductor

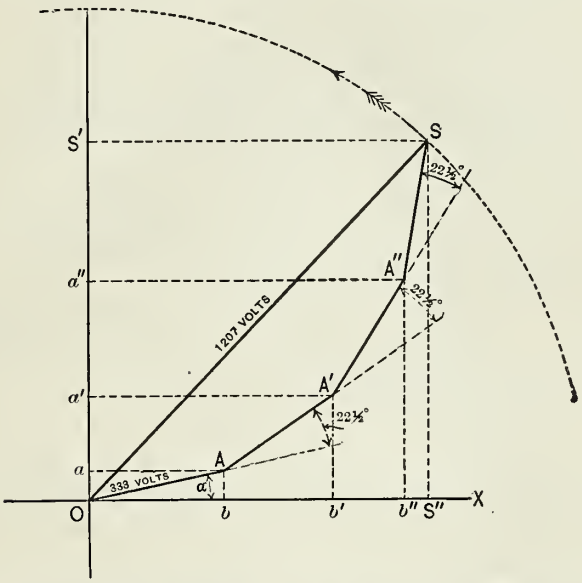


FIG. 33. — Graphical Representation of Voltage generated in an Armature having Four Slots.

voltages are in different phases and their resultant is therefore less in magnitude than their arithmetical sum.

The complex expression for these vectors is

$$\overline{OS} = (OS'') + j(OS') = [(Ob) + (bb') + (b'b'') + (b''S'')] + j[(Oa) + (aa') + (a'a'') + (a''S')],$$

or, $$(OS'') = (Ob) + (bb') + (b'b'') + (b''S''),$$

and $$(OS') = (Oa) + (aa') + (a'a'') + (a''S').$$

Having thus obtained the numerical values of the terms of the complex expression for the resultant, the numerical or scalar value of the resultant is obtained from the relations

$$(OS) = \sqrt{(OS')^2 + (OS'')^2}.$$

The value of Oa, aa', Ob, bb', etc., may readily be found by use of the trigonometrical table; for instance

$$\overline{AA'} = (bb') + j(aa') = AA' [\cos(\alpha + 22\tfrac{1}{2}^\circ) + j \sin(\alpha + 22\tfrac{1}{2}^\circ),$$

and therefore, as α, in this case, at the instant indicated in Fig. 33, is $11\frac{1}{4}^\circ$,

$$(bb') = 333 \cos 33\tfrac{3}{4}^\circ \text{ and } (aa') = 333 \sin 33\tfrac{3}{4}^\circ.$$

A winding which creates several equal elementary voltage curves following each other at equal angles, but added to obtain a resultant at the collector rings, is called a **Distributed winding**. Such windings are shown in Figs. 67–70, Chapter III.

Prob. 1. In an alternator armature there are five slots in a group and the angle covered by them is 120°. What is the value of the constant K for the group?

Prob. 2. The effective voltage set up in the conductors in each slot of problem 1 is 50 volts. Determine graphically the total voltage of the machine.

Prob. 3. Obtain the answer to problem 2 by the method of complex quantities.

Prob. 4. An armature has four slots in a group covering 60°. The conductors in each slot develop 400 volts. Determine the total voltage developed by each group, when the conductors are all connected in series, both graphically and by the method of complex quantities.

21. Most Economical Width of Pole Face. — The output of an alternator is proportional to the product $S\Phi = swbw'$, where s is the number of conductors per unit width of coil, b the number of lines of force per unit width of pole face, and w and w' are respectively the widths of coil and pole face. In order to economize material, the distance between pole tips may be taken as equal to the width of the coil, for the reason explained in the previous article, and this makes $w + w'$ equal to the pitch of the poles, which is constant; this makes the product ww', and hence the output of the machine, a maximum when w is equal to w', or when the width of coil and pole face are each equal to half the pitch. The result thus derived must be modified to suit practical conditions, since fringing tends to increase the width of field, and armature reactions tend to crowd the field

towards the trailing pole tip, thus narrowing the field. Experiment has shown that it is best to have the coils somewhat wider than half the pitch, and it is often advantageous to have the poles slightly less than half the pitch in width.

22. Polyphase Alternators. — As has been shown, only slightly over one half the armature surface is utilized in the

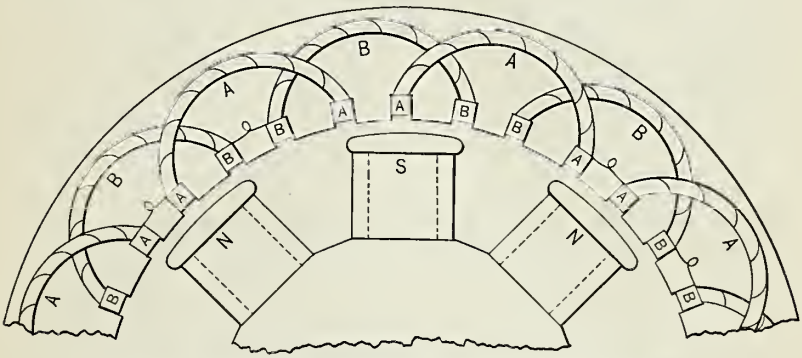


FIG. 34. — Diagram of a Simple Two-phase Armature Winding.

machines of the type which has been dealt with, and which are called **Single-phase machines** or **Single-phasers** from the fact that they give out a single current.

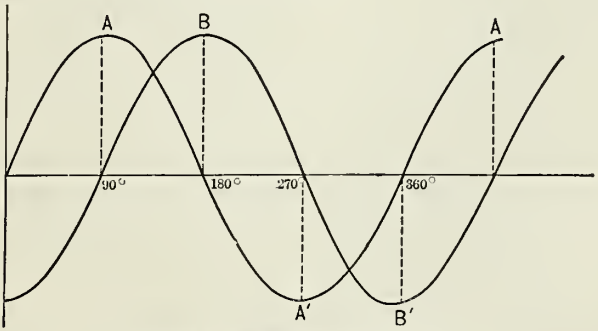


FIG. 35. — Sine Curves 90° Apart.

A second set of windings may be placed between the first, as in Fig. 34, which will give out from a second pair of collector rings or terminals another alternating current displaced 90° from the first. This is indicated in Fig. 35, where A, A', and B, B', represent the two currents. Such a machine is called a **Two-phaser.** Figures 36 and 37 show by diagram two methods, and c, c'

of connecting up the coils of two-phasers. In the former the phases are joined together at a central point called a **Neutral**

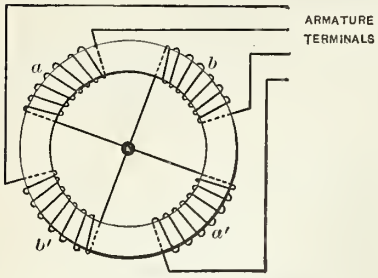


FIG. 36. — Two-phase Armature. Four Terminals.

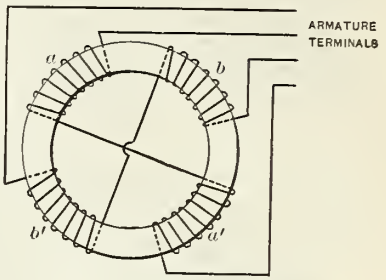


FIG. 37. — Two-phase Armature. Four Terminals.

point; while in the latter the two phases are entirely independent. The figures indicate a simple bipolar ring, to avoid the complexity that arises in diagraming the windings for multipolar machines, and represent either a stationary or rotating armature. Figure 38 shows the coil terminals passing to only three armature terminals, which is made possible by utilizing one wire as a common conductor for both circuits. The current in the common wire, as will be shown later,* is $\sqrt{2}$ times the current in either of the independent wires when the currents are equal in the two circuits and are of 90° difference of phase, which is the condition of a balanced two-phase system; and the voltage between the two independent wires is then $\sqrt{2}$ times the voltage between either of those and the common wire.

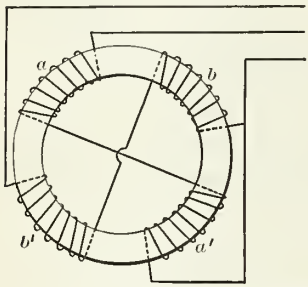


FIG. 38. — Two-phase Armature. Three Terminals.

Instead of two independent circuits being placed on the armature, the winding space may carry three circuits, as illustrated in Fig. 39, and the machine is then a **Three-phaser**. The three-phaser gives three voltage curves ordinarily differing from each other in phase by 120° as indicated in Fig. 40, where A, A', B, B', and C, C', are the three curves. Figures 41 and 42 illustrate such a winding for a simple Gramme ring in which a, a', b, b', c, c',

* Art. 100.

represent the coils of each phase or winding. In both figures the windings are so connected together that only three arma-

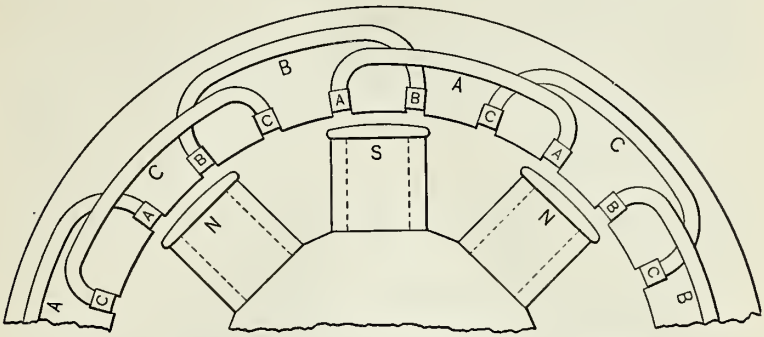


Fig. 39. — Diagram of a Simple Three-phase Armature Winding.

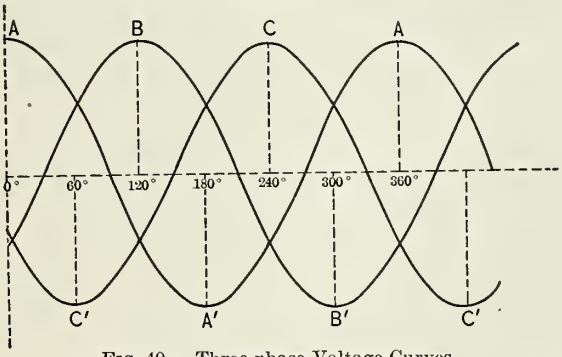


Fig. 40. — Three-phase Voltage Curves.

ture terminals are used instead of three independent pairs. A simple diagram of the connection of the coils of Fig. 41 is shown in Fig. 43. This winding is called a **Mesh** or **Delta(Δ) winding**. The currents which combine from two coils at the armature terminals (*A*, *B*, *C*) are $\sqrt{3}$ times greater than the current in a single phase,*

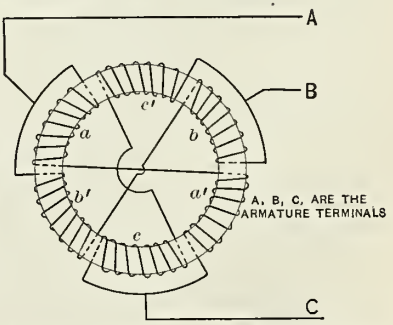


Fig. 41. — Three-phase Winding. Mesh Connection.

* Art. 100.

provided the currents are equal in the three windings and are 120° apart in their phases, which is the condition of a balanced three-phase system.

The arrangement shown in Fig. 42 is called a **Star** or **Wye** (Y) **winding**, and the connection of the coils is diagrammed in Fig. 44. The voltage measured between any two armature terminals connected to the Y winding is $\sqrt{3}$ times the voltage in the winding of a single phase,* provided the voltages are equal in the three windings and are 120° apart in their phases.

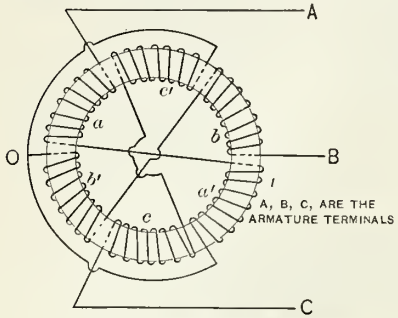


FIG. 42. — Three-phase Winding. Star Connection.

Machines having more than one phase or winding are called in general **Polyphase** and sometimes **Multiphase** machines, or **Polyphasers** or **Multiphasers**.

The methods of connecting up two-phase armature windings are very simple. If the armature is wound with independent concentrated coils, each coil may be connected to its individual armature terminals, in which case four terminals are required and the two circuits are entirely independent; or, one end of each coil may be connected to a common armature terminal and the other ends to independent terminals, in which case but three terminals are required and the circuits have a common point. If the armature has a direct-current or closed-circuit distributed winding, such as is treated in Art. 20, it may be made into a two-phaser by connecting collector rings to the windings at the ends of two diameters which are 90° apart, if the machine is bipolar; if the machine is multipolar, the connections must be made as described in Art. 28. Four rings are nec-

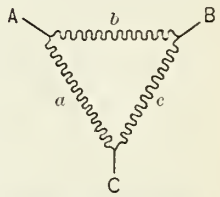


FIG. 43. — Diagram of Δ Connection.

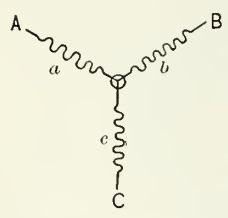


FIG. 44. — Diagram of Y Connection.

* Art. 100.

essary in this case, as the use of a common ring would cause the permanent short-circuiting of one quarter of the armature. A direct-current armature converted into a polyphaser (of any number of phases) in this manner has a capacity when used as a polyphase alternator which is nearly equal to its direct-current capacity, though it has already been shown that its capacity as a single-phaser is only seven tenths as great as its direct-current capacity.*

The manner of connecting three-phase armatures is not so immediately evident, but is perfectly simple. It is illustrated in Figs. 64 and 65. Three armature terminals are universally used, and if the armature is wound with three independent coils, these may be connected to the terminals in either of two ways: (1) one end of each of the coils may go to a common point, and the other ends go to independent terminals; or (2) the coil ends may be connected together two and two, forming a sort of triangle, and connections be carried to the armature terminals from these points. The latter arrangement makes each coil terminate at each end in an armature terminal, and, since there are six coil ends and three terminals, each terminal is connected with two coil ends. These two arrangements are illustrated respectively in Figs. 65 and 64, each of which shows the winding diagrammatically as developed and as projected. The following considerations make it perfectly easy to connect the coils in the proper order: Fig. 40 shows that when the current in one coil is at its maximum point, the currents in the other two are equal to each other and opposite to the direction of the first; then, considering the instant at which the conductors of one coil (such as *C* in the figure) are directly under the poles, if we connect its positive end to the common or neutral point the negative ends of the other two coils must be connected to the same point. Each of the free ends may then be connected to one of the armature terminals and the connection is completed according to the first or Y arrangement (Fig. 65). To make the second or Δ arrangement, the coils must be connected so that, at the instant considered, the current flows by two paths through the armature from the negative to the positive end of the first coil (coil *C* of the figure). Consequently the negative ends of the first and second coils go to one arma-

* Art. 20.

ture terminal, the positive ends of the first and third coils to another terminal, and the free ends of the second and third coils to the third terminal (Fig. 64).

If the armature has a closed-circuit or distributed winding, the connection is very simple, as the rings (terminals) are connected to points in the windings 120 electrical degrees apart. If the machine is multipolar, it must be remembered that one cycle, or 360 electrical degrees, is comprised within the space of twice the polar pitch.

In a three-phase machine, if the armature is Δ connected, the voltage between any two armature terminals is equal to the voltage developed in one coil, while the current leaving a terminal is composed of the vector currents in two coils; and, if the armature is Y connected, the voltage between the terminals is composed of the vector voltages developed in two coils, while the current leaving a terminal is equal to the current in a coil. The line currents with a balanced load are $\sqrt{3}$ times the current in a single coil in the Δ connection, and the voltages between lines or terminals are $\sqrt{3}$ times the voltage of a single coil in the Y connection, so that the capacity of a machine is independent of the way in which its armature is connected, but, for a given voltage and output, the windings will differ (though the weight of copper will be the same) for the two arrangements.*

Prob. 1. 100 volts are set up in each winding of a balanced Y connected three-phase alternator armature. What are the voltages between the terminals of the machine?

Prob. 2. 200 volts are generated in each coil of a balanced Δ connected three-phase alternator armature. What are the voltages between the terminals of the machine? And when 50 amperes flows in each coil, what is each line current?

Prob. 3. The two windings of a balanced two-phase armature are connected to three line wires. The voltage developed in each winding is 100 volts; the line wires are designated A, B, C, where B is the common return wire. What are the voltages between A and B, B and C, and A and C?

Prob. 4. In problem 4 the current flowing in each phase is 50 amperes. What is the current in the common return wire?

* Chap. VIII.

CHAPTER II

ELEMENTARY STATEMENTS CONCERNING TRANSFORMERS AND MEASURING INSTRUMENTS

23. The Fundamental Principle of the Transformer. — If a coil of wire is wound upon a closed iron core of high magnetic conductivity, *i.e.* low reluctance, as illustrated in Fig. 45, a small current passed through the coil will produce a powerful magnetic flux. If an alternating current is passed through the coil, the magnetic flux changes with the cycles of the current, and the changing number of lines of force linked with the coil create a **Counter voltage** in the coil which opposes the tendency of the voltage impressed at the terminals to send current through the coil. This action is in accord with a natural law which may be stated as follows: *When a current in a conductor is caused to change in value, or a conductor is moved in a magnetic field so as to cut lines of force, an induced electro-motive force is set up in the conductor which tends to oppose the change in current value or the movement of the conductor.* This statement is a corollary of the *Law of the Conservation of Energy.*

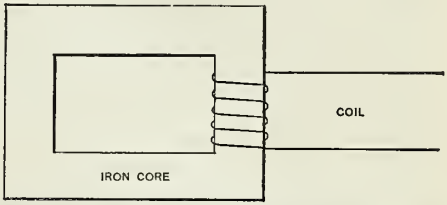


FIG. 45. — Coil Wound upon an Iron Core of Low Reluctance.

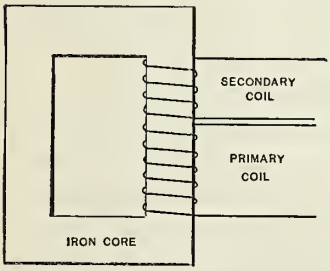


FIG. 46. — Low Reluctance Iron Core carrying Primary and Secondary Coils.

If another coil is wound upon the same core, as illustrated in Fig. 46, the magnetic flux set up by current in the first coil passes through the second coil also, and the changes in the strength of the field set up an induced voltage in this second coil similar to that in the first. An **Induction coil** of this character, used with

alternating currents, is called a **Transformer**. The coil which receives the current from an outside source is called the **Primary** coil and that in which voltages are induced by the magnetic flux set up by current in the primary coil is called the **Secondary** coil.

The phenomenon is called **Self-induction** when electro-motive force is induced in a coil as the result of changes of the current in its own conductors; and the phenomenon is called **Mutual-induction** when an electro-motive force is induced in a coil as the result of changes of the current flowing in a neighboring coil. When an alternating current flows through a primary coil, it is obvious that the magnetic flux set up in the iron core, such as is illustrated in the transformer of Fig. 46, must also be alternating and of the same period; and that the induced voltages in both the coils are also alternating and of the frequency of the primary current.

The maximum voltage induced per turn of wire in either coil is proportional to the maximum value of the alternating magnetic flux which links the coil and to the rate of change of the flux. The latter is proportional to the frequency of the current. Therefore, if all the turns of each coil are in series and the same magnetic flux links the two coils, the voltages induced in the primary and secondary coils are respectively proportional to their number of turns; or

$$\frac{E_1'}{E_2'} = \frac{n_1}{n_2},$$

where E_1' and E_2' are the induced voltages and n_1, n_2 are the turns in the primary and secondary coils, respectively.

In the case of commercial transformers, when no current is flowing in the secondary coil, the self-induced voltage in the primary coil almost equals the voltage impressed at its terminals. This is because only a very small **Exciting current** is required for the core magnetization necessary to set up an induced voltage equal to the impressed voltage, and the voltage used in sending the current through the resistance of the conductors of the primary coil is practically negligible. The instantaneous value of counter voltage at each instant must be smaller than the corresponding instantaneous value of the impressed voltage by an amount equal to the IR drop at that

instant. The IR drop in a commercial transformer is usually very small when the secondary circuit is open.

It may therefore be said that, in an ordinary unloaded commercial transformer, the ratio of the primary impressed voltage to the secondary induced voltage is substantially equal to the ratio of the numbers of turns in series respectively in the primary and secondary coils. That is, substantially,

$$\frac{E_1}{E_2} = \frac{n_1}{n_2},$$

where E_1 is the primary impressed voltage.

The induced voltage set up in the winding of the secondary coil is obviously in phase with the counter voltage of the primary coil, assuming them to be set up by the same magnetic flux, and this secondary voltage is practically opposite to the voltage impressed on the primary coil. Closing the circuit of the secondary coil through resistance (such as a number of incandescent lamps) allows a current to flow under the impulse of the secondary induced voltage, which current is opposed in phase to the current caused by the impressed voltage to flow through the primary coil. The result is that the magnetizing effect of the primary current is opposed by the magnetizing effect of the secondary current, and the primary current must increase sufficiently to overcome the magnetic effect of the secondary current and continue to magnetize the core as before. The ampere turns of the primary coil must therefore increase by an amount equal and opposite to the ampere turns of the secondary coil. That is,

$$n_1 i_1 - n_2 i_2 = n_1 i_u,$$

where $n_2 i_2$ are the instantaneous ampere turns of the secondary coil, $n_1 i_1$ are the ampere turns in the primary coil, and $n_1 i_u$ represents the part of the primary ampere turns required to set up the magnetic flux in the core at the corresponding instant.

The current in the primary coil, then, must be slightly greater than $\frac{n_2}{n_1} i_2$ in order to supply the core magnetization, but in commercial transformers this extra magnetizing current is small, and may usually be neglected; and the statement may therefore be made that the currents in the primary and secondary coils of the usual commercial transformers designed for constant

voltage service are almost inversely proportional to the number of turns in the two coils. That is, approximately,

$$\frac{i_1}{i_2} = \frac{n_2}{n_1}.$$

When currents flow in the secondary coil, the voltage lost in the resistance of the coils can no longer be neglected. This drop at full load amounts to from one to five per cent of the voltage at no load in the transformers of ordinary practice, and the ratio of the voltages at the primary and secondary terminals is no longer correctly represented by the ratio of the numbers of primary and secondary turns. Magnetic leakage, whereby the magnetic flux linking one coil becomes different from the flux linking the other coil, also affects the ratio of voltages. These deviations from ideal action are fully discussed in a later chapter.

A transformer may be defined as *a combination of coils and magnetic core for transforming alternating currents of a given voltage into proportional alternating currents of another fixed voltage without the intervention of physical motion.* The core of a transformer must be **Laminated**, that is, made up of very thin sheets of soft iron or steel laid parallel to the lines of force to prevent the flow of eddy currents. Figure 47 shows a skeleton view of a commercial transformer.

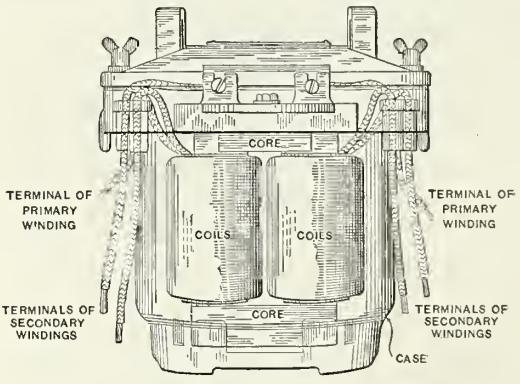


FIG. 47. — Skeleton View of a Transformer.

If there were no losses of power in the operation of a transformer, the power given out by the secondary coil would obviously be equal to the power received by the primary coil; that is, P_2 would be equal to P_1. But an actual transformer cannot be operated without I^2R losses in the conductors, hysteresis losses in the core, and eddy current losses in the core and perhaps other parts, so P_2 is always smaller than P_1.

The difference is only a few per cent of the input in the case of a well-designed modern constant voltage transformer operated at normal full load.

A modified transformer having the secondary mounted upon a rotating core and so arranged as to deliver mechanical instead of electrical power is called an **Induction motor**. A discussion of transformers and induction motors is to be found in Chapter XI.

Prob. 1. A certain transformer delivers to the external secondary circuit a power of 50 kilowatts. If the efficiency of the transformer is 96 per cent at this load, what power must be supplied to it?

Prob. 2. In a fully loaded transformer the voltage supplied is 1000 volts, the current 50 amperes, and the secondary voltage is 100 volts. What is the approximate value of the secondary current?

24. Alternating-current Amperemeters, Voltmeters, and Wattmeters.* — The three general classes of direct-reading instruments most suitable for making measurements in alternating-current circuits have already been given.† Working gear for each of the three most important of the instruments for commercial electrical measurements (namely, amperemeters, voltmeters, and wattmeters) may be made on the principles of either of the three types named in Art. 17. Paragraphs *a*, *d*, and *e*, following, deal with the first class.

a. Electro-dynamometers. — Large numbers of the best known alternating-current instruments depend upon the electro-dynamometer principle, *i.e.* the magnetic action of the current in one coil upon the magnetic field of a current in another coil. Such instruments must be constructed without masses of solid conducting material about them, or eddy currents may be set up and the readings of the instruments become affected by the frequency of the current to be measured. This is due to the dynamic effect which eddy currents (induced short-circuit currents) circulating in the metallic masses produce upon the

* Jackson's *Elementary Electricity and Magnetism*, Chaps. XIII, XVI, and § 231.

† Art. 17.

currents in the moving parts of the instruments.* Likewise, iron cores can be used only with the utmost caution in the construction of the working gear, even though laminated (built up of thin wires or sheets) to lessen eddy currents, as the hysteresis of the iron is likely to impair the accuracy of the readings.† If the instrument is to be used as a voltmeter, the self-induction and electrostatic capacity of the windings must be reduced to negligible values, or the readings may be affected by the circuit frequency. The elimination of self-inductive or capacity effects from the voltage coils of wattmeters is also very important, as is fully demonstrated later.‡

If any of these harmful elements are present, the instrument must be calibrated under the exact conditions of frequency, current, or power, with which it is to be used, or the readings may prove to be quite erroneous. On the other hand, such instruments, properly constructed, can be relied on to read with equal accuracy in alternating-current circuits having any of the commercial frequencies; and they may ordinarily be calibrated in a direct-current circuit (unless iron is used in the working gear). Figure 48 illustrates a long-used form of electro-dynamometer arranged for use as an amperemeter. This is often called the *Siemens Electro-dynamometer*. This form gave excellent service, though it is somewhat clumsy. One coil, *F*, in this instrument is fastened to the frame of the instrument, and the other coil, *M*, which stands at right angles to the first, is suspended by a heavy silk fiber, so that it is free to move. The ends of the wire composing the movable coil dip into little cups, *CC*, containing mercury, which are connected with the main circuit so that the current can enter and leave the coil. The movable coil is attached to a spring *G*, the other end of which is connected to a thumbscrew *T*, called a *Torsion*

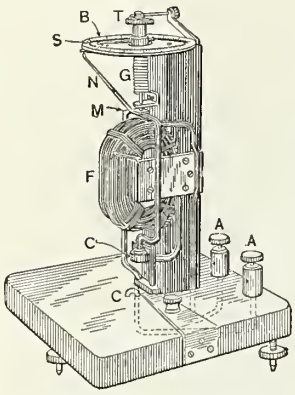


FIG. 48. — Siemens Electro-dynamometer and Diagram of Connections to Circuit.

* Arts. 90 and 111. † Art. 106. ‡ Art. 90.

head, by means of which the spring may be twisted. When a current flows in the coils, the magnetic force tends to turn the movable coil around so as to place it parallel with the fixed coil. This force is balanced by twisting the spring by means of the thumbscrew. The amount of twist, as shown by a pointer B attached to the screw, is proportional to the force exerted by the coils on each other. This force is proportional to the square of the current flowing in the coils, since the coils are connected in series and the magnetism set up by each coil is proportional to the current, and they act upon each other mutually. The pointer N indicates whether the movable coil is at its zero position. The "binding posts" for connecting the instrument into circuit are shown at AA.

According to the known laws of electro-dynamics, the torque acting at each instant to rotate the movable coil is proportional to the product of the simultaneous instantaneous values of the currents flowing in the coils, and is equal to this product multiplied by a constant of the instrument which is fixed by the relative dimensions, the numbers of turns, and the relative positions of the two coils. The average torque acting on the movable coil through each period of an alternating current is therefore proportional to the average (taken through the period) of the products of the corresponding instantaneous currents in the two coils; and when the two coils are in series relation, as in the electro-dynamometer type of amperemeter or voltmeter, the current at each instant is the same in the two coils, and the average torque is proportional to the average of the squares of the instantaneous values of the current. That is, the average torque is equal to $K(\text{av. } i^2)$, where K is the constant of the instrument fixed by the construction. This average torque is balanced by the spring attached to the torsion head in the Siemens electro-dynamometer; and if the spring is uniform, the movement of the torsion head required to hold the movable coil in its initial position is proportional to $(\text{av. } i^2) = I^2$.

Instruments of the electro-dynamometer type are more convenient if the movable coil carries a pointer which moves over a scale. This arrangement has largely displaced the Siemen's type. Such an instrument, of Weston make, is illustrated in Figs. 49 and 50. This instrument is like that of Fig. 48 in principle, but it has lighter coils connected in series with a large

non-inductive resistance JK to reduce the time constant of the instrument circuit, and make it practicable for use as a voltmeter.* The resistance of the commercial alternating-current voltmeters of this type is from about 20 to 40 ohms per volt of the maximum reading of the scale. The construction of the voltmeter shown in Figs. 49 and 50, which are lettered alike, is as follows: B is the movable coil, and AA is the stationary or fixed coil; M, N, O are binding posts; J, K are extra resistance coils, wound non-inductively; L is a special variable resistance used to correct the readings for variations of temperature; D is a push-button switch; C, C_1 are springs through which the current enters and leaves the movable coil; P is the needle or pointer; H is the scale, which is engraved with two sets of figures; and G is a thermometer with its bulb near the coils of the instrument, and its stem in view near the scale of the instrument. The instrument is usually boxed up so that only the scale H, H, over which the pointer moves, the end of the pointer, the dial of L, and the stem of the thermometer are

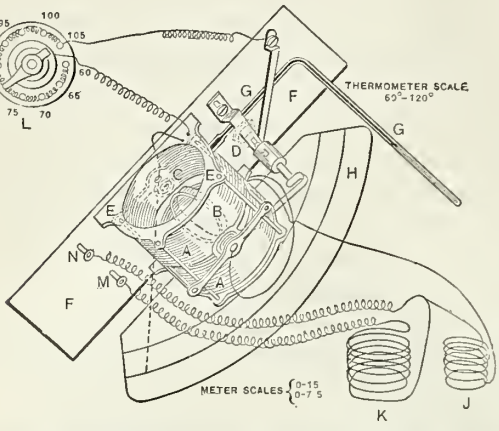


FIG. 49.—Perspective View of Interior of Weston Alternating-current Voltmeter from below.

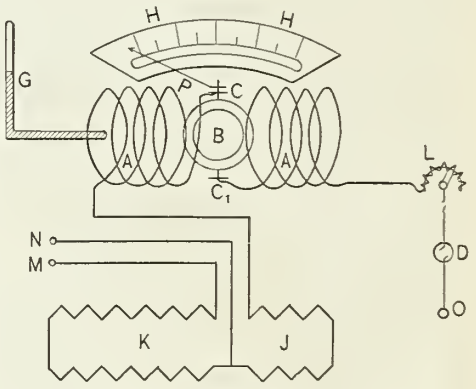


FIG. 50.—Diagram of Weston Alternating-current Voltmeter.

* Jackson's *Electricity and Magnetism*, pp. 250, 251.

visible. The thermometer is not always essential when the best modern resistance material of low temperature coefficient is used.

When the instrument is in service, the voltmeter is connected to the circuit by means of the binding post *O*, and either the binding post *M* or the binding post *N*. When the button *D* is depressed, current flows through the fixed and movable coils, and through the resistance $J + K$ or J alone (depending upon which binding post, *M* or *N*, is used), and the pointer is made to move over the scale by the movement of the movable coil, which action is caused by the electro-magnetic attractions between the current in its windings and the current in the windings of the fixed coil.

A pointer on the dial *L* is set at a mark which corresponds to the temperature indicated by the thermometer, and more or less of the resistance coils connected to the dial are thus included in the voltmeter circuit. In this way the resistance of the voltmeter, measured from binding post to binding post, may be kept uniform, regardless of the temperature of the instrument, and the readings are thus corrected for the variations of temperature. The resistance of the windings *AA* and *B*, added to the resistance of *J*, is just equal to one half of the resistance of the same windings plus the resistance of $J + K$. Consequently only half the voltage between the instrument terminals is required to cause a given movement of the needle when the binding posts *O* and *N* are used, as when the binding posts *O* and *M* are used. The scale which reads up to 7.5 volts, in the instrument illustrated, is therefore used in connection with binding posts *O* and *N*, and the scale which reads up to 15 volts, with the binding posts *O* and *M*. The movement of the coil and pointer is opposed by the springs *C*, C_1, and the scale is engraved so that the instrument is direct reading. As the springs offer an opposing force which is practically proportional to the extent of movement of the movable coil, and the torque acting to move the movable coil is proportional to the square of the current, it is obvious that the scale of the instrument must widen out from left to right. That is, the scale is a "square-root scale," but by proper design of the coils of such instruments it is possible to make the widest divisions at the most important part of the scale.

If an electro-dynamometer is to be used as a wattmeter, the movable coil is usually placed in series with a large non-inductive resistance and connected across the circuit, and the stationary coil is of low resistance and is connected in series with the circuit. This is illustrated in Fig. 51. Under these circumstances the instrument can evidently be calibrated to measure watts, since the torque is proportional to the average of the products of the instantaneous voltages with the corresponding instantaneous currents. Figure 52 shows a portable form of wattmeter with its cover removed so as to show the working parts, which are indicated by letters. *MM* and *OO* are binding posts, the former of which are terminals for the "voltage coil," and the latter terminals for the "current coil" (one of the latter is at the far corner of the instrument and is hidden); *A* is the stationary or fixed coil; *BB* is the movable coil; *P* is the pointer or "needle"; *H* is the scale; *K* is a non-inductive extra resistance coil which is placed in series with the voltage coil; *D* is a torsion head by means of which the spring *E* may be turned so as to bring the movable coil, which is attached to it, into zero position. This position is indicated by the pointer *P'*. When the pointer *P'* points to zero, the pointer *P*, which is attached to the torsion head, points to the reading of the instrument. *R* and *Q* are the wooden base and supports of the instrument. *CC* are flexible conductors for affording the current ingress and egress to and from the movable coil.

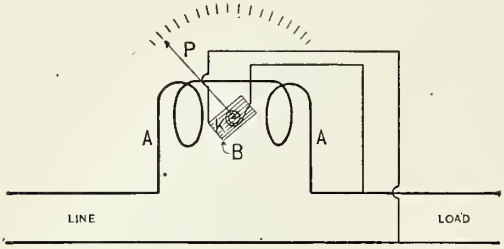


FIG. 51. — Diagram of Wattmeter Connections.

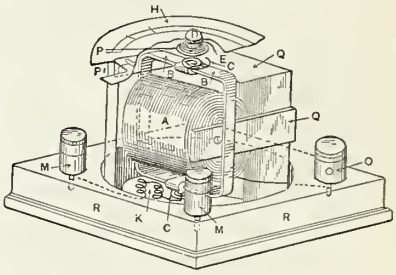


FIG. 52. — Partial Perspective View of Hoyt Wattmeter.

Wattmeters are usually made so that the needle *P* is directly attached to the movable coil in the manner illustrated in Fig.

50; and in that case the flexible conductors *CC* are replaced by conducting springs, also as illustrated in Fig. 50.

Electro-dynamometers are made with an endless variety of details. One type of special accuracy is the Kelvin balance, which is frequently used as a standard for calibrating less permanent instruments. Such a balance, illustrated in Fig. 53,

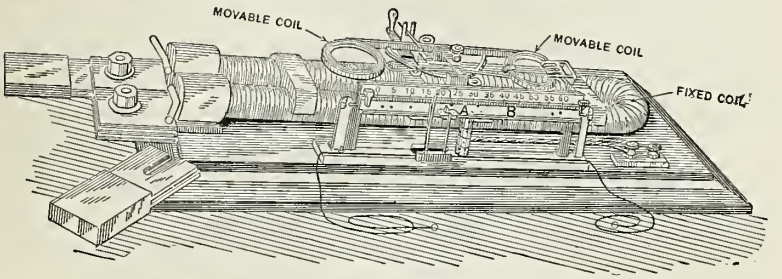


Fig. 53. — Kelvin Alternating-current Kilowatt Balance.

consists essentially of two electro-dynamometers which tend to turn an arm pivoted at the center between the movable coils. The fixed and movable coils in these instruments are parallel to each other and horizontal. The two movable coils are connected in circuit in series with each other so that the current in one circulates in the same direction as the current in the other. This balances the effect of any reasonably uniform external magnetic field (such as the earth's field) on the balance arm and eliminates such effects from the readings, which is very desirable when the instrument is used in connection with direct currents. When the instrument is used with alternating currents, the external fields would not, in any event, affect the readings unless these fields also were alternating, and of the same frequency. The force with which the movable coils tend to move when a current flows in the sets of instrument coils is directly balanced and weighed by means of a slider *A* moving on a scale beam *B*. Some of the balances having a large number of turns are not suitable for alternating currents on account of the coils being of too high self-inductance.

b. Hot-wire Instruments. — If a heated wire is carefully inclosed so that its temperature is not affected by air currents, it will rise a definite number of degrees in temperature for every current that is passed through it, and the rise is propor-

tional to the energy expended in the wire and therefore to the square of the current. The length of wire increases practically in direct proportion to its rise in temperature when it is heated, and the length again decreases when the wire is cooled. Consequently, when currents of different strengths flow through a wire, it will take up a corresponding length with each current, and measuring its length therefore measures the square of the current.

Hot-wire instruments evidently average up the instantaneous currents squared, since the heat developed in the wire is I^2R and (the wire being protected so that the coefficients of radiation and convection are constant) the temperature is proportional to the rate at which heat is liberated in the wire. Figure 54 shows a hot-wire switchboard instrument. The current passes through the platinum silver wire A and this fine wire is so arranged, by the use of special springs, that its elongation operates the pointer B, which is mounted on jewels. The wire is of too low resistance to adapt the instrument for direct use as a voltmeter on ordinary voltages, and a resistance coil, non-inductively wound, is put inside the case and connected in series relation with the wire to make a voltmeter. The wire is of small current-carrying capacity and must be shunted to adapt the instrument for use as an amperemeter unless the current is only a few milliamperes. Hot-wire instruments have not been used commercially as wattmeters.

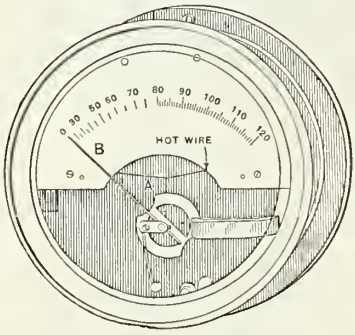


Fig. 54. — Hot-wire Instrument.

A *shunt*, except for very small currents, is ordinarily placed within the case and permanently attached in most types of portable amperemeters; unless the instruments are to be used for large currents or on switch-boards, when the shunt is separated from the instrument and they are connected in circuit as illustrated in Fig. 55, or, as is frequently the case, a

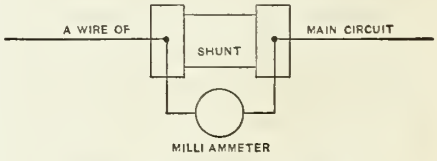


Fig. 55. — Ammeter with Shunt.

current transformer takes the place of the shunt. The shunt should be made of material having a low temperature coefficient and large surface and should be substantially non-inductive.

c. Electrostatic Instruments. — Electrostatic instruments are built on the principle of the electrometer. Figure 56 shows the plan of a quadrant electrometer. If the needle (which is indicated by the dotted line in the illustration) and a pair of opposite quadrants are connected to a point in an electric circuit and the other pair of quadrants to another point in the circuit, the needle will tend to deflect with a force proportional to the square of the difference of potential between the points. This must be so, since the force tending to move the needle is proportional to qq', where q and q' are the electric charges on the two elements of the instrument respectively, and since both these quantities vary with the difference of potential between the points of connection with the circuit. Evidently qq' may be written Aq^2, where A is a constant depending upon the shapes and dimensions of the parts of the instrument. If the movement of the needle is opposed by a uniform spring, as usual in electrical instruments, the deflections will be proportional to the torque and therefore the readings will be:

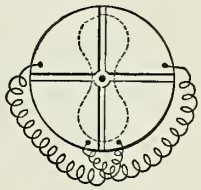


FIG. 56. — Plan of Quadrants and Needle for a Quadrant Electrometer.

$$D = B(\text{av. } q^2) = K(\text{av. } e^2),$$

where B and K are constants. Such an instrument can therefore be calibrated to read effective volts, but (like the electro-dynamometer voltmeter) it has a "square-root scale." Figure 57 shows a Kelvin electrostatic voltmeter arranged to be used commercially. It is seen from the figure that there are a large number of sets of quadrants and needles, one above the other. The needles receive their charges through a fine phosphor bronze wire which acts as a

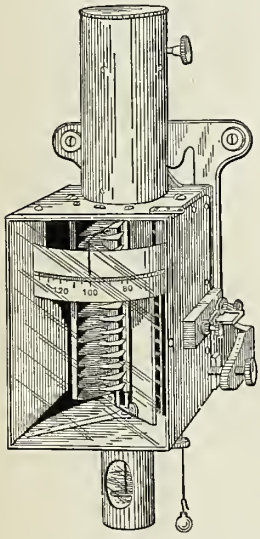


FIG. 57. — Kelvin Electrostatic Voltmeter.

supporting filament for the needles and also furnishes the necessary restraining force.

If an instrument of this class is to be used as an ammeter, it must be in connection with a shunt, though it is not well suited for this purpose. In this case the needle deflects proportionally to the square of the voltage drop in the shunt, and hence to the square of the current flowing. Special arrangements are made when electrostatic instruments are applied to power measurements, as will be explained later.*

d. Magnetic Vane Instruments. — Magnetic vane voltmeters and ammeters are instruments in which a very light vane or needle of soft iron is caused to turn under the influence of a magnetic field. The magnetic vane will theoretically have a torque exerted upon it proportional to the instantaneous current squared, since the torque is proportional to the product of the magnetic field due to the current in the coil and the field induced in the iron vane, which is in turn proportional to the current if the intensity of magnetization is low. Hysteresis will not materially affect the readings in connection with alternating currents if the vane is of very small volume. Such instruments cannot be used readily as wattmeters, and they are generally looked upon as less reliable than electro-dynamometer instruments for standard amperemeters and voltmeters. They are much used for switchboard instruments which may be expected to be subjected to currents of a single fixed frequency. An instrument of this type is illustrated, with its cover taken off so as to expose the working parts, in Fig. 58.

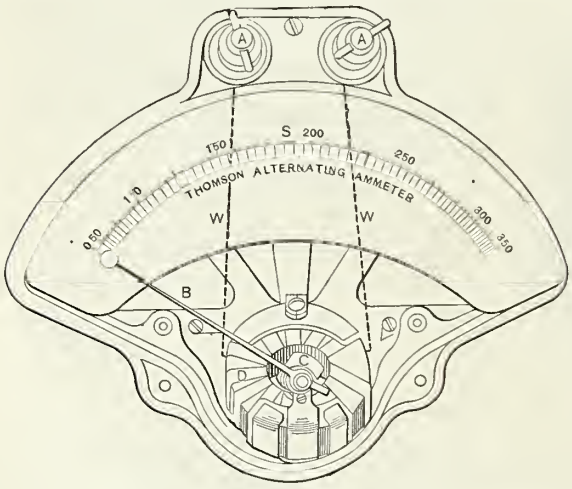


Fig. 58. — Plan of Thomson Alternating-current Amperemeter.

* Art. 99.

The parts of this instrument are indicated by the letters, where D is the current coil, C the thin movable iron vane, B the needle, S the scale, and AA the binding posts which are connected to the coil D by the wires WW.

e. Induction Instruments. — Induction instruments are made on the same principle as the integrating or recording instruments fully described later. In the usual form, they include coils which induce currents in other coils by transformer action. The magnetic fluxes due to the currents of the various coils combine to form what is called a rotating field. This in turn sets up eddy currents in a movable disk, or shell of metal, to which the pointer is attached, which cause a torque between the disk and coils.

25. Arrangement of Instruments for Measuring High Voltages or Large Currents. — In using a wattmeter where very high voltages are met, considerable difficulty is found in arranging a satisfactory non-inductive resistance for the voltage coil. This difficulty may be overcome by the use of a transformer. Instead of connecting the voltage coil of the wattmeter across the terminals of the test circuit, the primary of a transformer is so connected, and the voltage coil of the wattmeter is connected to the secondary of the transformer (Fig. 59). The constant of the wattmeter is then dependent upon the ratio of transformation of the transformer, which may be readily measured. This method gives fairly reliable results, since the phases of the primary and secondary voltages of a very slightly loaded transformer are almost exactly 180° apart. If the wattmeter constant is determined without the transformer, its constant when in use with the transformer must be multiplied by the ratio of transformation; but wherever practicable, the calibration ought to be made of the wattmeter and transformer as a unit. In like manner if the current is too great to conveniently pass through the instrument, a series transformer may be used, as shown in Fig. 60. In such a case if the wattmeter has a scale

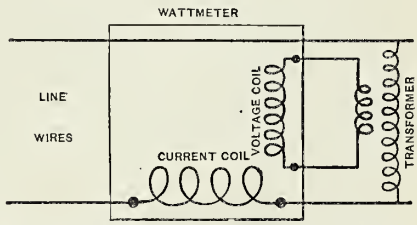


FIG. 59. — Wattmeter connected to High-voltage Circuit through a Voltage Transformer.

for use without the transformer, the readings must be multiplied by the ratio of transformation (the ratio of the main current to that in the transformer secondary). Changes of frequency change the reading slightly where a series transformer is used, so that it is necessary to calibrate the wattmeter with the transformer for the frequency of the circuit to be measured. Evidently amperemeters and voltmeters can have the current or voltage reduced in the same manner. For high-voltage switchboard instruments it is very desirable to use both series current and voltage transformers with the switchboard wattmeters, and indeed with all such switchboard instruments, as in this way the danger of accident from the high voltage can be greatly reduced.

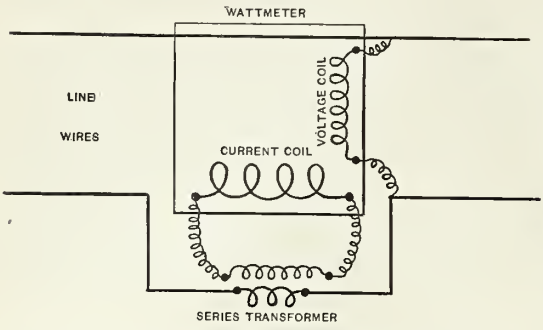


FIG. 60. — Wattmeter connected to a Circuit of Large Currents through a Series Transformer.

Non-inductive resistance coils can be used in series with the voltage coils of any instrument, as already described, but it is quite difficult to secure a satisfactory resistance device without appreciable self-inductance or capacity for voltages larger than a few hundreds of volts. Shunts may be provided for the current coils, but in the case of instruments of the electro-dynamic or magnetic vane type the shunts are apt to cause frequency errors, and in any event the dangers due to direct connection with the high-voltage circuits are not eliminated as when transformers are used.

CHAPTER III

ARMATURE AND FIELD WINDINGS FOR ALTERNATORS. MATERIALS OF CONSTRUCTION.

26. Classification of Armatures. — Dynamo armatures may be classified under three divisions :

1. Where a wire cuts lines of force by moving across them, as in the case of a slider or of a wire moving around a magnet pole.
2. Where a coil, or set of coils, is moved parallel to itself, or nearly so, between points of different field strength.
3. Where a coil, or set of coils, is wound on a ring or drum and given a rotary motion in a fixed magnetic field.

The first division includes only the so-called unipolar armatures, and requires no treatment here. Alternator armatures are in general classed in the second division.

It is most usual for the field to move instead of the armature, and in some cases neither the armature nor field coils move, but the magnetism linking the former is varied by the revolution of iron **Inductors**. Where the field rotates, or the variation of the magnetism is effected by moving inductors, the voltage generated in the armature windings is produced in exactly the same manner as if they moved, and such armatures therefore belong to the second division. That is, the generation of the voltage is dependent upon the *relative* motion of armature with respect to field, and this relative motion may be effected by the mechanical rotation of either.

The construction of the machines thus enumerated requires an additional classification into :

1. Alternators with moving fields.
2. Alternators with moving armatures.
3. Inductor alternators.

The armatures of alternators belonging to the first and second of these classes are almost always, in the United States,

Chord wound, *i.e.* the winding of a coil passes across the face of the armature under one pole face, and returns under the next pole of opposite sign without spanning an entire diameter of the armature. The arrangement is the same whether the armature revolves within the field or the field revolves within the armature. Such a disposition is termed **Drum winding**, in contradistinction to **Ring winding**, where the wires pass through and around a ring-shaped core, as in the armature invented by Gramme.

The windings may further be distinguished as **Bar** and **Coil windings.** In the former, rectangular bars are laid in slots across the face of the armature, and these are properly connected together at their ends by suitable welded, brazed, or bolted connections; and in the latter, coils of either small rectangular or circular wire are wound upon a former, insulated, and then placed in the armature slots.

The wires of a coil are sometimes bunched together in a single slot, as shown in Fig. 34, but are more commonly divided into several small coils, each occupying its own slot. The latter arrangement is called **Distributed winding**, because the active conductors are distributed more or less evenly over the face of the armature. (See Figs. 31, 67, 68, 69, 70.)

27. Examples of Armature Windings. — Essentially the same forms of windings are used for machines with revolving armatures and machines with revolving fields, and those described in the following paragraphs are applicable to both classes. In some of the illustrations, collector rings and an external crown of poles are shown, for convenience, for revolving armatures; but the same type of winding may as well be used for a stationary armature, in which case the crown of poles is internal and the armature terminals are stationary. The original type of armature winding used in America is like that

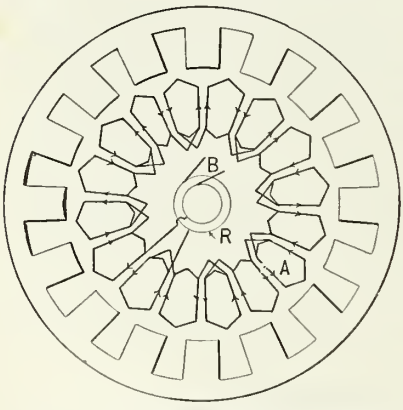


FIG. 61. — Coil Winding having as many Coils as Poles.

shown in Fig. 61. In this case the field magnet comprises a ring of poles of alternate polarity, and the armature consists of as many coils A laid on the face of the armature core as there are poles in the field magnet. The ends of the set of coils are connected to two collector rings R, on which bear brushes B, by means of which the current reaches the external circuit. In this figure the coils are shown for clearness as though lying in the plane of the page, but in reality they lie upon the face of the cylindrical armature. If another set of coils is placed 90 electrical degrees from those shown, and connected to additional collector rings, the winding is of the two-phase variety*; while if three sets of coils are placed so that the centers of the coils are 60 electrical degrees apart, the winding is three-phase.* In the latter case, the coils are almost invariably connected to three collector rings, and the connections must be made in the manner described in Art. 22. Referring again to Fig. 61, it is seen that adjacent coils move at any instant under magnet poles of opposite signs, and that the voltages developed in them are in opposite directions. They must therefore be connected in circuit with each other so that they are alternately right and left handed. Likewise, when an armature winding is composed of a number of coils equal to one half the number of poles, as illustrated in Fig. 62, the coils must all be connected in the same direction in the circuit.

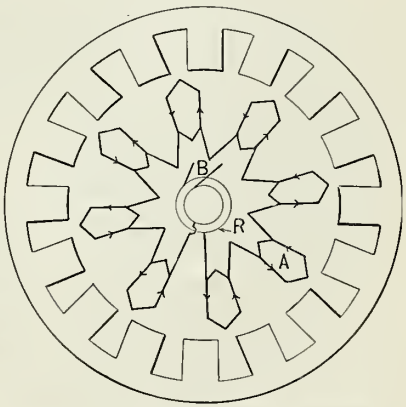


FIG. 62.—Coil Winding having half as many Coils as Poles.

To provide a winding similar to Fig. 62 for two or three phases involves adding the additional one or two independent sets of windings properly spaced and suitably connected to appropriate collector rings, as already explained for the winding of Fig. 61.

When the coil connections cross each other in their progress

* Art. 22.

around the armature, as in Fig. 61, or one half of Fig. 63 *a*, the windings may be called, after E. Arnold * and S. P. Thompson,† **Lap windings.**‡

Alternator armatures are frequently connected with the two halves of the windings in parallel, or, where the machine is to work at low voltage and deliver heavy currents, there may be several circuits in parallel in each phase. In this case, instead of being arranged as illustrated in Fig. 61, the coils must be connected in each half so that they are alternately right-handed and left-handed; and where the coils join the collector rings, both must have the same polarity. This is indicated in Figs. 63 and 63 *a*, which are diagrams for single-phase machines. Such a winding, having two circuits in each phase, is called a **Two-circuit winding**. When all the coils are connected in series, the first and last coils lie side by side; consequently, in armatures built for high voltages, a severe strain is thrown upon the insulation separating them, and the strain is especially troublesome if parts which differ largely in potential occupy the same slot. When the halves of the armature are connected in parallel, the

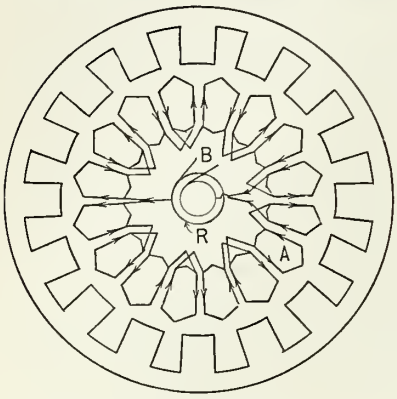


FIG. 63.—Coil Windings like Fig. 61, but with Halves connected in Parallel.

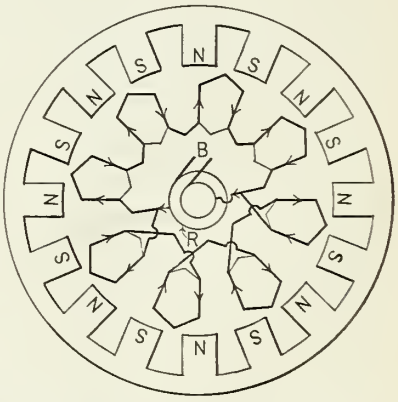


FIG. 63 *a*.—Coil Winding like Fig. 62, but with Halves connected in Parallel.

* *Die Ankerwickelungen der Gleichstrom-Dynamomachinen*, p. 13.

† *Dynamo Electric Machinery*, 7th ed., vol. 2, p. 168.

‡ For numerous diagrams of alternator windings, see Parshall and Hobart, *Armature Windings*, Chap. XII.

first and last coils in the armature circuit lie on the opposite sides of the armature, and effective insulation is therefore less difficult to maintain. In the latter case the voltage between the adjacent coils cannot be greater than the total machine voltage divided by half the number of coils. When the coils are all in series, the voltage between adjacent coils, except the two end coils, is equal to the total voltage divided by the total number of coils. When a coil winding of this type is connected with its halves in parallel, precautions must be taken to make the two halves of the winding alike, and have the field poles of equal strength; as, otherwise, the instantaneous voltage of one half of the winding is likely to differ from that of the other half, and injurious cross currents may occur in the winding. This precaution, of course, applies wherever there are several parallel circuits per phase.

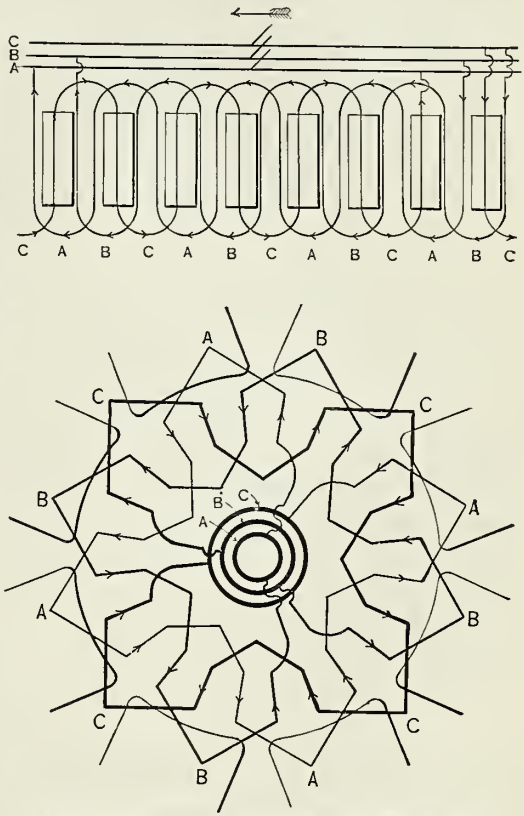


FIG. 64. — Three-phase Progressive Windings with Δ Connection.

Those forms of windings in which the end connections of any one phase need not cross each other at the ends of the armatures are called by S. P. Thompson* **Wave windings**, or according to more recent and acceptable nomenclature, **Progressive windings**. Figures 64 and 65 illustrate three-phase progressive windings.

* Thompson's *Dynamo Electric Machinery*, 7th ed., vol. 2, p. 168.

The first is connected in Δ and the second in Y. Figure 66 shows a winding of the same type for two phases. In these figures, with the exception of the developed views at the top of Figs. 64 and 65, the active conductors lying in the face of the armature are shown as radial lines, while the end connections are indicated by the diagonal lines connecting the active conductors together in such a way that the electro-motive forces of each circuit add together. Each line may represent a number of wires in the same slot, which may be arranged as coils in each phase in the manner indicated in Fig. 62.

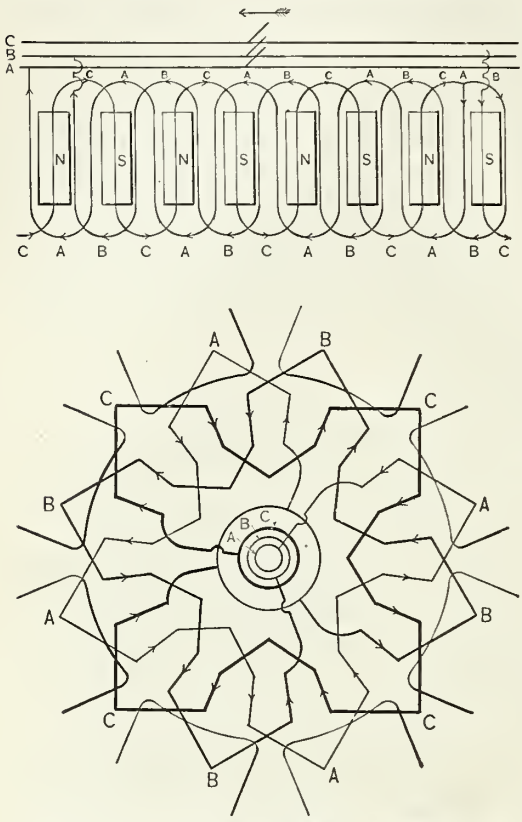


FIG. 65. — Three-phase Progressive Windings with Y Connection.

The principles of armature windings used in direct-current machines are, in general, applicable to alternators, but practical considerations make it often advisable to modify the windings for the latter machines. Synchronous alternator armatures often have **One-circuit windings**; that is, the conductors of each phase are connected in series, making one circuit per phase, inasmuch as high voltage is usually desired. Direct-current armatures, on the other hand, have necessarily two-circuit or multiple-circuit windings; hence, while direct-current windings may be adapted to alternating-current work, the reverse is not always true.

The possibility of converting a direct-current dynamo into a

double-current machine has already been referred to in Art. 20. A machine so constructed, with a direct-current commutator and alternating-current collector rings, may be used to convert a direct-current which is fed into its commutator end, and by which it is driven, into an alternating-current which is taken from the collector rings. Or, the conversion may be from alternating to direct currents, if the armature is properly synchronized so that it runs as a synchronous motor. Or, alternating-current and direct-current may be simultaneously produced or utilized in such machines, or either one of them alone. When these double-current machines are used for converting one kind of current into the other, they are called **Rotary converters.** The simple diagram, Fig. 66 *a*, shows the principle clearly, though such machines are ordinarily multipolar. (For a discussion of such machines see Chapter XII.)

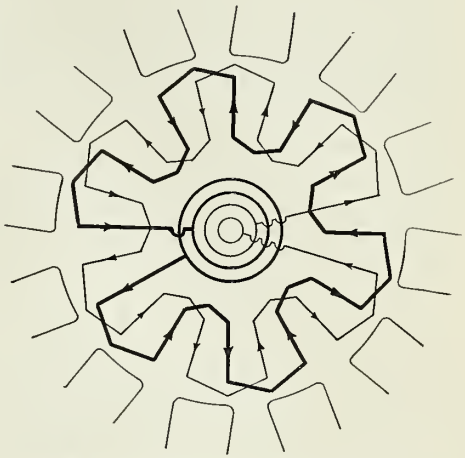


Fig. 66.—Two-phase Progressive Winding connected to Four Collector Rings.

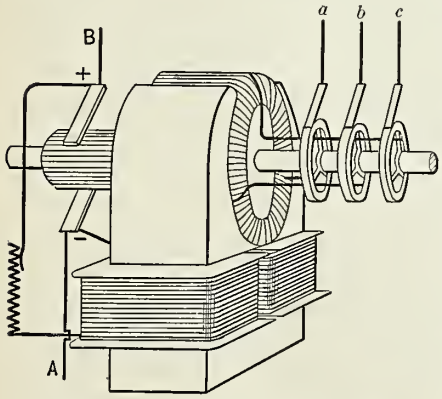


Fig. 66 *a*.—Double-current Dynamo.

It is possible in this manner to make a two-phaser to be used *with separate circuits* out of any direct-current machine with Gramme or Siemens armature, by arranging four collector rings on the shaft and connecting them to the armature windings at points which are 90 electrical degrees apart. It is also possi-

ble to make a three-phaser out of a direct-current machine by arranging three collector rings on the shaft, and connecting them to the armature winding, at 120 electrical degrees apart; and, in general, a polyphaser of any number of phases may be made out of a direct-current machine by providing a proper number of collector rings and connecting the rings to appropriate points on the armature winding. The taps for the different rings should connect with the armature windings at points $\frac{360}{n}$ electrical degrees apart, where n is the number of phases, except that, for single and two-phases, n must be taken equal to two and four respectively. Such machines may be used to transform direct currents into polyphase currents or vice versa. (See Fig. 66 *a*.)

In connecting the collector rings of such machines to the armature windings the relative angles corresponding to the current phases must be carefully distinguished. One complete revolution of an armature in a two-pole field corresponds to one complete period of the alternating current, and therefore 360 mechanical degrees correspond to 360 electrical degrees in that instance, but in multipolar machines a rotation of the armature equal to twice the angular pitch of the poles corresponds to one complete period, so that, in general, the relation of electrical degrees to mechanical degrees is $\frac{p}{2} : 1$, where p is the number of poles. Two-pole alternators of the form here described evidently utilize the whole of the armature winding with each collector ring connected to a single point, as the winding has only two paths for the current from commutator brush to commutator brush, and the same is true of multipolar machines with two path windings. If single connections to the collector rings are used in multipolar machines with multiple path windings, a proportion of the armature equivalent to 360 electrical degrees only is occupied in the delivery of alternating currents, and the armature capacity is therefore not fully utilized. To fully utilize the armature in this case, each collector ring must be connected to the winding at as many points as there are pairs of paths, the points being 360 electrical degrees apart; but the different collector rings must connect successively into points on the armature winding which are $\frac{360}{n}$ electrical degrees apart, as already stated.

In general, any form of winding used for direct-current machines, in which alternating voltages are induced in the coils, may be used for alternating-current machines by properly providing collector rings. The only forms of windings excluded from this are the so-called acyclic or homopolar machines, wherein the induced electro-motive forces are unidirectional.

28. Distributed Windings. — At the present day *distributed windings* are used almost universally, as they have a tendency to reduce the armature reactions, and also may be more readily arranged to create a curve of voltage which closely approximates the sinusoidal form.* The simplest form of distributed winding is of the coil type, distributed as in Fig. 67, which



Fig. 67. — Distributed Single-phase Progressive Coil Winding.

shows three partial coils per pole, or three slots per pole. Each coil division may be wound complete on a former. The formed divisions may then be placed in the slots and correctly connected together to make up the armature circuit. Windings of this type can be made for two or three phases by superimposing one or two more circuits at 90 or 120 electrical degrees apart respectively. For single-phase windings the

distances AB and CD in the figure are each usually made nearly equal to BC to reduce differential action.*

A common winding for polyphasers is a *distributed winding of the progressive type and is similar to the two-circuit chord windings such as are used on multipolar continuous-current machines.* This arrangement is sometimes called "barrel winding." Figure 68 shows such a winding for six poles having the Y connection. In this particular armature there should be either three or six slots per pole per phase, depending upon whether one or two

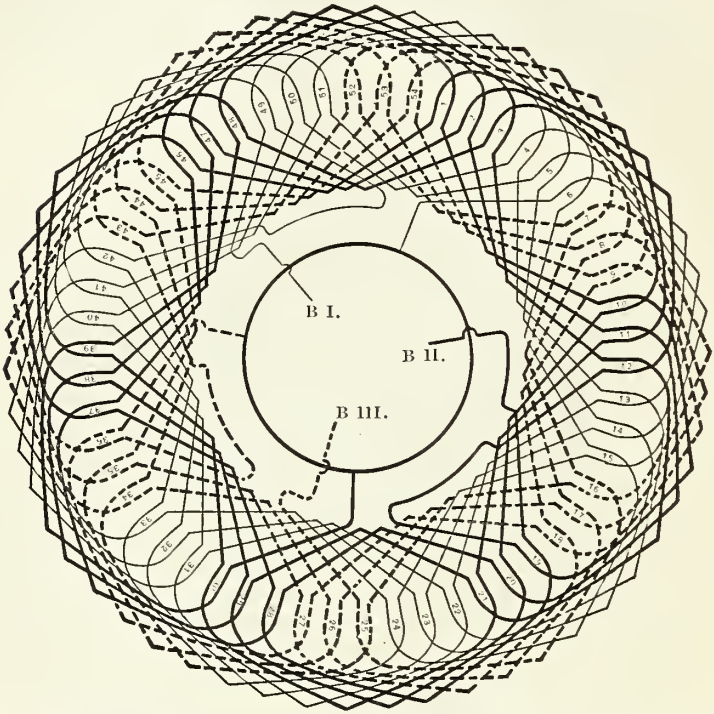


FIG. 68. — Three-phase Winding with Y Connection and Three Slots per Phase per Pole in a Six-pole Machine.

sets of conductors are placed in a slot; and all conductors in a phase are connected in series, thus making it a *one-circuit winding.* It is noticed that, in Fig. 68, after the winding of one phase has passed around the armature three times it doubles back upon itself and passes around the armature three times

* Art. 20.

again. This is to permit connecting the armature with its halves in parallel if desired. If the terminals of the coils, instead of passing to collector rings at *BI*, *BII*, and *BIII*, respectively, are properly connected into the windings to make them reëntrant, and the junctions of the interior connection wires are connected to commutator bars, the winding is suitable for a direct-current machine. In fact the alternator windings of the present day are frequently the same as those used for multipolar direct-current machinery. In tracing the winding

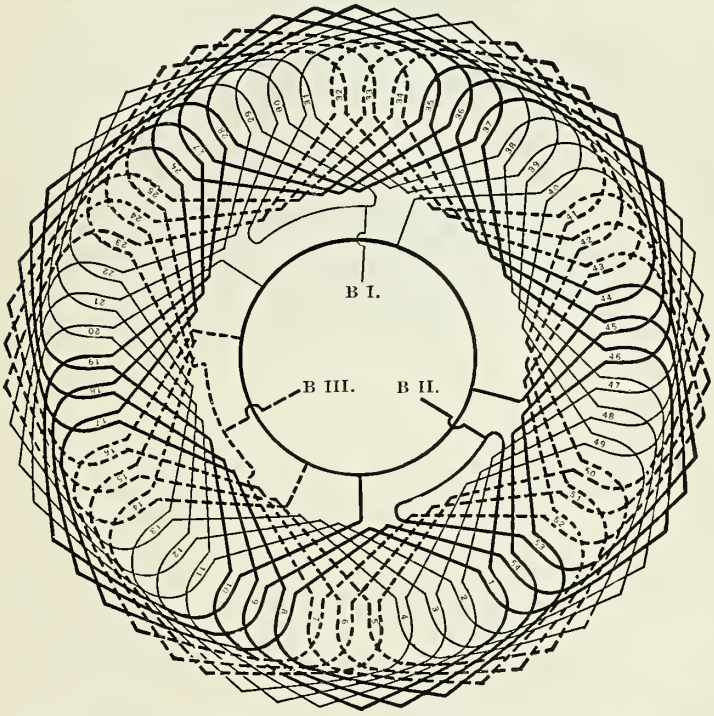


Fig. 69. — Three-phase, Two-circuit Winding with Y Connection for Six-pole Machine.

of one phase through its length it is necessary to follow around the armature six times for the winding illustrated in Fig. 68, which is for a six-pole machine, and the conductors in each complete path traced in going once around the armature generate a voltage wave which differs in phase from the voltage wave of the next set of conductors in the same circuit by an angle equal to the electrical angular distance between the

centers of adjacent slots. As the six slots in each group of Fig. 68 span 60 electrical degrees, the value of K will be 1.055 * approximately. This same winding can be used for a two-phase machine, when, instead of having three sets of windings using 60° of width each, the connections are so made that there are two sets using 90° of width each. In like manner a single-phase machine can be constructed by removing one of the two-phase windings.

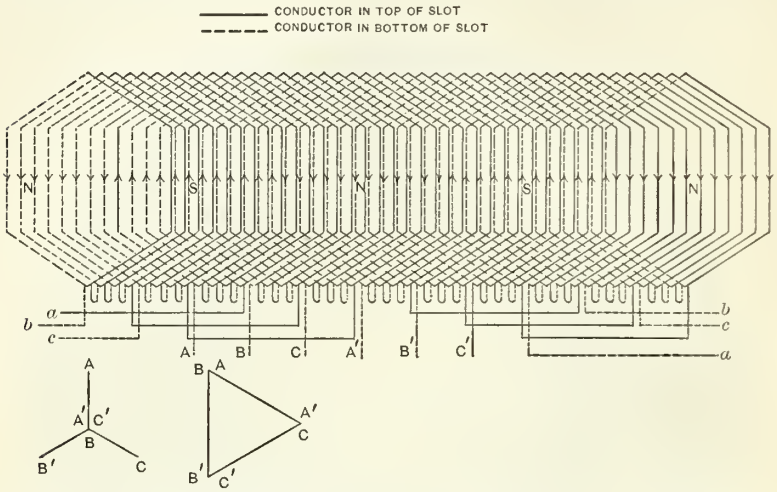


FIG. 70. — Distributed Winding with Four Slots per Pole per Phase connected in either Three-phase Δ or Three-phase Y Connection, for a Four-pole Machine.

Figure 69 illustrates the same armature as Fig. 68, with the halves of the winding of each of the three phases connected in parallel. In this figure, starting from any one of the three terminals *BI*, *BII*, or *BIII*, there are two paths to follow. Each of these paths passes entirely around the armature three times and occupies three slots. In the case of a three-phase winding, as shown, the six slots of a phase should occupy one third of the polar pitch on the armature circumference.

Figure 70 shows a distributed winding to occupy four slots per pole per phase and to be connected in either three-phase Δ or three-phase Y style, intended for a four-pole machine. In this figure the winding is shown in the developed form, *i.e.* the armature is supposed to be rolled out upon a flat surface.

* Art. 20.

The connections at the points a, b, c are brought together in the finished winding. The terminals A, B, C, and A', B', C' are the terminals which pass to collector rings, as shown by the Δ and Y diagrams.

Figure 70 a shows another form of distributed three-phase

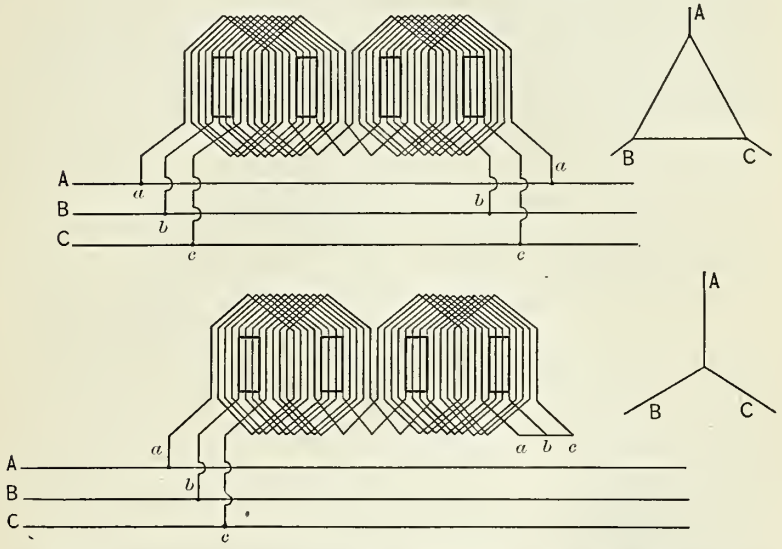


FIG. 70 a.—Distributed Winding with Four Slots per Pole per Phase, connected in either Three-phase Δ or Three-phase Y Connection, for a Four-pole Machine.

winding with four slots per phase per pole, which may be connected in either Δ or Y style. This is a distributed winding of the type shown for one phase in Fig. 62.

The advantage of distributing the winding in several slots, as pointed out in Art. 20, is the reduction of armature reaction and more particularly the closer approximation toward a sinusoidal wave form; but the larger number of conductors used to produce a given effective voltage results in increased self-induction in the armature. The question of the number of slots per phase is settled after balancing the influences of these several effects and the conditions of manufacture. Practice indicates that three or four slots per pole per phase form a satisfactory compromise, though cost of insulation and manufacture often make it desirable to reduce this number at the expense of the wave form. When a sine wave form is of prime

importance, a larger number than four slots per pole per phase is sometimes used.

29. Disk Armatures. — The disk form of armature for alternators is one of the earliest that came into service. The first commercial alternator was a magneto machine (*i.e.* with permanent field magnets) known as the Alliance Dynamo. This was originally devised as early as 1849, but was not developed into commercial form until after 1860. It had a ring armature.

In 1867 Wilde built an alternator with electro-magnets and a disk armature. From that time on, the disk armature received much attention in Europe and was an element of many successful machines designed by such eminent designers as Siemens, Ferranti, and Mordey. It has received less attention in America, and no machines of importance are at the present day being actually manufactured with disk armatures on either side of the Atlantic. This may be due to the essential peculiarity of the disk which admits of no substantial iron core and is therefore difficult to build in a solid and workmanlike manner. Also, it is probably not possible to build machines with disk armatures as economically as those in which the armatures are built upon substantial iron cores; at least when the two types are made of equal mechanical stability. Disk armatures may be wound either with lap or progressive windings. Figure 71 illustrates three coils of a coil-winding arranged to rotate between poles of opposite polarities, and a section through one pair of poles. Figure 72 shows a progressive winding. Either the armature or the field may revolve. (Examples: Ferranti

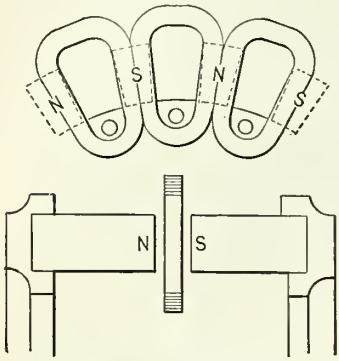


Fig. 71. — Disk Armature with Coil Winding.

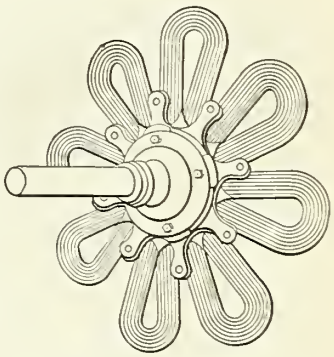


Fig. 72. — Disk Armature with Progressive Winding.

Alternator; Mordey Alternators; Brush Alternator.) The absence of iron in disk armatures reduces the losses due to hysteresis and eddy currents to a minimum, but it increases the depth of the air gap. Hence, a greater exciting current is apt to be required for the field magnets, and many turns must be placed upon the armature. That this objection may be overcome is shown by the small amount of energy required to magnetize the old Mordey alternators. In a 75-kilowatt machine of this type the I^2R loss in the armature is 2.3 per cent and in the field is 1.5 per cent, which compare favorably with the losses in other machines. The curve of voltage in disk armatures is in general quite near to that of the sine function. The first experimental determination of the form of the curve was made in 1880 by Joubert, who experimented upon a Siemens machine having a disk armature. The curve proved to be practically a sinusoid. This is also true of the curve of voltage developed by the Mordey alternator.

A diagram of a Mordey alternator is shown in Fig. 73. It is seen that in the construction of this machine only a single exciting coil F is required, and thus the expense of construction is in some degree reduced. In this machine all poles of the same sign lie upon the same side of the armature. The armature coils are arranged in a disk, which is stationary between the faces of the revolving poles.

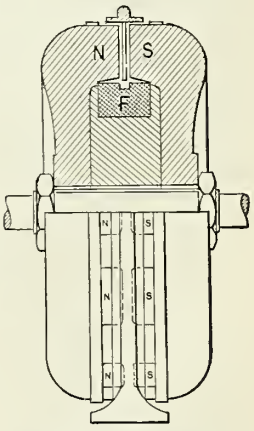


Fig. 73. — Diagram of a Mordey Alternator.

The Ferranti alternator has two crowns of pole pieces and magnet coils at either side of the disk, as shown in Fig. 71. The poles in this machine which lie on one side of the armature alternate in polarity. The Ferranti alternator is of especial interest on account of the Ferranti machines installed at Deptford Station near London, which was the first high-voltage, long-distance transmission plant of any importance. The Ferranti and Mordey machines are now substantially obsolete.

In iron-cored machines, armature reactions have a more distorting effect and therefore tend to modify the form of the vol-

tage curve. The variation is usually not great in machines having distributed windings, but where single-coil windings are used on single-phasers, the curves of voltage may deviate widely from the sinusoid, and sometimes become quite irregular.

30. Other Types of Armatures. — Ring-wound alternator armatures were early used with commercial success, and some of the older magneto machines of the De Meritens type with permanent field magnets and ring armatures are still in existence. The invention of the ring armature for alternating-current machines is usually ascribed to Gramme or Wilde, who independently patented the form in France and England in 1878. In America ring armatures have never been viewed with as great favor as have drum armatures, and they are not used, partially on account of the small mechanical stability of the ring and the greater self-induction of its windings, but more especially on account of the fact that the windings must be placed by hand and are of greater length, which conditions result in excessive expense. Figure 74 is a diagram of a segment of a revolving ring armature intended to revolve inside of an inwardly pointing crown of poles of alternate polarities. Such armatures can, of course, be mechanically arranged so that field magnets with outwardly pointing poles may be located inside the ring instead of the reverse, and either armature or fields may be arranged to revolve in any particular instance.

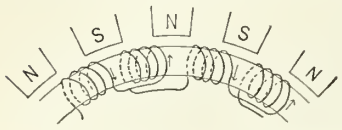


FIG. 74. — Diagram of a Revolving Ring Armature.

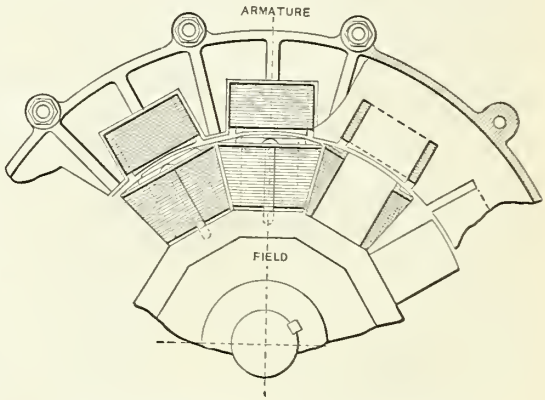


FIG. 75. — Alternator with Pole Armature.

In some of the older types of foreign alternators the armature coils were wound upon long projections similar to the field

cores, and these were called **Pole armatures.** On account of the great variation in reluctance of the magnetic circuit as the armature revolves, such armatures cause large losses in the pole pieces by eddy currents, unless the iron field is entirely laminated. One form of pole armature is shown in Fig. 75. In this machine the armature is stationary while the field revolves. (Example: early Ganz alternators and others.) In the form illustrated in the figure, the reluctance of the magnetic circuit varies less as the poles move past the armature coils, on account of the fact that the armature coils are really wound in deep slots, thus reducing the type to a close approximation of the drum windings on slotted cores, as already described.

31. Methods of Applying the Wires to Chord-wound Armatures. —The coils for alternator armatures are ordinarily wound on formers and then fastened upon the armatures after having been well insulated. At present the armature windings are nearly always placed in slots, which are properly formed in the surfaces of the cores, but most of the older American machines had their windings fastened upon the surfaces of smooth armature cores. In the old machines where surface windings were used, the coils were frequently arranged to bend down over the ends of the cores, as illustrated in Fig. 76, where they could be securely fastened by end plates or blocks of wood or fiber. It was usual to fill the spaces in the centers of the coils with wood blocks screwed to the cores or held by binding wires. (Examples: early Westinghouse and National alternators.) In some machines the coils were flat or pancake-like, and of the same length as the armature cores. In this case they were laid upon the cylindrical surfaces of the armature cores and securely bound with wire bands. (Examples: early Thomson-Houston and similar alternators.) The high peripheral velocities of alternator armatures make heavy bands essential to the preservation of surface windings, and all blocking must be securely

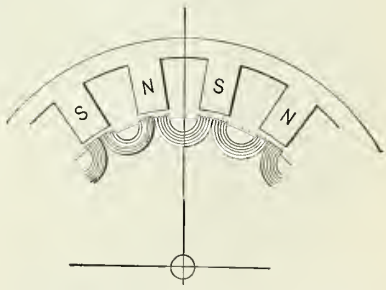


FIG. 76.— Surface-wound Alternator Armature.

fastened. The wood blocks which fill the center of the coils make excellent driving teeth, and therefore serve a good purpose if they are fastened securely to the core.

When imbedded coils are used, they may be made upon formers (lathe-wound), and then applied to the core, or the conductors may be threaded through the grooves which are provided with insulating linings. When the core teeth are T-shaped, the coils are sometimes wound of sufficient width to slip over the head, and when in place, they may be narrowed by squeezing. The coils for this purpose must be wound with the ends of such shape that they will bend without injury, as is shown, for example, in Fig. 77. The methods used in manufacturing the coils and applying them to slotted armature cores of direct-current machines may also be used in the construction of alternator armatures with distributed windings; but the alternator armatures are usually of much higher voltage, and the insulation must be carefully and specially designed. Construction of this character (namely, using small distributed coils) has largely superseded construction of the character illustrated in Fig. 77.

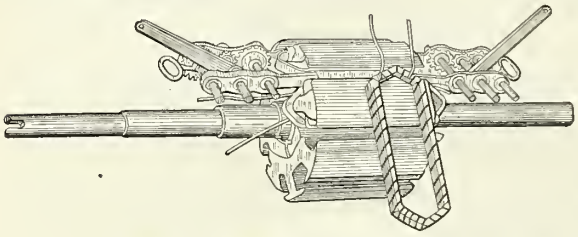


FIG. 77. — A Method of Inserting Coils over T-shaped Teeth.

Lathe-wound coils are of decided advantage for armatures designed for high voltages, as their insulation may be made particularly safe. Such coils are usually served with layers of japanned canvas, special fuller or press board made for insulating purposes, vulcanized fiber, and mica. The slots between the core teeth may be made of sufficient area to permit the use of any desirable thickness of insulation, and the teeth afford very complete mechanical protection for the coils. Toothed armatures with lathe-wound coils should therefore be thoroughly reliable. If the magnetic surface of the armature is fairly complete, the wires are protected from magnetic drag, which decreases the chances of the conductors chafing and injuring the insulation. The winding shown in Fig. 77 is not distributed, but grouped in one slot per pole, as shown also in the

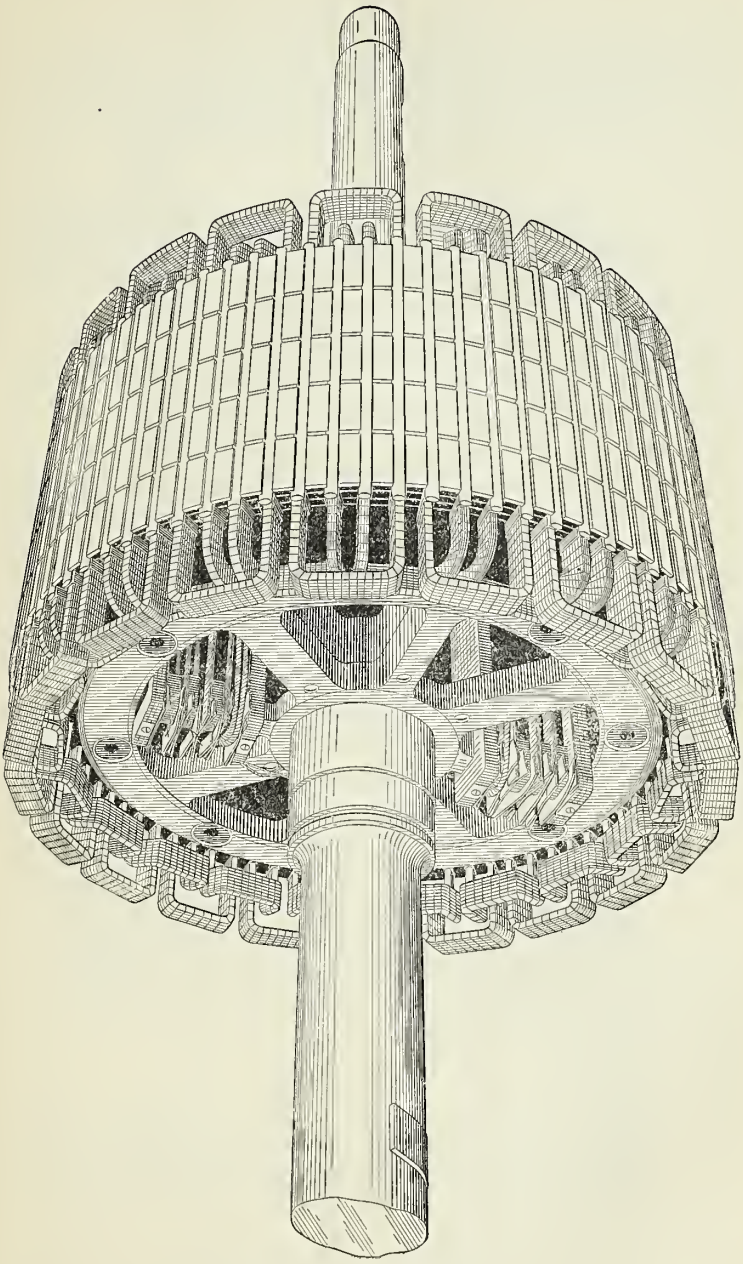


FIG. 78.—Two-phase Armature Construction.

diagrams of Figs. 61 to 63. Distributed windings have very largely superseded this type, as already explained.

The construction of a two-phase group-wound armature, having open rectangular winding slots in the core, is shown in Fig. 78. Each phase in this case forms a complete single-phase winding, as in Figs. 61 and 63. It is to be noticed that the forms upon which the coils of one phase are wound are of such a shape that the ends of the coils bend down over the ends of the armature, while the coils of the other phase are rectangular. This construction keeps the coils well apart at points of crossing. After the coils are properly insulated they are slipped into the rectangular slots, and are held in place by wedge-shaped strips of hard wood. Grooves are arranged near the upper edges of the iron teeth into which the wooden strips fit. In this manner the windings may be held very firmly without the aid of band wires. The four peripheral slots to be observed in the core are ventilating openings, maintained by fingered grids inserted between the adjacent core stampings.

The way in which coils are arranged for insertion into the partially closed slots of the modern distributed armature windings is shown in Fig. 79. One end of each coil, which in this case is composed of several turns of round wire, is left open. The coil is then slipped through the proper slots, and the several wires are joined to complete the coil. In this figure, each coil occupies four slots and is therefore a distributed winding. The coil terminals whereby the coils are connected with each other are shown as spirally insulated wires at the top of the figure. Another and similar winding for partially closed slots is shown in Fig. 80. This winding, as seen, is placed upon a stationary armature intended to surround a rotating field, and copper straps or bars are used for making the coils instead of round wire. To complete the winding, the open ends of the coils must be brazed together and the coils themselves be interconnected. Figure 81 shows how the straps are brought around ready for brazing, and also shows the completed joints after the final insulating wrapping has been applied. In the left-hand side of the figure may be seen the connectors between coils. The method of insulating the slots is shown in the lower right-hand corner of the figure; the two slots, which have not as yet received their coils, show the tubes of insulat-

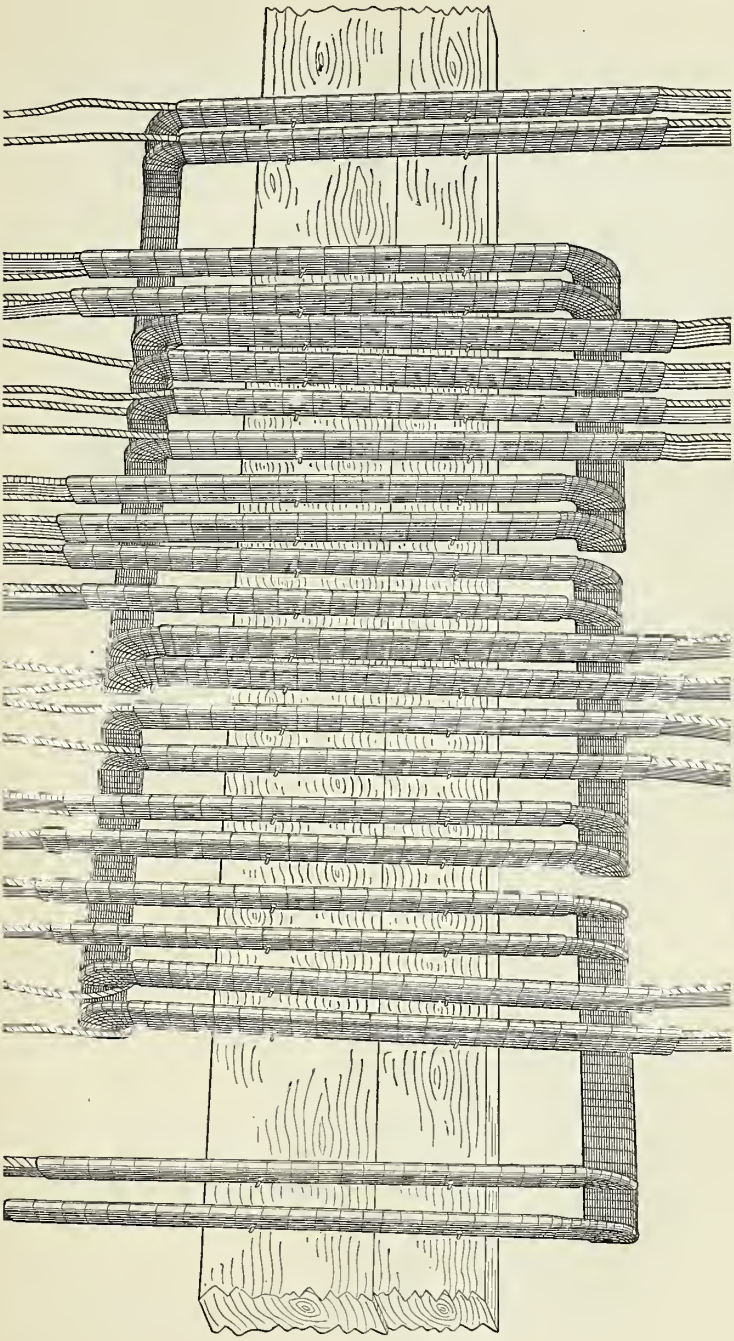


Fig. 79.—Grouping of Wire Coils for Partly Closed Slots.

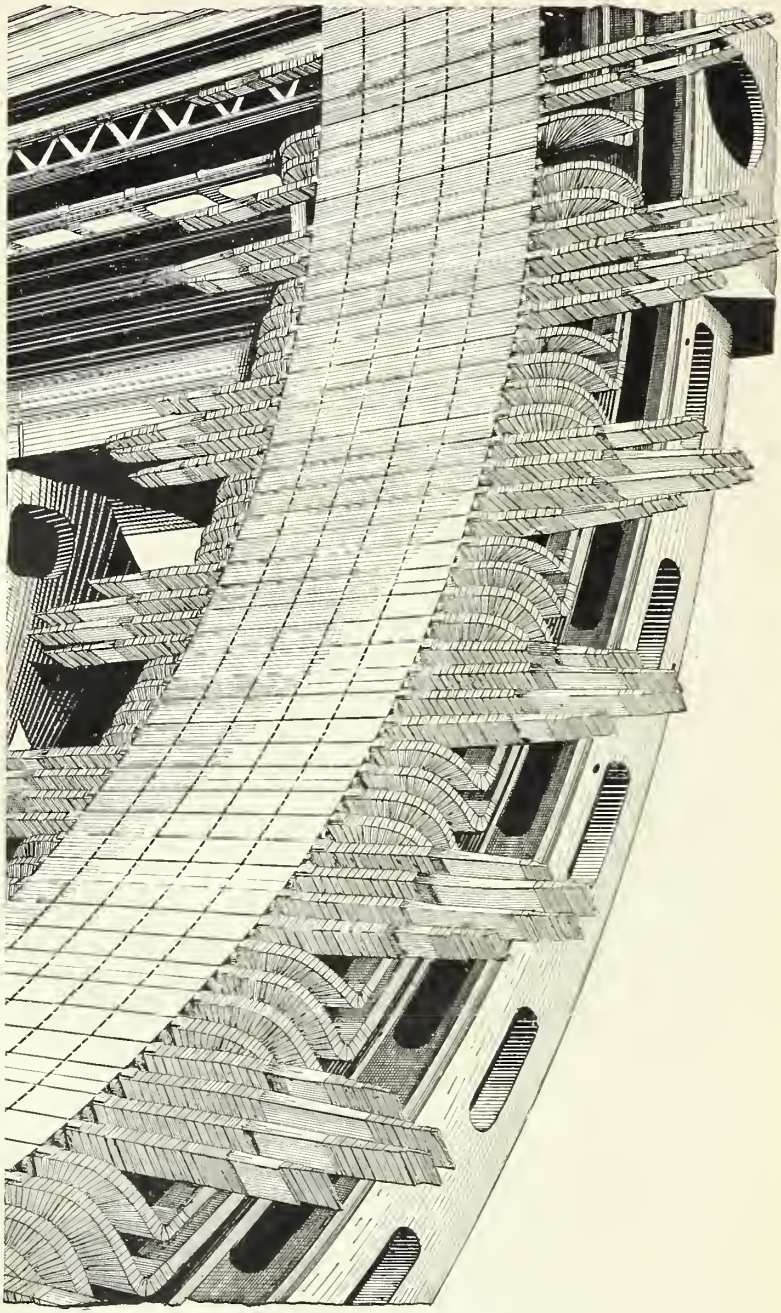


FIG. 80. — Strap or Bar Coils in Partly Closed Slots. Incomplete Winding.

ing material — micanite, fuller board, or other insulator — extending out a short distance from the core. The method of taping the coils to complete the insulation can also be readily seen. The conductors are held firmly in place by wooden wedges driven in the tops of the slots in the manner more plainly seen in Fig. 78.

In Fig. 82 is shown a three-phase, coil-wound, stationary armature having a distributed winding of four slots to the coil.

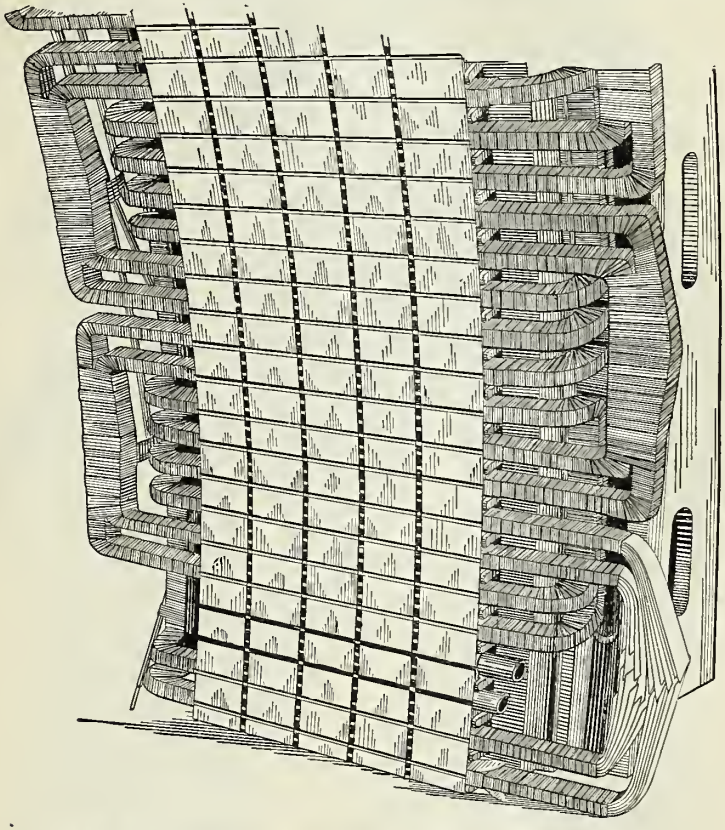


FIG. 81. — Method of Connecting Open Ends of Strap Coils.

The particular machine from which the photograph was taken is wound to generate a voltage of 13,000 volts, and is therefore very heavily insulated.

Figure 83 shows a model of a three-phase partially distributed winding applied to one of the 5000-kilowatt generators

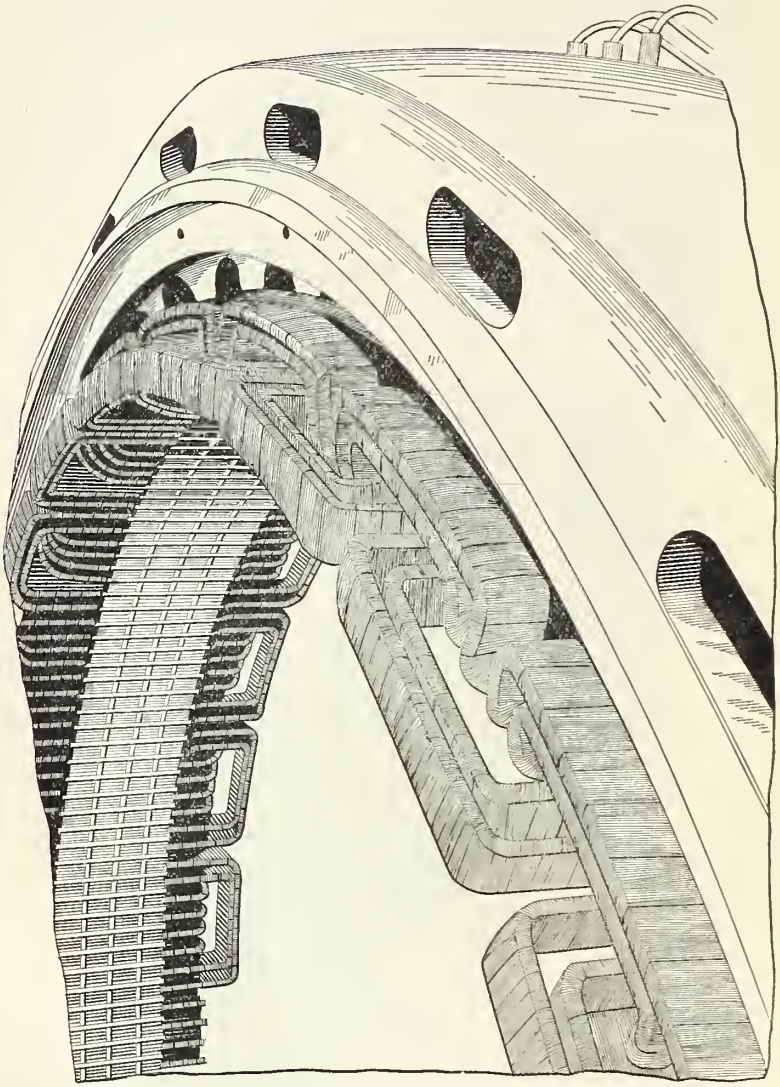


FIG. 82.—Three-phase Winding for 13,000 Volts.

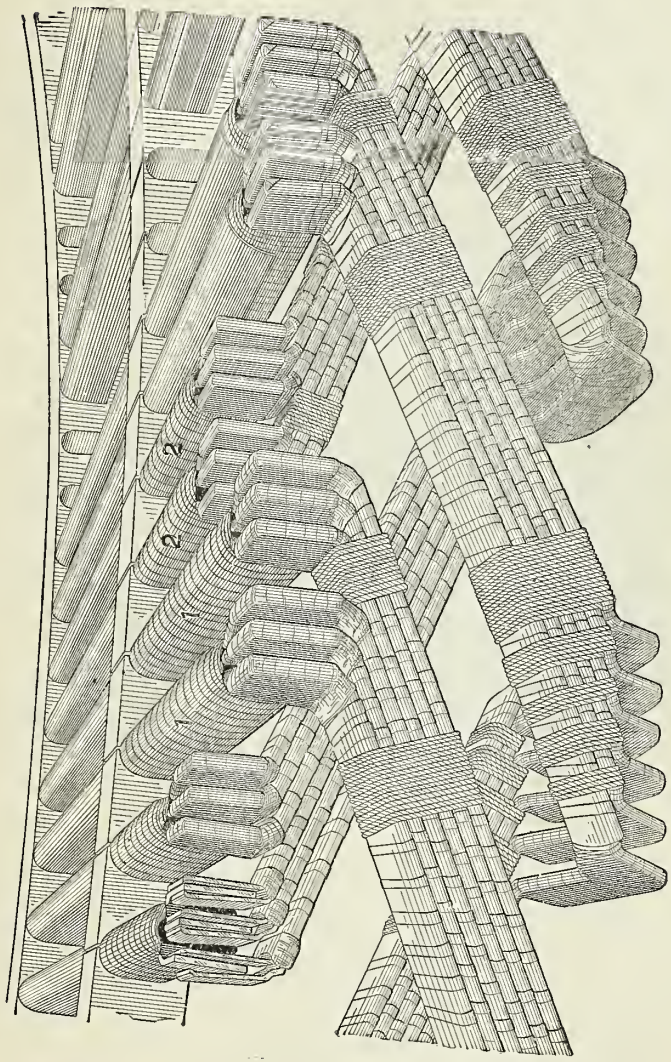


FIG. 83.—Model of Three-phase Partially Distributed Winding.

of the Manhattan Station in New York City. It is especially interesting as showing the manner in which the windings are arranged for crossing at the ends and the methods of construction and insulation.

The lap, progressive, and barrel windings given heretofore are suitable for any type of generator, including those used with steam turbines. Figure 83 *a* shows the stationary armature of a horizontal steam turbine generator of 1000-kilowatt capacity of Westinghouse type. It is evident from the con-

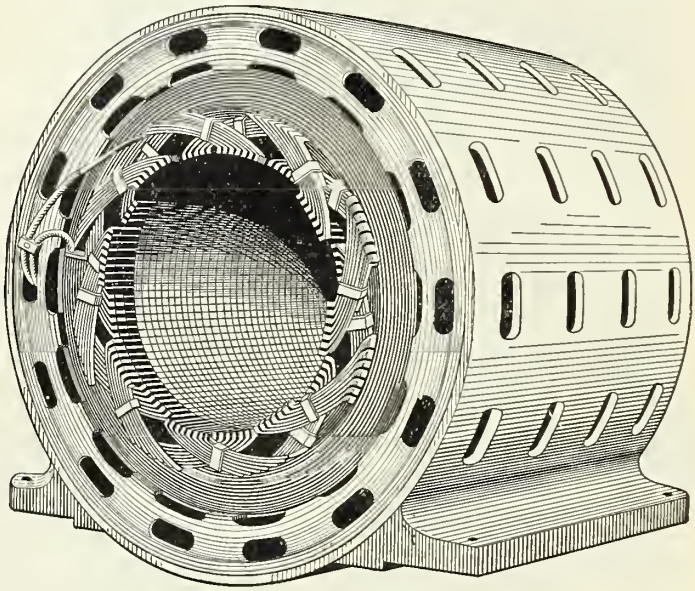


FIG. 83 *a*. — Turbine Generator Armature.

struction shown in the figure that this type of generator is of the same character as those driven in a different manner. The high speed of such machines makes it necessary to wind them for a comparatively small number of poles. The winding in the figure is two-phase, four-pole, of the distributed coil type.

32. Collectors. — In all the alternators heretofore described, either the field or the armature windings are arranged to revolve. In the case of inductor alternators to be described later it is possible to arrange the construction so that both the magnetizing and armature coils may remain stationary, but mechanical and other considerations ordinarily render this inad-

visable. It therefore may be said generally that either the armature or the field windings must revolve with the iron cores on which they are wound. This makes essential the use of some means for conveying the current to and from the revolving coils.

In either case no modification of the current is required and therefore plain insulated **Collector rings** serve the purpose when they are properly mounted on the shaft so that the brushes may be arranged to bear against them. Figure 84 is an illustration of collector rings for conveying the current to a multipolar revolving field. These rings, often called **Collectors**, are sometimes made of copper or bronze, though they are now commonly made of cast iron. In the older construction the rings were imbedded in cylinders of vulcanized fiber or the like keyed to the shaft of the rotating part. In modern construction the rings are usually placed upon a metal spider and hub. This makes it possible to do away with such a large bulk of insulation, and thus adds materially to the mechanical rigidity of the construction. The insulating materials used are ordinary pressed fiber, vulcanite, or some form of mica board. Figure 92 shows four rings mounted between discs of solid insulation, for a four-phase rotating armature; and Fig. 84 shows two rings on metal spiders affording air insulation.

Collection is obtained by means of brushes fastened to the brush holders and bearing upon the rings. These brushes are usually of carbon, electroplated with copper to increase their conductivity. The copper coating is cut away from the end of the brush which bears on the ring. As carbon has a high resistance it is not suitable for machines delivering a very large amount of current. In such cases it is common to use brushes made of fine woven copper wire gauze. Former practice employed brushes made of copper strips, but as it is difficult to make strip brushes which will not cut the rings, the substitution of carbon and woven wire has become almost universal.

A carbon brush can collect without undue heating from about 40 to 80 amperes per square inch of contact surface, while a woven-wire brush can collect over two or three times that amount. A number of brushes may make contact with each ring.

The brushes are mounted in brush holders, which in turn are



FIG. 84. — Rotating Field Magnet, showing Collector Rings, and also showing Exciter Armature mounted on Same Shaft.

fastened to the frame of the machine by some convenient arms or brackets; the circuit connections are often made to the brushes by means of flexible copper cords or strips soldered on, in addition to the contact of brush with holder. The brush holders are so arranged through the medium of springs that the brushes may be made to bear upon the collector rings with any desired pressure. Ordinary practice calls for a pressure of from 1 to $1\frac{1}{2}$ pounds per square inch. A greater pressure is apt to cause cutting of the rings and a less pressure may permit the brushes to vibrate sufficiently to cause sparking.

The safe limit for the amount of current that can be collected per square inch is fixed by the heating alone and therefore may be quite large without danger. If mechanical considerations required it, doubtless as much as 500 amperes per square inch of contact might be satisfactorily collected by means of copper gauze brushes and as much as 200 amperes by carbon brushes, but such extreme cases do not often arise and it is not generally advisable to exceed one fourth these amounts.

Instead of brushes the collection may be effected by means of flexible weighted copper bands hung from the rings. Figure 85 indicates such an arrangement. An arrangement somewhat similar to that of Fig. 85 was used on the Ferranti alternators of the famous Deptford Station already referred to.* By this construction a large collecting area may be gained without unduly increasing the size of the rings, but the arrangement of carbon brushes rubbing on flat rings is mechanically preferable. As modern alternators are usually of the revolving field type and the collectors are used for transmitting exciting current to the revolving field coils, the collectors need not be so highly insulated as would be necessary for a high alternating voltage. The field voltage seldom exceeds 220 volts. It is good practice to mount the two rings, for this

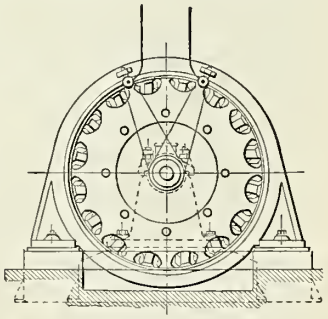


Fig. 85. — Copper-band Contactors.

* Art. 29.

purpose, upon separate spiders attached to one hub, so that there is an air space between the rings.

33. Field Excitation of Alternators. — The exciting current for synchronous alternators must be unidirectional, and it is ordinarily obtained from auxiliary direct-current shunt or compound dynamos called Exciters, as illustrated in Fig. 86. If the alternator field magnets are stationary, this exciting current is fed directly into the coils, but if the machine is of the revolving field type, the exciting current must be led into the coils through collector rings; in this case the armature current is led directly from the terminals of the stationary armature. The exciter is frequently driven from the same prime mover as is its alternator, and is sometimes even mounted upon the same shaft. In generating plants having large units, however, it is better practice to have the exciters driven by separate prime movers, in which case two or three exciters may be provided to furnish the exciting current for all the main generators. The regulation of the alternator voltage is secured by varying the resistance in the field circuit of the alternator or by varying the field strength of the exciter. These are the usual methods in America and are done both by

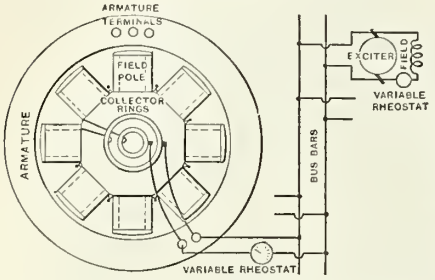


FIG. 86. — Diagram of a Separately Excited Alternator, with Bus Bars for Parallel Connection of Exciters and Alternators. Switches, etc., not Shown.

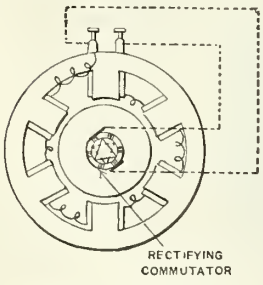


FIG. 87. — Self-excited Alternator.

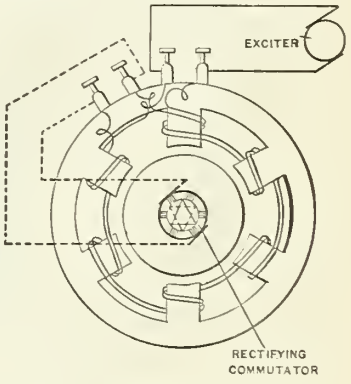


FIG. 88. — Compositely Excited Alternator.

hand or by an automatic controller, as is explained in detail in the chapter dealing with alternator operation. Self-excitation and composite excitation, as shown below, were much used earlier and have sufficient possibilities to warrent study.

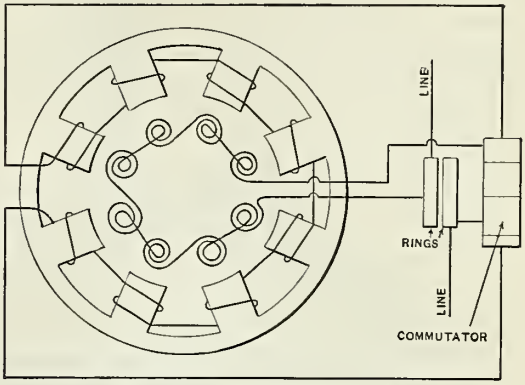


FIG. 89. — Diagram of a Series-wound Alternator.

By suitable commutation the alternator may evidently be made to furnish its own exciting current either wholly or in part, and the windings of the field magnets of alternators may be classified according to their arrangement in circuit. The principal divisions are three: **separately excited, self-excited, compositely excited**; so-called, respectively, when the exciting current is supplied from an external source (Fig. 86), when it is supplied through a rectifying commutator from the armature of the machine under consideration (Fig. 87), or when these two arrangements are combined (Fig. 88).* Self-excited alternators may again be divided into series-wound and shunt-wound, depending upon, first, whether the whole current is rectified and led through a comparatively small number of turns around the field magnets (Fig. 89), or, second, whether only a portion of the current is rectified and led through a

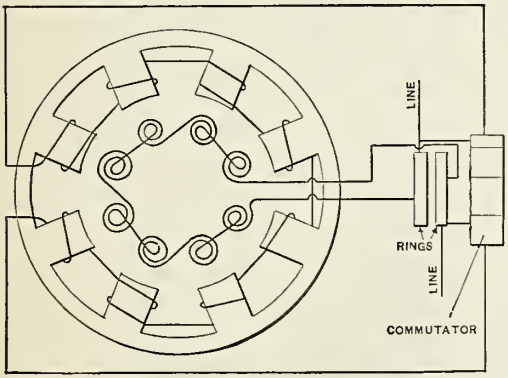


FIG. 90. — Diagram of a Shunt-wound Alternator.

* Compare Jackson's *Electromagnetism and the Construction of Dynamos*, p. 136.

shunt circuit many times around the magnets (Fig. 90). In the latter, either the whole voltage of the armature, or that of one or more coils, may be impressed upon the rectifying commutator either directly, as illustrated in Fig. 90, or by means of a transformer attached to the armature. Figure 91 illustrates the transformer arrangement when the fields rotate. Self-excited alternators are now quite uncommon, as the inconveniences accompanying the rectification of the current under usual conditions of operation more than outweigh any advantages derived from making the machines self-contained.

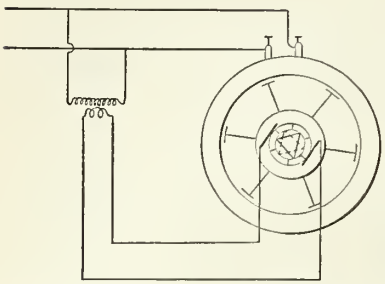


Fig. 91.—Diagram of a Shunt-wound Revolving Field Alternator, with the Voltage reduced for Commutation by a Transformer.

Composite Excitation. —Evidently another division might be added to those named in the preceding paragraph, which would comprise a compound winding in which both the shunt and series field currents are supplied by rectification. This, however, would require two rectifying commutators, which at the best are unsatisfactory, and for other reasons would not prove practical. To gain the result for which compounding is used in direct-current dynamos, the composite winding is used. That is, the alternator is *externally excited to its normal voltage on open circuit* and the internal losses *are compensated by series ampere-turns from self-excitation.* This method of excitation is common with machines having rotating armatures and is very desirable in small or medium-sized electric light plants or where the nature of the service calls for constant regulation with meager switchboard attendance. The series turns may be so arranged that they will add more than the voltage lost in the machine itself, thus providing some compensation for the voltage lost in the feeder wires and tending to maintain the voltage at the center of distribution approximately constant. Evidently, in the latter case the voltage at the machine terminals will rise when the load increases. Under such circumstances the machine is said to be **Over-compounded**, and the term **Compound-wound** has of recent

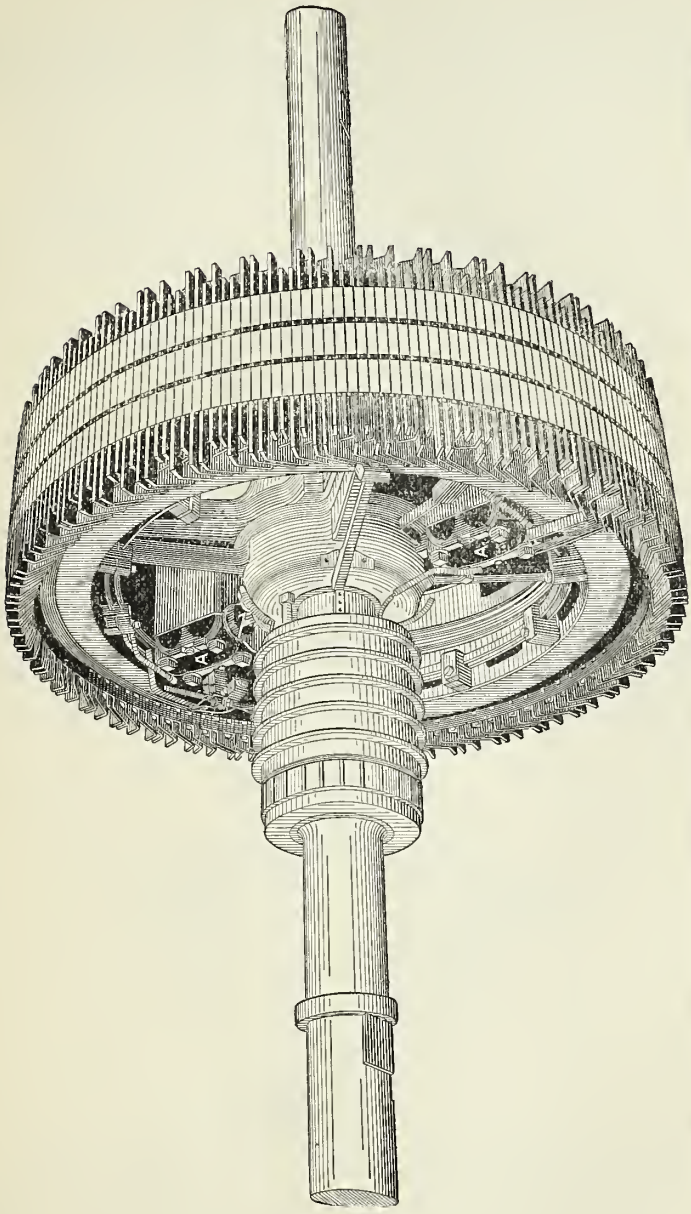


FIG. 92.—Revolving Armature arranged with Series Transformers for Composite Excitation. *AA*, Series Transformers.

years come into general use to designate these machines instead of the more distinctive term composite-wound.

The self-exciting circuit of the composite winding may be arranged in various ways. Thus the armature current may all be rectified for use in excitation (Fig. 93) or it may pass

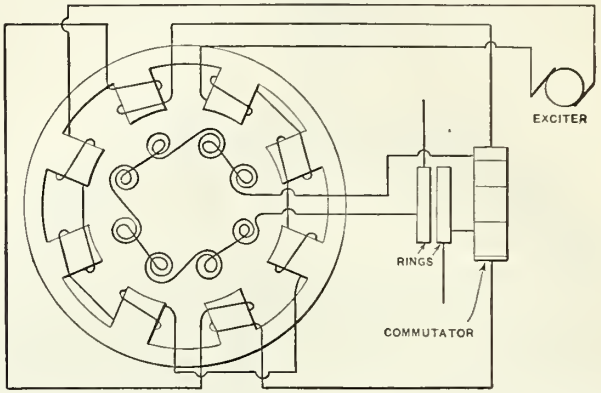


FIG. 93. — Diagram showing All of the Armature Current rectified for Field Excitation.

through special series transformers attached to the armature, and the secondary of these may then supply the current for rectification and self-excitation (Figs. 92 and 112). The cores of these transformers may be either independent of the arma-

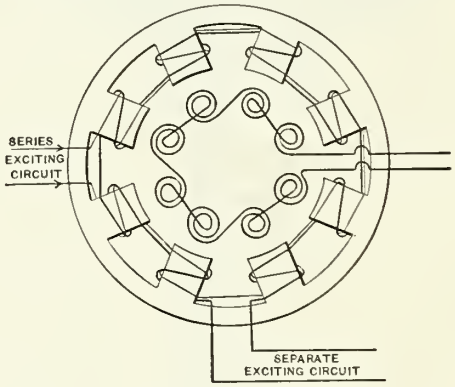


FIG. 94. — Diagram of Composite Excitation with Series Turns on All Poles.

ture core, as at A in Fig. 112, or may consist of the laminated spider or other portions of the armature core. Again, the rectified current may be passed through a few turns of wire

on each pole (Fig. 94), or all the necessary series turns may be concentrated upon one or two poles (Figs. 95 and 96). In the latter case, the series turns must always be equally divided between two poles with symmetrical positions, as in Fig. 96,

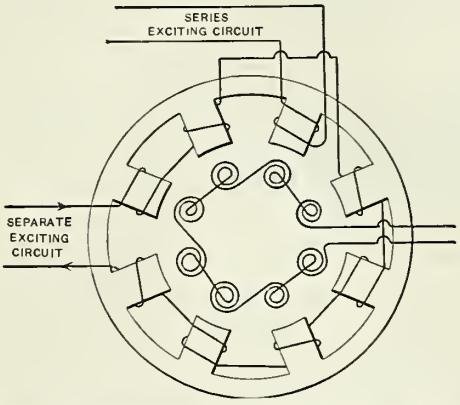


Fig. 95.—Diagram of Composite Winding with Series Turns on One Pole.

when the armature winding is connected with its halves in parallel. Composite windings may be arranged with the self-exci-

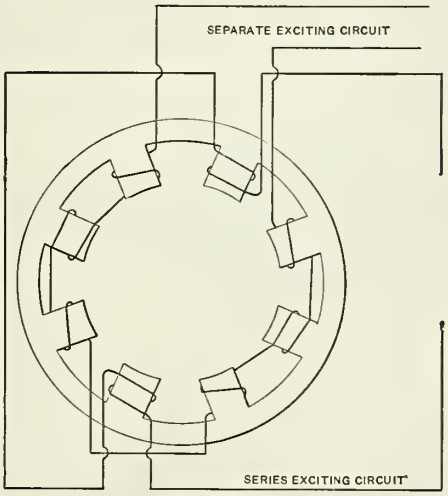


Fig. 96.—Diagram of Composite Winding with Series Turns on Two Poles.

tation in a shunt circuit, but no advantage is gained by this arrangement over complete self-excitation in shunt or over separate excitation, and the arrangement is never used. In some

old-type self-exciting alternators, a separate set of exciting coils was wound on the armature and connected to a rectifying commutator. These were wound either directly with the main armature coils, or across a chord of the armature core, as illustrated in Fig. 97, in which the coils A for generating exciting current are wound across a long chord of the armature, as shown in the upper diagram of the figure, while the main armature coils B are wound pancake fashion. The compounding may be effected in self-excited alternators by means of shunt and series transformers combined as in Fig. 98, which shows a machine with stationary armature. This arrangement pertains to some early European examples put out by Ganz and Co. The arrangement permits a self-excited alternator to be compounded by the use of only one commutator. The shunt transformer furnishes what is equivalent to the shunt current of an ordinary compound dynamo, and the series transformer adds additional voltage to increase the current as required for compounding. This latter voltage increases with the load on the alternator as the primary winding of the transformer is in series with one of the alternator leads. In the illustration, S represents the series transformer, T the shunt transformer, and R a variable rheostat to control the degree of excitation.

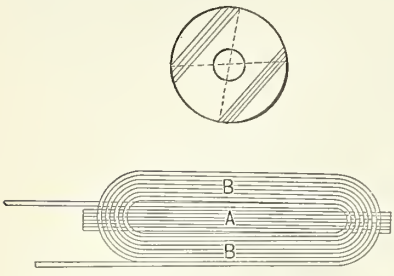


Fig. 97. — Method of Winding Special Alternator Exciting Coil.

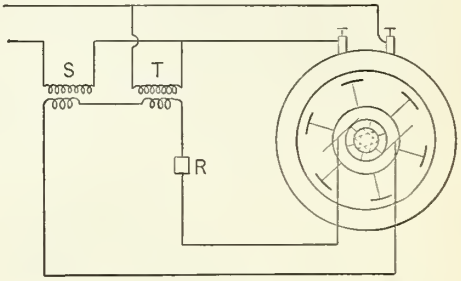


Fig. 98. — Diagram of Compounding by Means of Shunt and Series Transformers.

For the purpose of varying the magnetizing effect of the series turns, a variable shunt is often connected across their terminals in a manner similar to the usage with direct-current machines, as at A in Fig. 99, and a shunt is sometimes placed across the rectifier terminals in such a way that only a fixed

proportion of the total current is rectified and passes through the series field winding, as at B in Fig. 99. This latter shunt reduces the difficulties caused by sparking by reducing the current to be rectified.

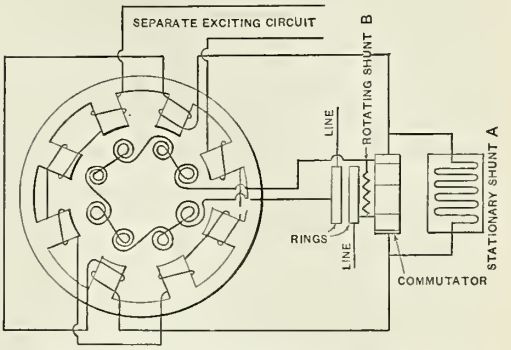


FIG. 99. Diagram of Composite Excitation having the Series Field and the Rectifier Shunted.

Polyphase alternators may be compounded in the manner described above by rectifying the current of one phase.

Another method of compounding is based upon the magnetic reactive effect of current in an armature upon the field. The revolving field magnet of such a machine, called a "compensated alternator," is shown in Fig. 100. Upon the same shaft as the revolving field magnet is placed the direct-current exciter armature and its commutator. Surrounding the revolving field magnet is a three-phase stationary distributed armature winding of the ordinary type, and supported by the same frame

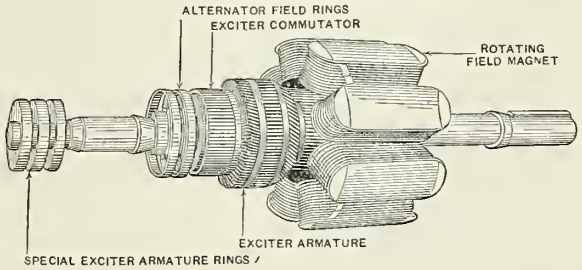


FIG. 100.— Revolving FieldMagnet and Exciter Armature of a Compensated Alternator.

is the exciter field, which has the same number of poles as the field of the alternator. Direct current is drawn from the commutator of the exciter, and part is taken through the alternator field rings to the revolving field magnet, and part goes through a rheostat to the stationary exciter fields. So far as described the arrangement corresponds with an ordinary

separately excited rotating field alternator having the exciter mounted upon its shaft. In order to procure the compounding effect, however, the alternating currents from the secondary windings of three series transformers placed in series with the three-phase alternator leads, are led by means of the special exciter armature rings into the exciter armature winding. As explained above, the exciter and alternator field magnets have the same number of poles, and hence the frequency of the electromotive force is the same in the exciter and alternator armatures, and therefore the alternating current will pulsate through the exciter armature with the peaks of its waves in a definite position with reference to the exciter pole pieces. By a proper organization, the reactions of the three-phase currents thus fed to the exciter armature can be caused to increase the strength of the exciter fields in a degree dependent on both the amount and phase position of the currents. The field magnet of the exciter is mounted so that it can be angularly shifted, slightly, so as to bring the impulses of the alternating current into proper relation with the polar positions. Such machines can be made for any number of phases.

34. Rectifying Commutators. — The rectifying commutator has as many segments as there are poles on the alternator, and alternate segments are joined in electrical connection, making two sets composed of alternate segments, as illustrated in Fig. 101. When series excitation is to be provided by the rectified current, one terminal of the armature winding and one terminal of the external circuit are attached respectively to these two sets of segments, as shown in the figure, or the secondary terminals of a series transformer are respectively attached to the two sets if the excitation is to be obtained through the medium of such a transformer, as shown in Fig. 92. Brushes bearing upon the commutator at non-sparking points (that is, under the circumstances considered, as nearly as practicable at points of zero current) then collect a rectified current. The brushes are shown in Fig. 101 bear-

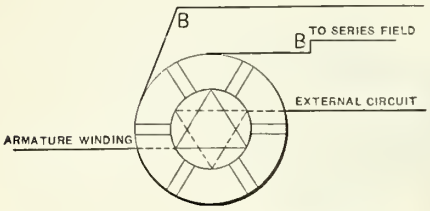


FIG. 101. — Rectifying Commutator.

ing upon adjacent commutator segments, but they may obviously be placed on the commutator so as to be any odd number of segments apart. It is evident that direct current will flow in the series field between the brushes B, B when the circuit connections are those indicated in Fig. 101, and the machine is in operation. The commutator possesses as many segments as there are poles in the field magnet, and if the brushes are properly placed, each brush will pass from one set of commutator bars to the other at each alternation of the current. That is, the polarities of the segments reverse, but the brushes simultaneously slip from one set of segments to the next, and therefore remain of fixed polarity. The current in the circuit of the series field is unidirectional, although it is electrically in series relation with the armature and external circuit, and the current is alternating in them.

To obtain the maximum effect, the brushes must pass across the insulation between the commutator segments at about the time when the current in the armature reverses. Minimum sparking will occur if the brushes are shifted to the point where the current is zero in the circuit at the instant each brush breaks contact with one segment as it slips on to the next. If the brushes are shifted in the forward direction, commutation will take place on a rising current, which is likely to cause quite serious sparking. The character of the current in the series field winding manifestly could be modified by shifting the position of the brushes, if sparking were not prohibitive. For instance, if the brushes could be maintained in the position where commutation takes place when the armature current is maximum, the current in the series field circuit would in each period change gradually from a maximum in one direction to a maximum in the other and be then reversed to a maximum in the first direction, and the resultant effect on the strength of the field magnet would be zero. On the other hand, if the brushes are upon the neutral points, that is, commutate at the instant the armature current is zero, the current is unidirectional in the field circuit.

Various devices have been employed to avoid sparking at the rectifying commutator, but in American machines no special precautions are taken. In a Zipernowsky alternator, built by Ganz and Co. of Budapest, the following arrangement of

the commutator was employed. Between the commutator divisions are inserted narrow metallic sectors which are connected together. Four brushes are used, two on each side of the commutator. One brush of each pair is set a little in the lead of the other, and the pair is connected together through a small resistance. The leading brushes are connected directly to the circuits. When the commutating point is reached during the rotation of the commutator, the trailing brushes move on to intermediate segments, while the forward brushes are still on main segments. Hence both the field circuit and supply circuit are short-circuited for an instant through the resistances connecting the brushes (Fig. 102). Upon short-circuiting the fields with this or an ordinary commutator, their self-inductance tends to uphold the current in the windings, and this, therefore, does not fall to zero at each commutation, as shown in Fig. 103, but the current curve becomes a wavy line more like that of Fig. 104. Picou says * that it is

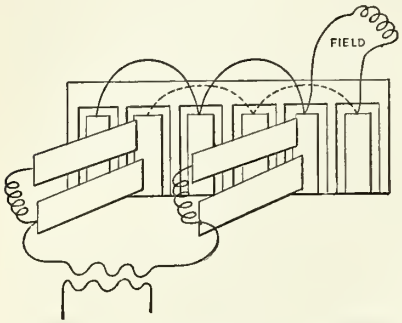


FIG. 102. — Special Rectifier for Minimum Sparking.

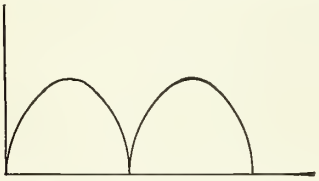


FIG. 103. — Current Impulses that would be produced by a Rectifier having no Self-inductance in Circuit.

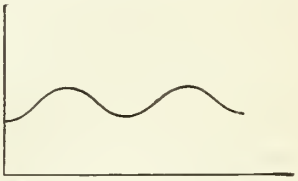


FIG. 104. — Curve showing the Wavy Current which passes from a Rectifier to the Alternator Fields.

preferable to place the brushes so that commutation occurs slightly earlier than the point of least sparking. In this case the spark is due to a decreasing current, and is thin and weak. With the commutation occurring later than the point of least sparking, the spark is due to a rising current, and it is of great

* *Machines Dynamo-Électriques*, p. 99.

magnitude. The advantage of a wavy current in the field, instead of a discontinuous one, is sufficiently well gained by the use of copper brushes of considerable thickness, on an ordinary rectifying commutator, which short-circuit the supply circuit and field circuit at the instant when they bridge over the insulation between two segments. Figure 104 very closely represents the form of current curve under these circumstances.

Figure 101 shows the circuit connections at the rectifying commutator of a compound-wound machine with rotating armature. For a machine with a rotating field magnet, the series field connections and the connections to armature and external circuit are interchanged so that the former go directly to the commutator segments and the latter to the brushes.

35. Inductor Alternators. — The windings of inductor alternators may be made entirely stationary, thus avoiding collecting devices. These devices, however, are of so little expense in construction and material that there is no marked advantage in suppressing them, and the mechanical and magnetic difficulties encountered in the design and construction of inductor alternators have not permitted them to come into general use, though there are several theoretical points of advantage presented in their design. In order to avoid excessive losses due to eddy currents and hysteresis it is important that the magnetic density in the field magnets be kept as uniform as possible. Since this cannot be fully accomplished, it is necessary to thoroughly laminate the iron in which the density varies. The inductor must be moved in such a way as to periodically short-circuit or break the lines of force which naturally pass through the armature coils. This may be accomplished as shown in Fig. 105, where the effective reluctance of the total magnetic circuit is fairly constant for all positions of the inductor. In this example it is seen that as an inductor travels from one pole piece to the next of opposite sign, the mag-

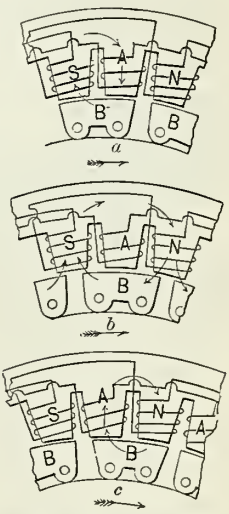


FIG. 105. — Diagram of an Inductor Alternator. *A*, *A* are the Armature Windings, *NS* the Field Windings, and *BB* Inductors.

netism through the armature coil changes from a maximum in one direction to a maximum in the other. In spite of the fact that the field poles are maintained at a fairly constant magnetic density it is evidently necessary, in order to sufficiently reduce the iron losses, to laminate all the iron of the magnetic circuit. The figure shows that the reluctance cannot remain entirely constant, and that the effective ampere turns in the magnetic circuits also vary with the position of the inductor. In Figs. 106 and 107 are shown two types of inductor machines in which no attempt is made to keep the magnetic circuit of constant reluctance. Each of these forms gains some economy in construction by uniting the coils.

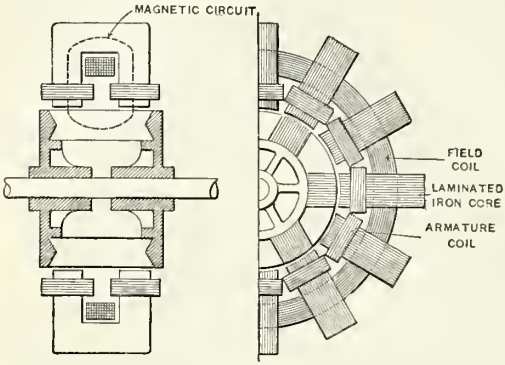


Fig. 106. — Diagram of Inductor Alternator having a Single Magnetizing Coil.

In the Stanley alternators which were at one time considerably used in this country, lines of force are caused by the motion of the inductor to sweep across the armature coils, while the total magnetism in the inductor remains fairly constant. The field windings, which are arranged in the form of a single coil, embrace the inductor core, and though this coil is stationary the machines are not true inductor alternators. Figure 108 shows the so-called inductor and field coil of a 2000-kilowatt alternator of the Stanley type. The field coil rests in the frame, as seen in the figure, and does not rotate with the in-

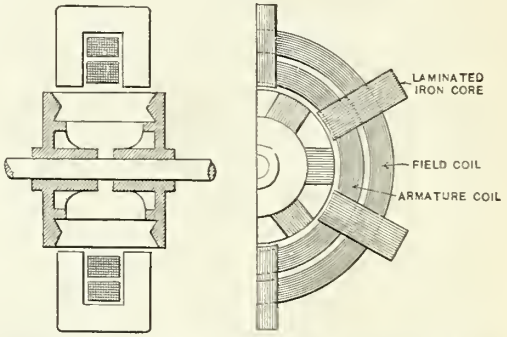


Fig. 107. — Diagram of an Inductor Alternator having a Single Magnetizing Coil and Single Armature Coil.

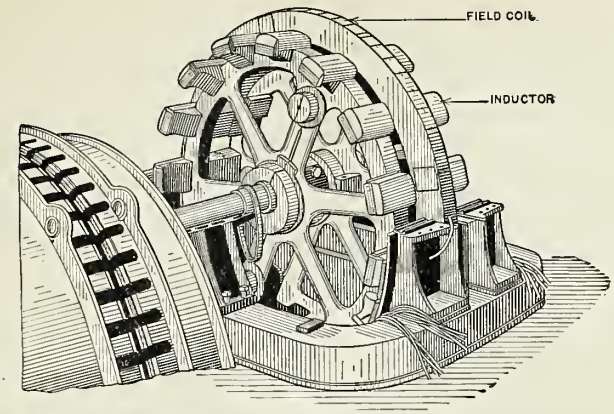


FIG. 108. — Inductor and Field Coil of a 2000-kilowatt Stanley Alternator.

ductor. This coil magnetizes the inductor so that the crown of inductor projections on one side of the coil are of one sign and those on the other are of the opposite sign. The armature coils are usually of the distributed type, as shown in Fig. 109, and in modern machines are always chord wound. The armature windings are also in two crowns; one surrounding each crown of inductor projections. There are in each phase twice as many coils in each crown as inductor projections, and in order that they shall add their voltages the coils must be connected into circuit alternately right and left handed.

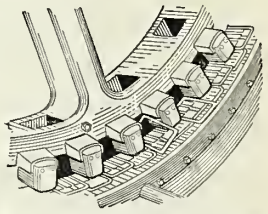


FIG. 109. — Illustration showing the Armature Windings of a 2000-kilowatt Inductor Alternator. Two Coils are partly Removed.

The Warren alternator was very similar to the Stanley machine. It had, however, only one set of inductors and armature coils. Figure 110 shows the inductor. The armature windings were placed on the inner surface of a laminated stationary iron frame surrounding the projections at the left hand of the figure; and the magnetizing coil surrounded the cylindrical portion at the right. The magnetic circuit was completed through the frame.

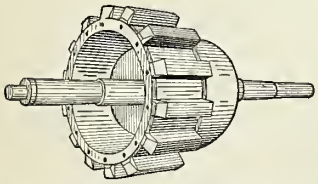


FIG. 110. — Inductor having a Single Crown of Projections.

36. Armature Insulating and Core Materials and Construction. — Before the conductors are placed on an armature core it is usual to insulate it, very much as in the case of a direct-current armature,* but more thoroughly on account of the high voltages usually produced in alternator armatures. For this purpose mica, micanite, mica paper and cloth, shellacked canvas, fuller board, oiled paper and cloth, sheets of vulcanite, vulcabeston, vulcanized fiber, and similar insulating materials, are used. Mica, micanite, vulcanite, vulcabeston, bonsilate, vulcanized fiber, asbestos paper, and similar materials are also used to insulate collecting rings and brush holders, and for insulation between the armature coils. The wire used for high-voltage alternator armatures is often triple-cotton covered, and is thoroughly japanned during the process of winding. Vulcanized fiber is made from paper fiber by a chemical process and is furnished in sheets and tubes. Its convenient form, cheapness, and ease of working have brought it into extensive use. It unfortunately absorbs moisture when exposed to the air, which causes it to expand and contract to a remarkable degree. The moisture also reduces its insulating qualities to a large extent. It is therefore unsafe to place entire reliance upon it where continuously high insulation is required. Of the various available insulating materials mica is the only thoroughly reliable one, but it is unduly expensive and of poor mechanical qualities. On the latter account it is generally used in combination with other materials. When made up in the form of micanite by treating with varnish, laying in overlapping scales between paper or cambric, and subjecting to high pressure and heat, its mechanical qualities are somewhat improved, and it may be formed into sheets and tubes or molded as desired. Materials such as vulcabeston and bonsilate are advantageous for insulating collector rings and similar details, since they can be molded into any desired form. Vulcabeston, which is a compound containing rubber and asbestos, is also manufactured in sheets and has quite satisfactory mechanical and electrical properties. Bonsilate is not sufficiently tough for very general service. Boxwood and paraffined maple or hickory are frequently used for insulation where considerable bulk is required and their mechanical properties will serve, though, as they are liable

* Jackson's *Electromagnetism and the Construction of Dynamos*, p. 104.

to be more or less affected by the surrounding condition of humidity, they must be used with care. A special material called asbestos wood is now manufactured in sheets and plates adapted for use as insulating material in some of the situations now unsatisfactorily filled by other materials.

Tubes or troughs of micanite, varnished fiber, or some other insulator that will meet the conditions, are ordinarily slipped into the armature slots before the wires are inserted. The coils, after careful taping, are then placed within the tubes. The method of insulating armature conductors is illustrated in Figs. 79 to 83. When a coil is made up of many turns of wire, it is sometimes necessary to also insulate between layers, in addition to the insulation afforded by the cotton covering of the wire itself. The field coils of alternators are insulated with the same materials as the armature windings, though the lower voltages usually employed in the field circuits call for a less thickness of insulation.

The field coils are generally wound upon formers, and insulating sheets are placed between layers. The wire used is often double cotton covered; but a flat strip (sometimes bare) is not uncommonly used. This latter may be wound on edge with insulating strips between the conducting strips. Winding a single layer of strips on edge is thought to improve the mechanical and electrical stability of the field coil and improve its heat-dissipating qualities. The taped coil may be placed upon an insulating spool and then slipped upon the pole piece. Formed conductors are often given a coating of enamel for insulation; and coils are frequently heated to very high temperature during construction, while under the high pressure of a former, to drive off volatile matter and leave the conductors embedded in a compact and stable mass of insulation.

In insulating alternators care must be taken to use such materials as will not deteriorate either electrically or mechanically under the rather high temperature which the machines attain when running. If the insulating material softens perceptibly, it may permit points of high difference of potential to come close together with a resultant short circuit; or the same failure may result in the permanent lowering of the specific resistance of the insulating materials.

The **Dielectric strength**, or the property of an insulator to resist

electric puncture, is the basis upon which to calculate insulation, rather than the electrical resistance, though the latter often serves as an indication of the former. The following table gives the specific resistance, or the resistance in megohms per centimeter of length and square centimeter of cross section, of familiar insulating materials at ordinary temperatures (namely, about 70° F.). The values are only approximate, as different samples of the same material are apt to have quite different characteristics.

TABLE

SPECIFIC RESISTANCE OF INSULATORS (in megohms)

Mica	10^6 to 10^8
Paper	10^4 to 10^5
Paraffine oil	10^6 to 10^8
Rubber	10^8 to 10^9
Shellac	10^9 to 10^{10}
Vulcanized fiber	10^6 to 10^8

The dielectric strength of any insulating material is quite dependent upon the character of the particular sample under test, the character of the electrodes, the duration of the stress, and other conditions surrounding the tests; and therefore the data from tests should only be used for rough estimates or checks. Dr. Steinmetz * has worked out a number of special formulas from the results of tests which may be used in this way. In the following table of these formulas, d is the thickness of dielectric in milli-centimeters through which a disruptive discharge will occur under the impulsion of E kilovolts. The given voltage is the maximum value of a sinusoidal voltage wave between flat electrodes.

TABLE

DIELECTRIC MATERIALS	FORMULAS
Air	$d = 36\,(\epsilon^{-1.3E} - 1) + 54\,E + 1.2\,E^2$
Mica	$d = .24\,E + .0145\,E^2$
Vulcanized Fiber	$d = 7.66\,E + 2.3\,E^2$
Dry Wood Fiber	$d = 7.66\,E$
Paraffined Paper	$d = 3\,E$
Melted Paraffine	$d = 12.4\,E$
Copal Varnish	$d = 30\,E$
Crude Lubricating Oil	$d = 60\,E$
Vulcabeston	$d = 28\,E$
Asbestos Paper	$d = 23\,E$

* Steinmetz on Dielectrics, *Trans. Amer. Inst. Elect. Eng.*, vol. 10, p. 85.

In obtaining data upon which these very rough formulas are based, alternating currents of about 100 cycles per second were used, and the voltage was varied from two to thirty thousand volts. In general, thin layers of dry air withstand about six thousand volts per millimeter when the spark occurs between needle points; oils from twelve hundred to a hundred thousand volts, and specially selected oils from one quarter to a million volts per centimeter between needle points, depending upon the quality; and mica from several hundred thousand to several million volts per centimeter between flat electrodes pressing the material between them.

The wire used for winding the fields and armatures of alternators is almost always double or triple cotton-covered copper wire or strip of high conductivity. The standard of conductivity adopted by the American Institute of Electrical Engineers in 1893 is 9.59 ohms per mil-foot at 0° C.* The average rate of change of resistance per centigrade degree of change of temperature within ordinary ranges may be taken as .42 per cent of the resistance at 0° C. This makes the resistance at any temperature, in comparison with the resistance at 0° C.,

$$R_t = R_o(1 + .0042\,t),$$

where R_t is the resistance at the required centigrade temperature, R_o at zero temperature, and t is the required temperature. The copper conductors used in alternator windings should fall little below this standard. For winding armatures and fields the copper is made up in the form of bars (Fig. 83), strap (Figs. 80 and 81), or the ordinary round wire (Fig. 79), as the conditions make desirable. In large machines bar or strap copper is used almost exclusively.

The armature cores themselves may be made of punched iron disks, iron punchings of special shapes, iron wire, or iron tape. In American machines the punchings are almost universal. The old Kapp machines had armature cores made of iron tape.† In the larger sizes of American machines, which are built to connect directly to steam engines or large turbines, and there-

* *Trans. Amer. Inst. Elect. Eng.*, October, 1893. See also Standardization Rules of 1907, Par. 260.

† Kapp's *Dynamos, Alternators, and Transformers*, p. 467.

fore have armatures of large diameters, the armature cores are usually built up of segmental punchings put together in such a way that the segments of alternate layers break joints

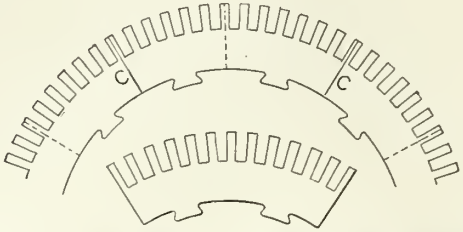


FIG. 111.—Punching for Rotating Alternator Armature.

(Fig. 111). In stationary armatures the teeth are, of course, inside and the keys outside.

Figure 112 shows an old-style single-phase machine with rotating armature constructed with one armature slot per field pole. At A' may be seen one of the punchings which go to make up the laminations of the armature. The field pole pieces are solid in this machine, which is common, though they

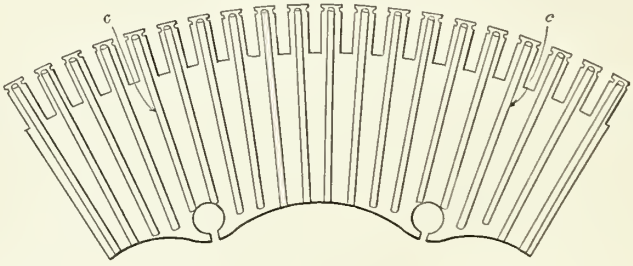


FIG. 111 *a*.—Core Stamping with Distance Pieces for Procuring Ventilation.

are also sometimes made up of punchings secured in the frame in numerous ways.

It is usual to provide ventilating ducts between core laminations, and these ducts are provided for in the assembling by the insertion of distance pieces. The distance pieces *c*, *c*, illustrated in Fig. 111 *a* are usually ribs of brass or T angle iron which may be one half inch or more in depth in large size machines and are riveted to core stampings. These are placed at the desired distance apart in the core and make annular ventilating ducts at intervals of a few inches in the length of the core. The

ventilating ducts are to be plainly seen where they come to the face of the armature core in various illustrations of the text, such as Figs. 80 to 84. The sectional view in Fig. 113 also shows the ventilating ducts.

Figure 113 is a partial side elevation and section of a rotating-field alternator having a three-phase armature winding with one slot per pole per phase, in which the general method of mechanical construction is shown. Much care must be taken in the selection of iron for the magnetic circuit of alternators, in order to keep the iron losses at a minimum. This subject will be discussed at some length in a later chapter.*

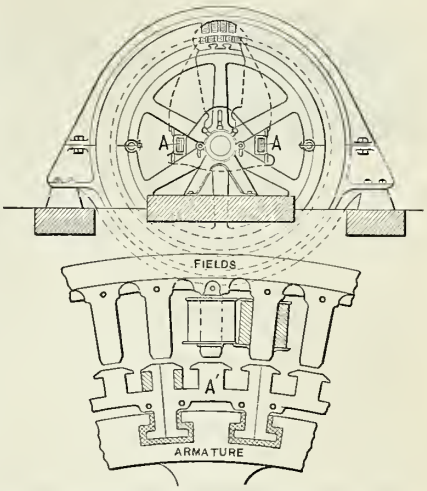


FIG. 112. — Side Elevation of Alternator, showing Armature Punchings.

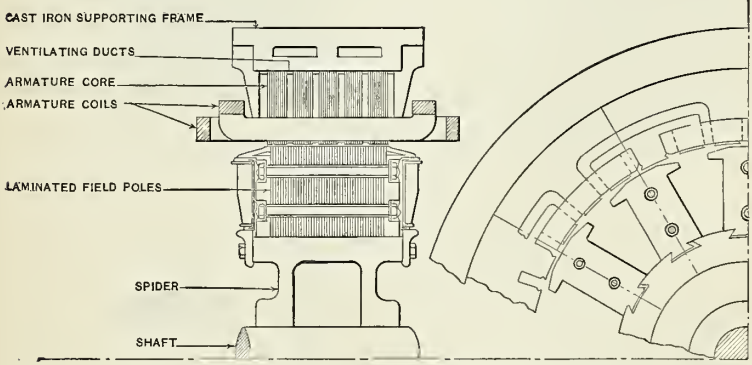


FIG. 113. — Side Elevation and Section of Rotating Field Alternator, showing General Construction.

In high-speed turbine generators the rotating field magnet must be made very substantial to withstand the strains due to

* Chap. IX.

the high circumferential velocity. Figure 113 *a* shows a Westinghouse two-pole rotating field magnet with the lower half of the field partly wound. After the coils have been wound in the slots and insulated, as shown in the lower half of the figure, they are fastened firmly in place by heavy metal strips driven through the keyways shown near the outer ends of the lugs. Thus the field is made entirely metal-clad. Ventilation is obtained by means of the holes seen in the ends of the core and by the circumferential slots. The core itself is built up of steel disks about .015 of an inch thick, which are machined for the windings and ventilation.

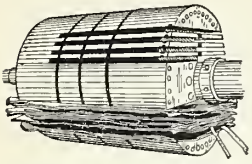


FIG. 113 *a*.— Two-pole Rotating Field for a Turbine Generator.

Figure 113 *b* shows an alternator of the General Electric Co. on a vertical shaft, driven by a Curtis steam turbine. This figure shows the machine in one half section. The nozzles and guide vanes of the turbine are within the lower cylindrical portion. The shaft, which carries the multipolar rotating field magnet of the alternator at its upper part and the turbine runners at its lower part, is supported on a step-bearing supplied with water under high pressure as a lubricant.

The fields and armature of the Westinghouse alternator illustrated in Figs. 83 *a* and 113 *a* are intended to be driven by a horizontal steam turbine of the Westinghouse-Parsons type which runs at a high speed, and the field has fewer poles, — in this case it is bipolar.

* See Arts. 106 and 113.

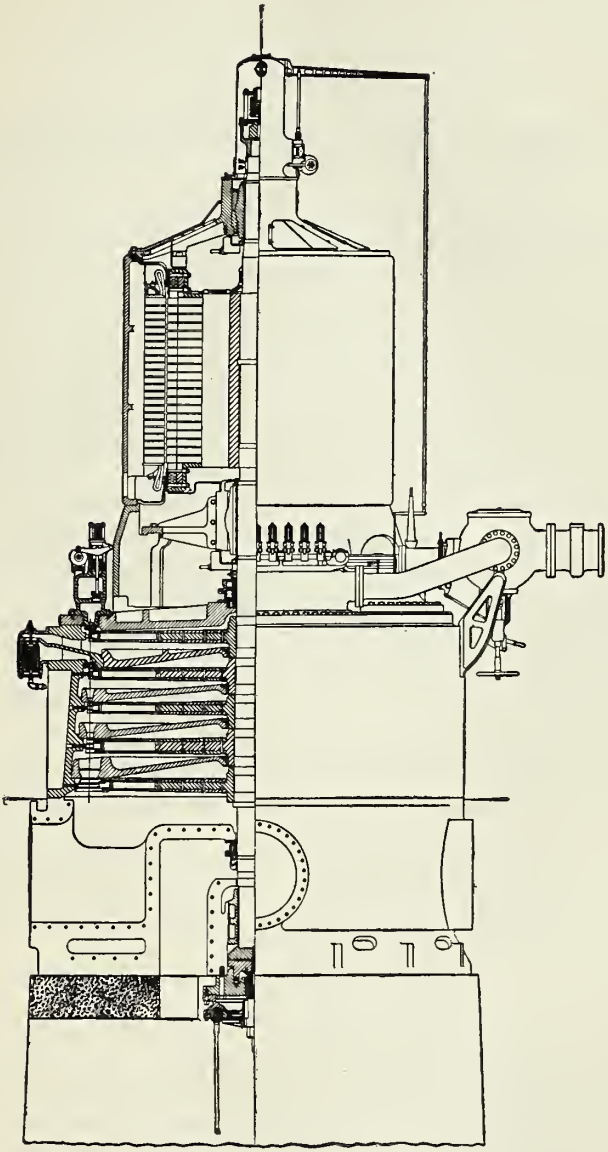


FIG. 113 *b*. — Alternator on Vertical Shaft of a Steam Turbine.

CHAPTER IV

SELF-INDUCTION, CAPACITY, REACTANCE, AND IMPEDANCE

37. Self-induction. — Before proceeding with the development of the design and operation of alternating-current machines, it is essential to discuss the relations which exist between voltages and currents in circuits which carry alternating currents. It is to be remembered that the current produced by cutting lines of force sets up in turn magnetic force, which is in opposition to the original field. Again, when a current is introduced into a circuit, it produces a magnetic flux the rise of which causes a counter-voltage. These conditions are necessary under the law of conservation of energy and its corollary, Lenz's Law.* This counter-voltage is called the **Voltage** or **Electro-motive force of Self-induction**, and the phenomenon as a whole is called **Self-induction.**

Self-induction, therefore, may be defined as the inherent quality of an electric circuit whereby the electro-magnetic induction set up by an electric current in the circuit tends to impede the current's own introduction, variation, or extinction in the circuit.

The voltage in a circuit which at any instant is due to self-induction is evidently proportional to the rate of change of the magnetic flux set up by the current of the circuit; therefore, the voltage of self-induction is proportional to the rate of change of current in the circuit, provided that the reluctance of the magnetic circuit remains constant; or

$$e_l \propto -\frac{d\phi}{dt} \propto -\frac{di}{dt},$$

where e_l is the instantaneous self-induced voltage in the circuit, ϕ the magnetism, and i the current, respectively, at the same instant of time t. The minus sign is used because the voltage induced at each instant is in a direction to impede the changes of current and magnetism.

* Jackson's *Electro-magnetism and the Construction of Dynamos*, p. 77.

If the circuit exhibits the phenomena of electrical resistance and self-induction, and none other, as would be true of a simple coil of wire, the voltage impressed on its terminals must at each instant be sufficient to overcome the counter-voltage e_l and supply the iR drop corresponding to the current at the particular instant. That is, the voltage impressed on the terminals of the circuit, or **Impressed voltage**, is made up of two components, one equal and opposite to the counter-voltage and the other equal to the drop of voltage through the resistance of the circuit. The latter component represents the portion of the force driving the current through the circuit which results in expending power therein.

When the instantaneous values of current and voltage are considered, the iR drop, the counter-voltage e_l, and the impressed voltage e are scalar values, and therefore $iR = e - e_l$; but a little consideration will show that the **Phase** of the alternating voltage of self-induction is not in unison with that of the current, although their periods are the same; because the counter-voltage is proportional to the rate of change of magnetism caused by the current, and, supposing the magnetism to be alternating, its rate of change is zero when it is at a maximum, and the voltage of self-induction is therefore zero at the same time. Manifestly, therefore, a vector relation must be recognized when the effective values of current and voltage are considered.

When the curve of current is a sinusoid, the rate of change of its ordinates at any point is

$$\frac{di}{dt} = \frac{i_m d(\sin a)}{dt} = i_m \cos a \frac{da}{dt} = i_m \sin (a + 90°) \frac{da}{dt},$$

i_m being the maximum value of current; and the counter-electromotive force e_l is therefore proportional to $-\dfrac{\sin (a + 90°) da}{dt}$, assuming that the reluctance of the magnetic circuit is uniform. The curve $\sin (a + 90°)$ is manifestly a curve which occupies a position 90° earlier than (or in the **Lead** of) the curve $\sin a$, and the curve $-\sin (a + 90°)$ is 180° later than $\sin (a + 90°)$ and therefore 90° later than the curve $\sin a$. Hence the curve of the counter-voltage of self-induction is a sinusoid, the phase of which **Lags** 90° behind (that is, is 90° later than) the phase

of the current setting it up. It is evident also that this must be the case, since when sinusoidal current is passing through the zero value, rising in the positive direction of flow through the circuit, its rate of change is a maximum and the self-induced voltage is at maximum value in a direction opposing the rise of current; and when the current has risen to a maximum (a quarter period, or 90°, later), its rate of change is zero, and the self-induced voltage is zero and about to change in direction from negative to positive. Thus it is seen that the self-induced voltage goes through its cycle in a phase which is 90° later than the current.

38. Vector Diagrams representing the Voltage Relations in an Inductive Alternating-current Circuit. — Suppose the line OC (Fig. 114) represents the maximum value of the iR drop in a circuit carrying a sinusoidal current. If the line is uniformly rotated around the point O, its end C describes a circle and its projection on the vertical axis describes a simple harmonic motion. (In this book, such vectors are assumed to rotate in a counter-clockwise direction.) At each instant the projection of the line on the vertical axis proportionally represents the magnitude of the instantaneous active voltage (*i.e.* iR drop) corresponding to the angular advance of the line. The projection of the line OD, which is 90° behind OC, likewise represents the instantaneous counter-voltage at each instant. The algebraic sums of the instantaneous projections of OC and CA (the reverse of OD) are al ways equal to the simultaneous projections of OA, where OA is the impressed voltage. The impressed voltage OA has the magnitude and phase relation of the resultant of the active voltage (iR drop) and the opposite of the self-induced voltage. This may be expressed in still another way, which may possibly give a more useful conception: *The impressed voltage applied to an alternating-current circuit must have such magnitude*

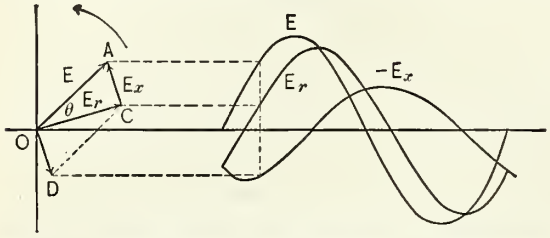


FIG. 114. — Phase Diagram of Voltages in a Self-inductive Circuit.

and phase relation that it will neutralize all other live voltages and at the same time furnish the active voltage which drives the current through the resistance of the circuit. From the relations given it is seen that

$$(OA)^2 = (OC)^2 + (CA)^2, \text{ or } OA = \sqrt{(OC)^2 + (CA)^2}.$$

The lengths of the lines in the figure have been assumed to be proportional to *maximum* values of the voltages, but the *effective* values of the voltages hold the same relations since each effective value is equal to the maximum multiplied by .707. Consequently, the effective value of the impressed voltage operating in a circuit with resistance and self-inductance in series is equal to the square root of the sum of the squares of effective values of the active voltage and the inductive voltage reversed. Or when E, E_r, and E_x are respectively effective values of impressed voltage, active voltage, and self-induced voltage reversed,

$$E = \sqrt{E_r^2 + E_x^2}.$$

The triangle of voltages representing these conditions is shown in Fig. 115. (Such triangles, in this book, usually have the active voltage line laid out in a horizontal direction.)

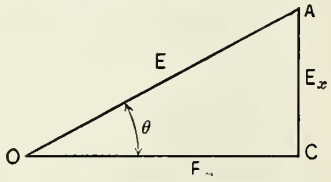


FIG. 115. — Vector Polygon of Voltages for Series Self-inductive Circuit.

If the heights of the points A, C, and D of the uniformly rotating vectors OA, OC, and OD are plotted as ordinates to the corresponding values of time or of α laid out on the axis of abscissas as illustrated in Figs. 3 and 4, three sine curves are traced like those at the right hand of Fig. 114. These are marked respectively E, E_r, and $-E_x$. They represent the waves of impressed voltage, IR drop, and inductive voltage. The instantaneous value of the impressed voltage at each instant is equal to the corresponding instantaneous values of E_r and E_x added together algebraically.

The instant of time in the sine wave plot which corresponds to the time when the rotating vectors arrive at the position indicated by the left-hand part of Fig. 114 is shown by the dotted vertical axis cutting through the sine curves. If the zero of time is assumed at the instant when the impressed voltage

is passing through zero from the negative to the positive direction, this would be indicated by the phase diagram of rotating vectors at the instant when OA lies in the positive direction along the X-axis. The relations of the voltage waves at that instant are indicated at the left-hand terminus of the plot of sine curves. It is obvious that the inductive voltage must be equal to the iR drop at that instant, to maintain the current i, since the then instantaneous value of the impressed voltage is zero.

Prob. 1. Construct a vector polygon of voltages for a circuit containing a self-inductive sinusoidal voltage of 50 volts and an IR drop of 50 volts.

Prob. 2. Construct a vector polygon of voltages for a circuit containing an IR drop of 100 volts and a self-inductive sinusoidal voltage of 5 volts.

Prob. 3. Construct a vector polygon of voltages for a circuit containing 20 volts IR drop and 500 sinusoidal self-inductive volts.

Prob. 4. What are the values of the impressed voltages in problems 1, 2, and 3?

Prob. 5. Draw the curves of voltage in a circuit having an IR drop (effective value) of 100 volts and a self-inductive voltage (effective value) of 50 volts, on the supposition that the curves are sinusoids.

Prob. 6. In problem 5 find the instantaneous values of the self-inductive and impressed voltages when the IR drop is 0; find the impressed voltage and IR drop when the self-inductive voltage is 0; find the IR drop and self-inductive voltage when the impressed voltage is 0.

Prob. 7. Find the instantaneous values of the three voltages in problem 5 when $\alpha = 0$, 45°, 90°, 135°, 180°, 225°, 270°, 315°, and 360°; find these values graphically from the rotated vector polygon and from the curves of voltage.

39. Angle of Lag. — The angle θ between the lines OA and OC in Figs. 114 and 115 shows the amount by which the phase of the active voltage (which is in phase with the current) lags behind that of the impressed voltage. The figure makes it evi-

dent that this lag is caused by the position of the self-inductive voltage, and that its magnitude depends upon the relative lengths of the line $CA(=-OD)$ and the line OC. The tangent of the angle is

$$\tan\theta = \frac{CA}{OC} = \frac{\text{Inductive voltage reversed}}{\text{Active voltage}}.$$

The angle θ is called the **Angle of lag.** The relations are plainly shown in Fig. 115.

Prob. 1. What are the angles of lag in problem 1, problem 2, and problem 3 of Article 38?

Prob. 2. Find the angle of lag in a circuit which has an impressed sinusoidal voltage of 100 volts and an active sinusoidal voltage (IR drop) of 30 volts.

Prob. 3. Find the angle of lag in a circuit which has an impressed sinusoidal voltage of 300 volts and a self-inductive sinusoidal voltage of 150 volts.

40. Sinusoidal Voltage in a Self-inductive Circuit expressed as a Complex Quantity. — The complex quantity which expresses the vector representing the impressed voltage is

$$\overline{E} = E_r + jE_x,$$

or

$$\overline{E} = E(\cos\theta + j\sin\theta),$$

where E is the numerical or scalar value of $\overline{E}$, E_r the scalar value of IR drop and E_x the scalar value of inductive voltage reversed.

Also

$$E = \sqrt{E_r^2 + E_x^2} \text{ and } \tan\theta = \frac{E_x}{E_r}.$$

If there are a number of circuits in series having impressed voltages E_1, E_2, E_3, etc., the combined voltage across the outside terminals is represented by the vector sum

$$\overline{E} = E_r + jE_x = (E_{r_1} + E_{r_2} + E_{r_3} + \text{etc.}) + j(E_{x_1} + E_{x_2} + E_{x_3} + \text{etc.})$$

in which

$$E_r = E_{r_1} + E_{r_2} + E_{r_3} + \text{etc. and } E_x = E_{x_1} + E_{x_2} + E_{x_3} + \text{etc.}$$

The angle of lag is equal to

$$\theta = \tan^{-1}\frac{(E_{x_1} + E_{x_2} + E_{x_3} + \text{etc.})}{(E_{r_1} + E_{r_2} + E_{r_3} + \text{etc.})} = \tan^{-1}\frac{E_x}{E_r}.$$

The angle θ is the angle of lag measured between the phase of

the current flowing in the circuit and the voltage impressed upon the entire circuit.

Prob. 1. The IR drop in a circuit is 50 volts and the self-inductive voltage 25 volts. Find the impressed voltage and the angle of lag of the current. (In this and the following problems all voltages are assumed to be sinusoidal.)

Prob. 2. The IR drop in a circuit is 50 volts and the self-inductive voltage 500 volts. Find the value of the impressed voltage and the current lag.

Prob. 3. The angle of lag in a circuit is 30° and the impressed voltage 200 volts. What are the values of IR drop and self-inductive voltage?

Prob. 4. The IR drop and self-inductive voltage of two circuits are respectively 50 and 30 volts, and 40 and 20 volts. What will be the resultant voltage if the two circuits are placed in series, and what will be the angle of lag between the current-flow and the voltage impressed between the main terminals?

Prob. 5. Two circuits in series have voltages impressed upon them of 100 and 200 volts and angles of lag of 30° and 45° respectively. Calculate the total voltage across the two circuits and the angle of lag of the current with respect thereto.

Prob. 6. Three voltages in a circuit of 100, 200, and 300 volts are in series, the angles of lag in the three parts of the circuit being respectively 20°, 40°, and 60°. What are the resultant voltage and angle of lag? Also what are the IR and self-inductive components of the total voltage?

41. Self-inductance. — The magnitude of the voltage of self-induction is dependent upon the number of lines of force inclosed by the circuit per unit current flowing in it, the number of turns composing the circuit, and the current flowing therein; which follows from the fundamental equation for voltage*; the formula is

$$e_l = -\frac{n d\phi}{10^8\, dt},$$

where e_l is the voltage of self-induction for the given rate of change of magnetization, n is the number of turns in the coil,

* Art. 11.

and, as before, $\frac{d\phi}{dt}$ is the rate of change of magnetism. But in a long or closed solenoid, the magnetizing force is *

$$M = \frac{4\pi ni}{10},$$

where i is the value of the current at the instant under consideration; and the reluctance of the magnetic circuit is †

$$P = \frac{l}{\mu A},$$

where l and A are respectively the average length and cross section of the solenoid and μ is the permeability, assumed in this paragraph to be uniform. Therefore, the number of lines set up by a current i is †

$$\phi = \frac{M}{P} = \frac{4\pi niA\mu}{10\,l}, \qquad d\phi = \frac{4\pi nA\mu}{10\,l} \times di,$$

and

$$e_l = -\frac{nd\phi}{10^8\,dt} = -\frac{n}{10^8} \times \frac{4\pi nA\mu}{10\,l} \times \frac{di}{dt} = -\frac{4\pi n^2 A\mu}{10^9\,l} \times \frac{di}{dt}.$$

The numerical value of

$$\frac{n}{10^8} \times \frac{4\pi nA\mu}{10\,l}$$

is called the **Self-inductance** or the **Coefficient of self-induction** of the coil, and is usually represented by the capital letter L. The number of lines of force passing through the solenoid when the current is one ampere is equal to

$$\frac{4\pi nA\mu}{10\,l}.$$

Hence *the value of the self-inductance of a long solenoid of uniform permeability may be defined as* $\frac{1}{10^8}$ *times the product of the number of turns in the circuit by the number of lines of force inclosed by the circuit when carrying one ampere of current.* That is, in the case of a long solenoid in a medium of uniform permeability,

$$L = \frac{n}{10^8} \times \frac{\phi}{i}.$$

* Jackson's *Electro-magnetism and the Construction of Dynamos*, p. 15.

† Jackson's *Electro-magnetism and the Construction of Dynamos*, pp. 6 and 7.

Since the number of lines of force developed by a coil is proportional to the number of turns composing it, **its self-inductance is proportional to the square of its turns.** Also, it will be observed that the formula for self-induced electro-motive force reduces to

$$e_l = -L\frac{di}{dt}.$$

42. The Henry. — The Chicago Electrical Congress formally assigned the name **Henry** to the unit in which self-inductance is measured, after Professor Joseph Henry, the notable American discoverer of electro-magnetic phenomena. Since the volt is 10^8 times as large as a C. G. S. unit of electro-motive force and the ampere is $\frac{1}{10}$ as large as the C. G. S. unit of current, it is apparent from the formula that the **Henry** is 10^9 times as large as the corresponding C. G. S. unit of self-inductance.

43. Self-inductance of a Short Coil; Circuit containing Variable Permeability; Examples. — The definition of the henry is above developed for a long solenoid in which all the lines of force pass through all the turns. If the circuit is not so constructed, the definition still holds, but the summation of the number of lines of force passing through each turn individually must be taken, since the number of lines passing through any turn is a variable which depends upon the position of the turn in the coil. Thus, suppose Fig. 116 represents a short solenoid of eight turns in which are developed ten lines of force when one ampere flows through the coil. Assuming the distribution of the lines shown in the figure, the self-inductance is calculated as follows:

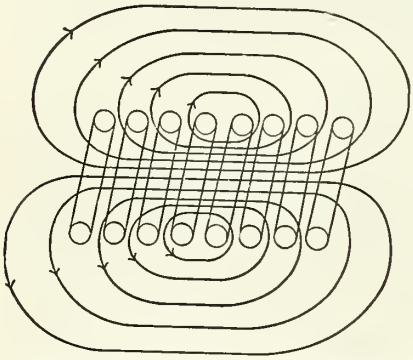


Fig. 116. — Solenoid and Magnetic Field.

$$\frac{\overline{10 \times 2} + \overline{8 \times 2} + \overline{6 \times 2} + \overline{4 \times 2}}{10^8} = \frac{56}{10^8} \text{ henrys.}$$

If an iron core is now placed in the coil, the number of lines of force is increased directly as the reluctance of the magnetic

circuit is decreased. Hence, assuming the distribution of the lines to remain unchanged, the self-inductance becomes $\frac{56\,P'}{10^8\,P}$ henrys, where $\frac{P'}{P}$ is the ratio of the reluctance before and after the iron core is inserted.

In the case of a long solenoid

$$L = \frac{n}{10^8} \times \frac{4\,\pi n A}{10\,l}$$

when the permeability is unity, but when the permeability of the magnetic circuit taken as a whole is μ, the number of lines of force per ampere, as shown before, is

$$\frac{4\,\pi n A \mu}{10\,l},$$

and therefore

$$L = \frac{n}{10^8} \times \frac{4\,\pi n A \mu}{10\,l}.$$

In general, where the magnetic circuit is composed wholly of non-magnetic material, the self-inductance is

$$L = \frac{n}{10^8} \times \frac{\phi}{i}$$

(where ϕ represents the number of lines of force for current i), and the inductance is a constant for all values of i. However, when iron or other magnetic material is included in the magnetic circuit, the value of the self-inductance varies with the value of i because μ varies with ϕ. As before,

$$L = \frac{n\phi}{10^8\,i},$$

but this may have a different value for each value of i since

$$\phi = \frac{4\,\pi n A \mu i}{10\,l},$$

and μ varies with ϕ. In this case the inductance for any value of i is μ times as great as when no magnetic material is included in the magnetic circuit, *the value of μ taken being that corresponding to the particular value of i.*

Therefore, *the self-inductance of a long solenoid which contains an iron core, when carrying a certain current, may be defined as*

$\frac{1}{10^8}$ *times the number of turns in the solenoid multiplied by the number of lines of force set up by the current and divided by the number of amperes of the current.* It is shown later that the self-inductance found by this relation is not necessarily equal to the apparent self-inductance found by the use of measuring instruments because the disturbing effects of eddy currents and hysteresis in iron cores sometimes mask the results.

As an example of the calculation of the value of L, consider a uniform ring of wrought iron 100 centimeters in mean circumference and 20 square centimeters in cross section. Suppose a coil of 2500 turns is uniformly wound on the ring, and a current of two amperes is passed through the magnetizing coil. Taking μ as equal to 250, which is a fair value,

$$\phi = \frac{4\pi nA\mu I}{10\,l} = \frac{4\pi \times 2500 \times 20 \times 250 \times 2}{10 \times 100} = 314{,}200;$$

hence, $$L = \frac{n\phi}{10^8 I} = \frac{2500 \times 314{,}200}{10^8 \times 2} = 3.93 \text{ henrys.}$$

If the current in the magnetizing coil is taken as $1\frac{1}{2}$, and it is supposed that the value of μ becomes roughly 300, then

$$L = \frac{2500 \times 282{,}744}{10^8 \times 1.5} = 4.71 \text{ henrys.}$$

That is, the permeability having increased with the reduction of the magnetic density in the iron core, the magnetism per ampere is greater with the smaller current and the self-inductance is therefore larger, although the total magnetism is smaller.

If a similar winding is put on a ring of the same dimensions but of brass or other non-magnetic material, the magnetism set up per ampere is:

$$\phi = \frac{4\pi \times 2500 \times 20 \times 1}{10 \times 100} = 628.3,$$

and the self-inductance is:

$$L = \frac{2500 \times 628.3}{10^8 \times 1} = .0157 \text{ henry.}$$

In this case the self-inductance is not affected by the strength of the current, because the permeability of the medium is fixed and independent of the current flow.

According to the foregoing definitions, the number of henrys of self-inductance possessed by any circuit, whatever may be the current flow, is equal to $\frac{1}{10^8}$ times the change of the summation of the linkages of lines of flux through or around the conductors of the circuit which would be caused by a change of the current by one ampere, assuming that the magnetic reluctance of the surrounding medium does not change at the same time. In the case of the long solenoid or the ring solenoid, all of the lines of flux link through the circuit as many times as there are turns in the coil, and the summation of linkages may be found by multiplying the total number of lines of flux by all of the turns of the coil; but in other instances some of the lines of flux make a different number of linkages with the circuit from others, and then the summation of the linkages can be determined only by tracing the linkages of the lines individually and summing them up as in the example on page 136. The self-inductance ordinarily required in dealing with alternating-current circuits, is that calculated upon the basis of the circuital magnetic permeability obtaining for the maximum instantaneous ordinate of the current under consideration.

In the usual practical problems that are met by the engineer, the conformation and numerical constants of the magnetic circuits and their windings are often unknown or are so irregular that the self-inductance cannot be determined by calculation, and experimental determination must then be resorted to; but the same definitions apply to the units and the same physical conceptions may be cultivated.

The formula $e_l = -L\frac{di}{dt}$ obviously only applies to conditions in which L is constant, which is not the case when the magnetic circuit includes an iron core. In the latter case, the formula becomes:

$$e_l = -\frac{d(Li)}{dt} = -a\frac{d(\mu i)}{dt} = -a\left(\mu\frac{di}{dt} + i\frac{d\mu}{dt}\right).$$

If the winding under consideration is a long solenoid on an iron core, the constant a in this equation is expressed by

$$a = \frac{n}{10^8} \times \frac{4\pi nA}{10\,l}.$$

It will be observed that the first term of the last member of the foregoing equation for e_l is equal to $L\frac{di}{dt}$. The second term of the same member shows the effect on the induced voltage which is produced by the change of the permeability of the iron in the magnetic circuit as the current and magnetic density change. The algebraic sign of the first term is fixed by the direction of change of the current, $\frac{di}{dt}$; but the second term may be either positive or negative with respect to the algebraic sign of $\frac{di}{dt}$, depending on which limb of the permeability curve comes into the conditions concerned, — that is, depending upon whether the permeability increases or decreases as the current rises.

The induced voltage e_l is therefore numerically equal to $L\frac{di}{dt}$ only when L is constant, and it may be numerically either larger or smaller than $L\frac{di}{dt}$ when the permeability of the magnetic circuit concerned varies as the magnetic density and current change. It is numerically larger than $L\frac{di}{dt}$ when the permeability varies similarly as the magnetic density and current (*i.e.* increases when they increase and decreases when they decrease); and it is numerically smaller than $L\frac{di}{dt}$ when the permeability varies inversely to the magnetic density and current, as it does when the magnetic density is larger than that producing the maximum permeability in the iron.

Prob. 1. A closed ring of iron contains a flux of 2,000,000 lines when a current of 5 amperes flows through its magnetizing coil of 1500 turns. What is the self-inductance of the coil? In this and the following problems the effect of magnetic leakage is assumed to be of negligible magnitude.

Prob. 2. A closed ring having a constant permeability of 1000 units and a cross section of 200 square centimeters and a length of 150 centimeters, has wound upon it a coil containing 2000 turns of wire. What is the self-inductance of

the coil? (Aid: remember that the flux $\Phi = \frac{M}{P}$, where M is magneto-motive force and P reluctance; that $M = \frac{4\pi nI}{10}$; and $P = \frac{l}{A\mu}$, where l is the length of the magnetic circuit and A is its cross section.)

Prob. 3. In problem 2 the ring is cut and its ends pulled apart until there is an air space included in the magnetic circuit having a length of a half centimeter and an effective cross section of 280 square centimeters. What is the self-inductance of the coil, supposing that the permeability of the iron has not changed?

Prob. 4. A ring of iron wire 100 centimeters long and 400 square centimeters in cross section is surrounded by a magnetizing coil of 200 turns. When the magnetic density in the iron is 2000 lines of force per square centimeter, $\mu = 1000$; when the magnetic density is 5000, $\mu = 1200$; when the magnetic density is 10,000, $\mu = 800$; when the magnetic density is 20,000, $\mu = 300$; when the magnetic density is 40,000, $\mu = 50$. What is the self-inductance of the coil for each value of the five magnetizations?

Prob. 5. The field windings of a bi-polar direct-current dynamo contain 2000 turns carrying a current of $2\frac{1}{2}$ amperes, the field magnet has an average length of 150 centimeters, an average cross section of 500 square centimeters, and an average permeability of 800; the armature has the same average magnetic cross section and a length of 30 centimeters with a permeability of 1500; and the two air spaces each have a length of 1 centimeter and a cross section of 600 square centimeters. What is the self-inductance of the exciting winding?

44. Examples of Self-inductances. — Ordinary practical experience in electrical measurements and in handling wires soon gives a capacity for estimating the values of resistances; in the same way facility is soon gained in roughly estimating electrostatic capacities, or the current which may be safely carried by a wire, or even the ampere turns required to produce a given magnetization in a magnetic circuit. Ordinary practice, however, gives little clue to estimating the self-inductance in a circuit.

It is true that, as already shown, the self-inductance is dependent upon the magnetism inclosed in the circuit and the number of turns thereof, but experience in dealing with coils and magnetic circuits is not usually regarded in such a way as to aid in estimating self-inductances. The following values of self-inductance are therefore presented here to give a foundation for judgment.

The range of self-inductances met in practice is very wide. The smallest which are practically met are in the doubly wound resistance coils used for Wheatstone bridges and similar devices. Since the wire in these is doubled back upon itself, the magnetic effect of the current is almost neutral and the inductance is often less than a microhenry (one millionth of a henry). The inductance of a certain electric call bell of 2.5 ohms resistance has been found to be 12 microhenrys; a telephone call bell of 80 ohms resistance, 1.4 henrys; a modern short-core telephone bell of 1000 ohms resistance, 5.5 henrys; the armature of a magneto calling generator of 550 ohms resistance, from 2.7 henrys when the plane of the coil lay in the plane of the pole pieces, to 7.3 henrys when the plane of the coil was perpendicular to the plane of the pole pieces; more modern magneto generator armatures, one measuring 540 ohms resistance, 1.6 henrys with plane of coil in plane of pole pieces, 2.4 henrys with plane of coil perpendicular to plane of pole pieces; one measuring 125 ohms resistance, .51 and .74 henry; and one measuring 110 ohms resistance, .89 and 1.17 henrys; a Bell telephone receiver measuring 75 ohms resistance, with diaphragm, 75 to 100 millihenrys (thousandths of henrys), without the diaphragm about 35 per cent less; a modern bipolar telephone receiver with 70 ohms resistance, 35 millihenrys; mirror galvanometers vary with their resistance from a few millihenrys to 10 or 12 henrys; a mirror galvanometer for submarine signaling of 2250 ohms resistance, 3.6 henrys; astatic mirror galvanometers of 5000 ohms resistance average about 2 henrys. The single coil of a Thomson galvanometer of 2700 ohms resistance measured 2.56 henrys; the coil of another Thomson galvanometer having 100,000 ohms resistance measured 70 henrys; the coil of an Ayrton and Perry spring voltmeter, without iron core, measured 1.462 henrys. This coil had a length of 2.88 inches, an external diameter of 3 inches, was wound on

a brass tube .58 inch in external diameter, and had a resistance of 333.5 ohms. Each of the above measurements was made with a current of a few milliamperes. The following are measurements of telegraphic apparatus:

TABLE

Polarized Relays of Various Types

Type	Resistance in Ohms	Self-inductance in Henrys	Testing Current in Milliamperes
1	419	1.99	6.3
2	423	1.89	6.3
3	413	1.69	6.3
4	413	1.31	6.3

All armatures were 4 mils (thousandths of an inch) from the magnet poles.

A common Morse relay of 148 ohms resistance measured 10.47 henrys with the armature against the poles, and 3.71 henrys with the armature 20 mils from the poles, the measuring current being 6.3 milliamperes. In ordinary working adjustment the inductance of a Morse relay is about 5 henrys. Telegraph sounders with bobbins, respectively, $1\frac{1}{4}$ by 1, and $1\frac{1}{2}$ by $1\frac{1}{4}$ inches, each wound to 20 ohms resistance, measured 191 and 150 millihenrys, the armatures being 4 mils from the poles and the measuring current being 125 milliamperes. A single coil of a Morse sounder with a resistance of 32 ohms, and having an iron core .31 inch in diameter and 3 inches long, the bobbin being .94 inch in diameter, was found to have a self-inductance of 94 millihenrys. A complete sounder with a core like that of the preceding coil, but with bobbins of 50 ohms resistance having a diameter of 1.25 inches, was found to have a self-inductance of 444 millihenrys. The self-inductance of a complete sounder of 14 ohms resistance measured 265 millihenrys; a modern telephone relay with 62 ohms resistance, 300 millihenrys; another with 2000 ohms resistance, 840 millihenrys; a supervisory relay for telephone service measuring 12.5 ohms, approximately 1 millihenry; a modern telephone induction coil, primary coil resistance 17 ohms, self-inductance 117 millihenrys, secondary coil resistance 27 ohms, self-induc-

tance 74 millihenrys, mutual inductance approximately 91 millihenrys; a modern telephone repeating coil with 46 ohms resistance in each coil, 1.10 henrys self-inductance in each coil and 1.09 henrys mutual inductance. These measurements of telephone apparatus were made with alternating current of a frequency of 800 periods per second.

A single phase transmission line of No. 2 B. and S. gauge copper wires spaced 36 inches apart is calculated to have a resistance of about 1.62 ohms and a self-inductance of about 3.78 millihenrys per mile; No. 6 copper wire under similar conditions to have a resistance of about 4.09 ohms and a self-inductance of about 4.08 millihenrys per mile.

Bare No. 12 B. and S. gauge copper wire erected on a pole line about 23 feet from the ground is calculated by Kennelly to measure about 8.5 ohms and 3.15 millihenrys per mile; No. 6 copper wire under similar conditions is calculated to measure about 2.1 ohms and 2.95 millihenrys. A quadruplex telegraph line, with all instruments in circuit, measures approximately 10 henrys.

The largest self-inductances met in practice are usually in the windings of induction coils or of electrical machinery. The secondary winding of an induction coil capable of giving a 2-inch spark and having a resistance of 5700 ohms, measured 51.2 henrys. The primary winding of an induction coil which is 19 inches long and 8 inches in diameter, measured .145 ohm and 13 millihenrys, while its secondary measured 30,600 ohms and 2000 henrys. The inductance of dynamo fields is likely to vary from 1 to 1000 henrys; direct-current dynamo armatures measure between the brushes from .02 to 50 henrys; the fields of a shunt-wound Mather and Platt direct-current dynamo built for an output of 100 volts and 35 amperes measured 44 ohms and 13.6 henrys at a small excitation; the armature of the same machine measured .215 ohm and .005 henry; a Mordey alternator armature of the disk type, with a capacity for 18 amperes at 2000 volts, measured 2 ohms and .035 henry; a Kapp alternator armature of the ring type, with a capacity of 60 kilowatts at 2000 volts measured 1.94 ohms and .069 henry; another Kapp machine, 30 kilowatts, 2000 volts, measured 7 ohms and .0977 henry; the fields of a Ferranti alternator measured 3 ohms and .61 henry, while the armature of the

same machine built for an output of 200 volts and 40 amperes measured .0011 to .0013 henry, with no current in fields; the armature of a British Thomson-Houston alternator, three-phase, 850 kilowatts, 5000 volts, measured .332 ohm in resistance and .061 henry in self-inductance per phase; the armature of a General Electric alternator, three-phase, 2500 kilowatts, 6500 volts, measured .41 ohm in resistance and .035 henry in self-inductance per phase; while another British Thomson-Houston machine, a turbo-alternator, 1500 kilowatts, 1000 volts, gave an inductance per phase of .0403 henry; the primary and secondary windings of transformers measure roughly from .001 of a henry up to 50 henrys, depending upon their output and the voltage for which they are designed.

An electro-magnet designed by H. DuBois* had, when carrying a current of 45 amperes, a self-inductance of 180 henrys. This electro-magnet had a core of iron made of two half rings butted together, which had a radius of 25 centimeters and a cross section of 78.5 square centimeters. The core was wound with 12 coils of 200 turns each. The self-inductance given is for all coils in series.

The effect of the field magnets upon the self-inductance of a disk-alternator armature is shown by some measurements taken by Dr. Duncan † on a small Siemens eight-pole alternator, the results of which are given in the following table.

TABLE

SELF-INDUCTANCE OF ARMATURE IN PLACE

CURRENT IN FIELD	POSITION OF ARMATURE		
	0°	$11\frac{1}{4}$°	$22\frac{1}{2}$°
0 ampere	.120	.112	.100
2.5 amperes	.125	.115	.108
4.5 amperes	.128	.115	.106

Self-inductance of armature removed from field, .082 henry; resistance of armature, 7 ohms; pitch of the poles, 45°.

Professor Ayrton found that the self-inductance of an unex-

* *The Magnetic Circuit*, H. DuBois, p. 264.

† *Electrical World*, Vol. 11, p. 212.

cited Mordey alternator armature varied between .033 and .038 henry, and that this decreased about 10 per cent when the fields were excited.*

45. Conditions of Establishment and Termination of Current in a Circuit containing Resistance and Self-inductance in Series. — *a. Rise of Current under Constant Impressed Voltage.* — If a circuit containing constant self-inductance and resistance is suddenly connected to a source of constant voltage, the current is retarded so that its rise is along a logarithmic curve. The voltage E of the source is absorbed at each instant in supplying the iR drop and overcoming the counter-voltage $-L\frac{di}{dt}$. Hence the equation

$$E = iR - \left(-\frac{Ldi}{dt}\right) = iR + \frac{Ldi}{dt}$$

and from it is given

$$i = \frac{E - \frac{Ldi}{dt}}{R},$$

which represents the instantaneous value of the current flowing at any moment while the voltage E, constant for the time under consideration, is applied to a circuit of constant inductance L. To find the value of the instantaneous current at any particular time t, we have from the same equation, by transposition,

$$\frac{di}{E - iR} = \frac{dt}{L};$$

whence

$$\frac{1}{R}\int_0^i \frac{Rdi}{E - iR} = \int_0^t \frac{dt}{L}$$

which gives

$$-\frac{1}{R}\log\,(E - iR)\Big]_0^i = \frac{t}{L}\Big]_0^t,$$

or

$$\log\left(\frac{E - iR}{E}\right) = -\frac{Rt}{L},$$

and finally

$$i = \frac{E}{R}(1 - \epsilon^{-\frac{Rt}{L}}).$$

* *Jour. Inst. Elect. Eng.*, Vol. 18, p. 662; also *ibid.*, p. 654.

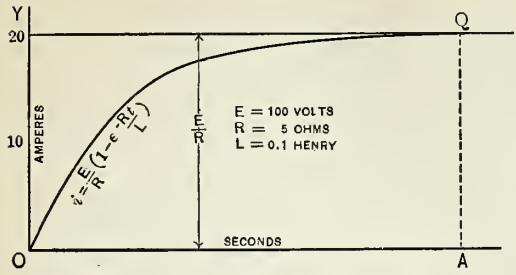


FIG. 117. — Logarithmic Curve of Rising Current.

where ϵ is the base of the Naperian system of logarithms. Figure 117 shows the logarithmic curve of rise of current in a particular circuit and Fig. 117 *a* shows the rate of change $\frac{di}{dt} = \frac{E}{L}\epsilon^{-\frac{Rt}{L}}$ of the same current.

It will be observed that, since in this case the counter-voltage of self-inductance, $-\frac{Ldi}{dt}$, is equal to $-(E - iR)$, its instantaneous value is represented by the exponential expression $-E\epsilon^{-\frac{Rt}{L}}$, where t is the numerical value of the time for the instant at which the ratio $\frac{di}{dt}$ is taken.

b. Fall of Current on Withdrawal of Impressed Voltage. — Likewise, when the voltage is withdrawn, $E = 0$, and if the circuit constants are not altered,

$$0 = iR + \frac{Ldi}{dt},$$

or $$\frac{di}{iR} = -\frac{dt}{L}.$$

Integrating between the limits I and i, and 0 and t,

$$\frac{1}{R}\int_I^i \frac{Rdi}{iR} = \int_0^t -\frac{dt}{L},$$

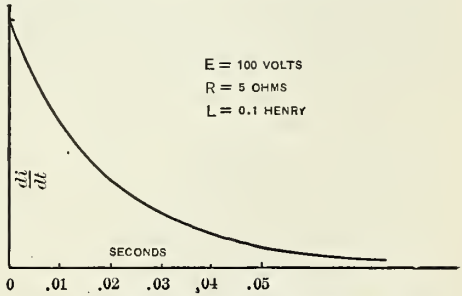


FIG. 117 *a*. — Logarithmic Curve, showing Rate of Change of Rising Current.

or $$\frac{1}{R}(\log i - \log I) = -\frac{t}{L},$$

and $$\log\frac{i}{I} = -\frac{Rt}{L},$$

in which I is the value of the current equal to $\frac{E}{R}$.

Hence, $$i = \frac{E}{R}\epsilon^{-\frac{Rt}{L}},$$

which gives the instantaneous value of the current at any instant during its fall, after the voltage is withdrawn. In this case $-\frac{Ldi}{dt}$ is the voltage of self-induction causing the current to flow, and it is obviously equal at each instant to $iR = E\epsilon^{-\frac{Rt}{L}}$.

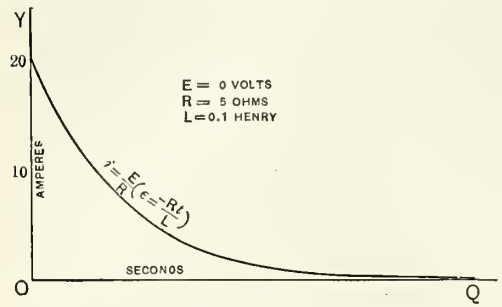


FIG. 118. — Logarithmic Curve of Falling Current.

Figure 118 shows the logarithmic curve of fall of current for the circuit already referred to by Fig. 117, and Fig. 118*a* shows the rate of change of the falling current.

c. Current Value when the Impressed Voltage is a Sine Function of the Time. — Under these conditions the voltage equation of the circuit becomes

$$e = f(t) = e_m \sin \omega t = Ri + \frac{Ldi}{dt}.$$

In this expression $\omega = 2\pi f$ and $\omega t = \alpha$. This is a linear differential equation of the first order, the solution of which is *

$$i = \frac{e_m}{\sqrt{R^2 + 4\pi^2 f^2 L^2}} \sin(\alpha - \theta) + A_1 \epsilon^{-\frac{Rt}{L}},$$

in which $\theta = \tan^{-1} \frac{2\pi fL}{R}$.

After a short time the exponential term in the right-hand member of the equation becomes inappreciable since the exponent is negative and t is an increasing quantity. It may be neglected for the present. Its effect will be discussed later.† The current now becomes a sine function of the time

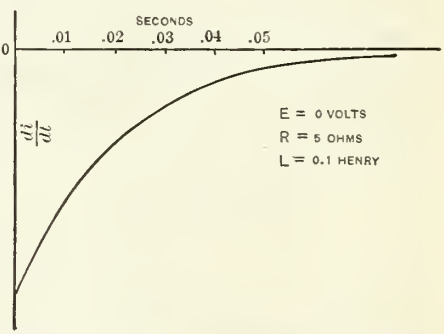


FIG. 118*a*. — Logarithmic Curve, showing Rate of Change of Falling Current.

* Murray's *Differential Equations*, p. 26.

† Art. 59.

and *lags* behind the impressed voltage by an angle θ whose tangent is $\frac{2\pi fL}{R}$. It has the same frequency as the voltage.

It will be observed that the sinusoidal current has a maximum value of

$$i_m = \frac{e_m}{\sqrt{R^2 + 4\pi^2 f^2 L^2}},$$

and since the effective values bear a fixed ratio to the maximum values, we have

$$I = \frac{E}{\sqrt{R^2 + 4\pi^2 f^2 L^2}},$$

in which I is the effective value of the current and E the effective value of the sinusoidal impressed voltage. The denominator of this expression is measured in ohms and is called the **Impedance** of the alternating-current circuit. This impedance is composed of the square root of the sum of two terms. The first of these terms is the square of the electrical resistance, and the second of the terms is the square of the expression $2\pi fL$. This latter expression is called the **Reactance** of the circuit. The vector relations of the voltages of a circuit containing resistance and self-inductance are shown in Fig. 119, which corresponds to Fig. 115.

46. Transference of Electricity during the Transient State.— When the current in an inductive circuit has reached its full value, a smaller quantity of electricity has passed through the circuit during the interval since the start of the current than would have passed if the retardation, or momentum effect, had not been present. This decrease in the quantity of electricity is proportional to the area OYQ between the curve of the current and the horizontal line YQ (Fig. 117). The ordinates of the curve representing instantaneous current strengths in amperes and the abscissas representing time in seconds, the area referred to obviously represents quantity of electricity measured in coulombs. This area may be found in terms of L, I, and R as follows:

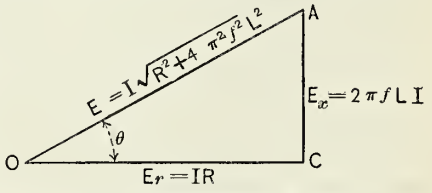


FIG. 119.—Voltage Relations in Circuit containing Resistance and Inductance.

The area is equal to $\int_0^T \left(\frac{E}{R} - i\right) dt$, where T is the duration of time from the instant of the introduction of the source of voltage, E, into the circuit to the instant at which the current reaches its full or ultimate value $I = \frac{E}{R}$. Also, since $iR = E - L\frac{di}{dt}$ and $i = \frac{E}{R} - \frac{Ldi}{Rdt}$, it is manifest that

$$\int_0^T \left(\frac{E}{R} - i\right) dt = \int_0^I \frac{Ldi}{R} = \frac{LI}{R}.$$

The area $OYQA$, which is equal to $\frac{E}{R}T$, represents the number of coulombs which would have been transferred in the circuit during time T had the current instantly come to its full or ultimate value of $I = \frac{E}{R}$; and the area OQA, which is equal to $\frac{ET}{R} - \frac{LI}{R}$, represents the number of coulombs that are actually transferred through the inductive circuit while the current is rising over the logarithmic curve to its full value. The quantity $\frac{LI}{R}$ coulombs is equal to the difference between the quantity of electricity which actually flows through the circuit in the period during which the current is rising to its permanent value $I = \frac{E}{R}$ and the quantity that would flow in the same time if the current immediately rose to its full value.

If the voltage is suddenly reduced from E to zero without breaking the circuit, the current does not stop immediately, but falls off along a logarithmic curve, and the quantity of electricity passing through the circuit is increased on this account. The increased quantity is proportional to the area OYQ in Fig. 118, which shows a curve of falling current in an inductive circuit corresponding with the curve of rising current shown in Fig. 117. That this quantity, proportional to area OYQ of Fig. 118, which is transferred through the circuit after the voltage E is removed, is equal to the deficit of electricity during the starting of the current in the same circuit is shown thus: the counter-voltage is, as before, $-L\frac{di}{dt}$,

and $i_l = \frac{e_l}{R} = -\frac{Ldi}{Rdt}$. But the area OYQ of Fig. 118 is equal to

$$\int_0^T i_l\, dt = -\int_I^0 \frac{Ldi}{R} = \frac{LI}{R},$$

where T is the time in seconds measured from the instant of removing the source of voltage E to the instant at which the current reaches zero value, and I is the value $\frac{E}{R}$ of the current at the instant of removing the source of voltage.

The current, which is thus maintained by the disappearing magnetic field after the external source of current has been removed and which conveys the quantity of electricity $\frac{LI}{R}$ coulombs, was formerly called the *extra current* of self-induction. It was then thought that the self-induction caused an increase of the flow of electricity through the circuit; but the foregoing demonstration of the manner in which the current rises and falls in a self-inductive circuit shows that the *extra current*, which occurs when the external source of current has been removed from the circuit and the current falls from I to 0, conveys a quantity of electricity, $\left(\frac{LI}{R}\right)$, which is exactly equal to the deficit of coulombs which was caused at the time that current was introduced into the circuit, — the aforesaid deficit being due to the fact that the self-induction causes the current to rise gradually instead of instantly when the source of voltage is switched into the circuit, and the *extra current* being due to the fact that the self-induction maintains the current while the magnetic field is dying away after the external source of voltage is removed. If the value I' is substituted for the value 0 in the integrations, it will be observed that the deficit coulombs upon the current being increased from I' to I and the extra-current coulombs upon the restoration of the current to I', assuming these operations accomplished without changing the circuit constants, are each equal to $\frac{L(I-I')}{R}$, and the principle may be stated as follows:

If a current is started or changed by changing the voltage in a circuit having only resistance and self-inductance, and these of fixed values, and the voltage is then restored to its

original value and the current allowed to return to its original or initial value without changing the resistance or self-inductance of the circuit, the total number of coulombs transferred through the circuit during the cycle are equal to the number that would be transferred if the circuit were without self-inductance.

The foregoing demonstration is founded on the proposition that the self-inductance of the circuit is fixed in value, and that the magnetic linkages existing at the respective ends of a cycle of changes are proportional to the initial and final values of the current; but if the electric circuit has an iron core, the density of the magnetism may not fall to its initial value upon restoring the current, and the energy restored is not then equal to that absorbed in building up the magnetic flux. *The difference in the energy remains stored in the magnetic flux in the form of magnetic linkages maintained by residual magnetism.* Under the circumstances here referred to, the permeability of the magnetic circuit usually varies as some complex function of the current, and the current therefore does not rise and fall in plain logarithmic curves because the value of L in the equation $i_x dt = \frac{Ldi}{R}$ is a function of i of greater or less complexity. In this case the integration along the curves of rise and fall may not be readily accomplished, but it always remains a fact (provided the circuit is affected only by resistance and self-inductance and the resistance is of fixed value) that the coulombs conveyed through the circuit by the *extra current* upon restoring a current to its initial value are equal to the deficit occurring during the rise of the current, provided the magnetism and the current both return to their initial values. The effect of hysteresis must be negligible in order that this condition may occur.

In case a current is started by switching a source of voltage into a circuit surrounding an iron core which is already magnetized, the current will rise more rapidly than if the core were not previously magnetized, if the magnetic force of the current is in the direction of the already existing magnetism; and the rise of the current will be retarded if its magnetizing effect is opposite to the already existing magnetism. This is an obvious corollary from the foregoing demonstrations.

Prob. 1. A steady current of 10 amperes is flowing through a coil having a self-inductance of 100 henrys and a resistance of 10 ohms. How many coulombs of electricity will be transferred through the circuit after the voltage is withdrawn without breaking the circuit or changing its resistance? Assume magnetic leakage and hysteresis to be negligible in this and the following problems.

Prob. 2. A magnetic circuit in the form of a ring of 200 centimeters mean length and 100 square centimeters in cross section, with a fixed permeability of 1000 units, has a coil wound upon it of 1000 turns and 20 ohms resistance. If a steady current of 10 amperes is flowing in the coil, what will be the number of coulombs of electricity which will be transferred through the circuit after the voltage is withdrawn without breaking the circuit or changing its resistance?

Prob. 3. Draw the curves of quantity and current when the voltage is introduced and withdrawn in the circuits in problems 1 and 2, assuming no change of resistance or self-inductance in either instance.

Prob. 4. What is the value in amperes of the current flowing in the circuit of problem 2, at an instant $\frac{L}{R}$ seconds after the removal of the voltage?

47. Energy stored in a Magnetic Field associated with an Electric Circuit. — The effect of self-inductance is manifested, as already explained, by a counter-voltage which tends to retard a rising current and to sustain or continue a falling current. That is, when a current is introduced into a circuit, the inductive effect of the rising magnetic flux created by the rising current prevents the current from immediately coming to a value equal to the impressed voltage divided by the resistance of the circuit. Likewise, when the voltage is withdrawn from the circuit, the dying out of the magnetic flux prevents, by its inductive effect, the current from immediately disappearing. The inductive effect of the magnetic flux is directly proportional to the rate of change of the summation of the number of linkages around the conductors of the circuit that are made by the lines of force of the magnetic flux. Each such magnetic linkage set up in connection with a cur-

rent in a circuit represents a definite amount of stored energy; and the phenomena relating to the setting up of the magnetic field (that is, these linkages) about a circuit, thereby storing energy by introducing a current in the circuit, and the phenomena relating to collapsing the linkages and recovering the energy by withdrawing the current, have close analogies to the phenomena relating to the energy of moving bodies.

If a mechanical force is applied to a movable mass or body, the velocity of the body does not immediately rise to its full or ultimate value, but is prevented from doing so by the inertia of the mass, until an amount of energy proportional to the product of the mass and the square of the ultimate velocity has been stored in the body. Likewise, when the impelling force is removed from a moving body, it will continue to move until all the energy that was stored in bringing it up to speed is dissipated in overcoming resistance to its movement. In case of tangible matter $\frac{Mv^2}{2}$, Mv, and $M\frac{dv}{dt}$ are respectively the energy, momentum, and rate of change of momentum (that is, counter-force or pressure caused by inertia) of the mass M when moving at a velocity v; while in the case of the electric circuit, $\frac{Li^2}{2}$, Li, and $L\frac{di}{dt}$, as will be seen later, may be called the energy, momentum, and rate of change of momentum (counter-electric voltage) of its magnetic flux. In such analogies, electric current has its counterpart in velocity or rate of motion, voltage in mechanical force or pressure, electric resistance in mechanical frictional resistance, and self-inductance in mass or inertia.

When a source of constant voltage is switched into a circuit with self-inductance and resistance, the current gradually rises to its final value of $I = \frac{E}{R}$. Also, a counter-electric voltage of self-induction is produced in the circuit by the effect of the rising magnetic flux set up by the current. This counter-voltage at any instant may be represented by the expression $-L\frac{di}{dt}$, assuming the permeability of the magnetic circuit to be constant.

The impressed voltage contains a component equal and

opposite to this counter-voltage at each instant and also supplies the iR drop, or, as has already been pointed out,

$$E = iR + L\frac{di}{dt}.$$

This equation manifestly is equally accurate when E is of constant value or when it represents the instantaneous value of a variable impressed voltage, provided the circuit contains only resistance and self-inductance (that is, the circuit is not subject to the effects of mutual induction or electrostatic capacity). The rate at which work is being delivered from the external source of voltage and energy and expended in the circuit is, in watts,

$$Ei = i^2R + Li\frac{di}{dt},$$

and the work delivered and expended during a time interval dt is, in joules,

$$dw = Eidt = i^2Rdt + Lidi;$$

and the joules delivered by the source and expended in the circuit during a period of T seconds in which the current rises from 0 to I amperes is

$$W = \int_0^T Eidt = \int_0^T i^2Rdt + L\int_0^I idi.$$

The first term of the right-hand member of this equation represents the aggregate heat produced by the current flowing through the resistance of the wire. The second term of the right-hand member is equal to $\frac{LI^2}{2}$ and represents the work done by the external source of voltage and energy in overcoming the counter-electro-motive force, caused by the magnetic flux while the current is changing. This energy is stored in the magnetic linkages of the flux with the turns of the electric circuit and is maintained as long as the current continues at its value I.

As we know the value of current at every instant during its rise,* it is practicable to compute the value of the energy,

* Art. 45, *a*.

$W = \frac{1}{2} Li^2$, stored in magnetic linkages at each instant during the rise of current. The curve exhibited in Fig. 120 shows the value of the energy at each instant corresponding to the conditions illustrated in Fig. 117. It is also practicable to compute the rate of the rise of energy, $\frac{dw}{dt} = Li\frac{di}{dt}$, in the magnetic linkages; and the curve exhibited in Fig. 120*a* shows the value of this rate at each instant, corresponding to the conditions illustrated in Fig. 117.

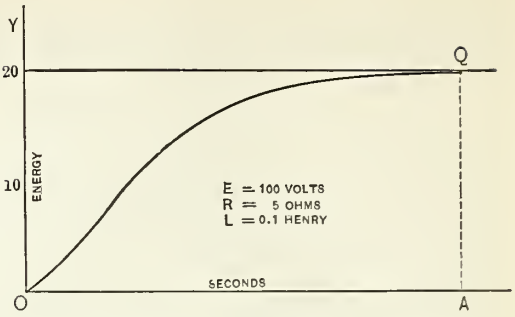


FIG. 120.—Energy Stored in Magnetic Linkages at Each Instant during Period of Current Rise.

The equation shows that the energy delivered to the circuit by a battery, dynamo, or similar source, switched into a circuit having resistance and self-inductance is equal to the sum of the energy converted into heat by the current flowing through the resistance of the conductors and the energy stored in the linkages of the magnetic flux with the conductors of the circuit. If the source supplies a constant voltage, the latter portion of energy is fixed by the character of the circuit and the magnitude of the voltage, and is supplied by the source while the current is rising to its steady value of $\frac{E}{R}$. As long as the current continues of that value the magnetic flux is maintained without further expenditure of energy, and the source then furnishes work equal to that converted into heat only. It will be observed that these deductions are founded on the original premise that the phe-

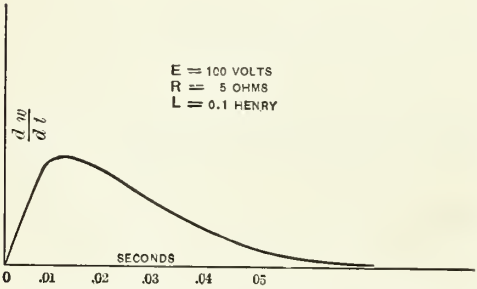


FIG. 120*a*.—Rate of Change of Stored Energy at Each Instant during Period of Current Rise.

nomena of hysteresis, eddy currents, and other effects of mutual induction and the like are absent, and the deductions are limited to conditions in which the phenomena of steady resistance and self-inductance alone occur.

From the formula $E = iR + L\frac{di}{dt}$ it is also to be observed that

$$idt = \frac{E}{R}dt - \frac{L}{R}di.$$

The expression idt obviously represents the quantity of electricity (coulombs) transferred through the circuit by the current flow during the period of dt seconds at the instant of time when the current is equal to i amperes. If the current came instantly to its full value of $I = \frac{E}{R}$ upon introducing voltage E into the circuit, the coulombs transferred through the circuit in any period of T seconds would be IT; but the current does not rise instantly to its full value in a circuit containing resistance and self-inductance, on account of the electro-magnetic inertia of the magnetic linkages. During the time of T seconds which is taken by the current in rising to its full value of I amperes, the number of coulombs transferred is less than IT because the current has had a smaller average value than I amperes. The actual number of coulombs transferred during this transient period of the current rise may be determined by integrating the foregoing formula from zero of time and current through the elapsed time of T seconds during which the current rises to its full value of I amperes, thus,

$$\int_0^T idt = \frac{E}{R}\int_0^T dt - \frac{L}{R}\int_0^I di,$$

or

$$Q_T = \frac{E}{R}T - \frac{LI}{R}.$$

Since $Q = \frac{E}{R}T = IT$ is the number of coulombs that would be transferred through the circuit in T seconds of time if the current had its full value of I amperes all of the time T, the formula $Q_T = \frac{E}{R}T - \frac{LI}{R}$ shows that during the rise of current to its full value of $I = \frac{E}{R}$ amperes, there is a deficit of coulombs

transferred through the circuit, compared with an equal time of full current flow; and that this deficit has a numerical value of $Q' = \frac{LI}{R}$ coulombs, as already pointed out.*

The total energy expended in the circuit during the time of T seconds while the current is rising to its full value of $I = \frac{E}{R}$ amperes is $$Q_T E = EIT - LI^2$$ and, of this, $\frac{LI^2}{2}$ joules goes into storage in the magnetic field. The energy expended during the transient period of the rise of current is therefore less than the energy (EIT) expended in the circuit during an equal length of time with the current at its full value. This is analogous to the energy expended on a moving body. When a given force is externally applied to bring the body in T seconds of time from rest to full speed at which the force is just balanced by frictional opposition, the distance moved by the body and the total work (joules or foot-pounds) done in the time of T seconds are less than would have been the case if the body had moved at its maximum velocity throughout the period of T seconds.

If the external source of energy is switched out of the circuit without changing the resistance and self-inductance when the current is of value I amperes, the energy equation becomes

$$0 = \int_0^T i^2 R dt + L \int_I^0 i di.$$

The last term may be written $-\frac{LI^2}{2}$; and $\frac{LI^2}{2}$ is equal to $\int_0^T i^2 R dt$, which shows that the magnetic field (under the circumstances considered) discharges energy equal in quantity to that stored at the time the current was started, and that this energy is now all converted into heat during the flow of the "extra current."

The energy stored in the magnetic field falls off as the current falls, and the values of the stored energy corresponding to the conditions illustrated in Fig. 118 are shown by the curve in Fig. 121. The rate with which the stored work is given out by the magnetic field at each instant for corresponding conditions is shown by the curve in Fig. 121*a*.

* Art. 46.

The voltage formula also becomes

$$0 = iR + L\frac{di}{dt},$$

or,

$$idt = -\frac{L}{R}di.$$

The expression idt represents the coulombs of electricity transferred through the circuit during dt seconds by the discharge of the energy from the magnetic field, after the external impressed voltage has been removed. That is, the current does not fall instantly to zero under the circumstances when the external voltage is switched out of circuit without changing the resistance or self-inductance, but the discharge of the energy from the magnetic field tends to maintain it briefly.

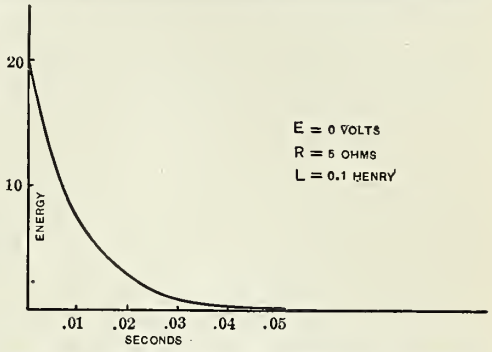


FIG. 121.—Energy Stored in Magnetic Linkages at Each Instant during Period of Falling Current.

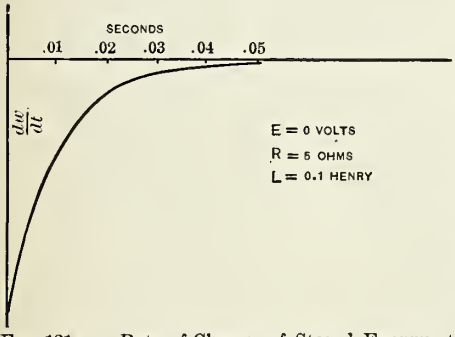


FIG. 121 *a*.—Rate of Change of Stored Energy at Each Instant during Period of Falling Current.

The total coulombs conveyed through the circuit by this "extra current" may be obtained by integrating the foregoing formula from the instant when the external voltage is switched out of the circuit (that is, from zero of time and I amperes of current) through the period of T seconds during which the current falls to zero, thus

$$\int_0^T idt = -\frac{L}{R}\int_I^0 di,$$

or,

$$Q'' = \frac{LI}{R}.$$

It will be observed from this that $Q'' = Q'$, which is to say that the number of coulombs conveyed through the circuit by the "extra current" is exactly equal to the deficit in the coulombs conveyed through the circuit during the time of the current rise, as previously pointed out,* so that the number of coulombs conveyed through the circuit as a result of the cycle of rise and fall of current does not differ from the number that would be conveyed through the circuit if L were zero or had any other value, provided R remained unchanged in value. The time occupied by the cycle consisting of the rise of the current to its full fixed value of $\frac{E}{R}$ and its succeeding fall to zero upon switching out the external voltage E is affected by changing the value of L, but the number of coulombs transferred through the circuit during the cycle is not changed.

The gradual decay of the electric current under the conditions here described, accompanied by the consumption of the energy stored in the magnetic field, is analogous to the decay of the velocity of the moving body referred to above. Upon the removal of the external force after the body has come to its full speed, the body would continue in motion with gradually falling speed and finally come to rest when the energy stored in its moving mass at the maximum speed had all been consumed in overcoming the frictional resistance and been converted into heat.

The work expended in a self-inductive circuit is therefore manifested by (1) the iR drop relating to the conversion of energy into heat and (2) the counter-voltage of self-induction relating to the storage of energy in the magnetic flux by a tendency to retard a rising current and accelerate or continue a falling current. The effects are in many respects analogous to the inertia of tangible matter, as already pointed out, in which $\frac{Mv^2}{2}$, Mv, and $M\frac{dv}{dt}$ are respectively the energy, the momentum, and the rate of change of momentum (force) acting in the mass M when moving with the velocity v; while in the electric circuit, $\frac{Li^2}{2}$, Li, and $L\frac{di}{dt}$ may be called the energy, momentum, and rate of change of momentum (counter-voltage)

* Art. 46.

of its magnetic field. The manifestations of the electro-magnetic inertia are exerted through the linkages of the magnetic lines of flux with the conductors of the electric circuit.

When the value of L varies with the current, as when iron is magnetized by the electric current under consideration, the energy stored in the magnetic linkages is measured by the value of L corresponding to the current flowing at the instant under consideration; and the value of L corresponding to $I=\frac{E}{R}$ is the correct value to apply in the expression $\frac{LI^2}{2}$ when it is desired to compute the energy stored during the rise of current from 0 to $\frac{E}{R}$ amperes in a circuit affected by a constant impressed voltage, since the total number of linkages only is required to obtain the stored energy and not the instantaneous rates at which they are created or disappear. The phenomena of hysteresis prevent the magnetic field from fully discharging its energy as the current falls. For instance, if the current is caused to increase from I_1 to I and the self-inductance changes from L_1 to L, the energy that is delivered to the magnetic field by virtue of the increase of current is equal to $\frac{LI^2}{2}-\frac{L_1I_1^2}{2}$. If the current now returns to the value I_1 amperes, hysteresis may prevent the magnetic field from assuming its first value, so that the value of the self-inductance has the different value of L_2. In this case, the cycle of the rise of current from I_1 to I and its restoration to I_1 has caused an increase of the energy stored in the magnetic field which is equal to $\frac{(L_2-L_1)I_1^2}{2}$. If the current changes in recurrent cycles between $-I$ and I, the conditions in the magnetic field also become cyclic, and a certain amount of energy is converted into heat in each cycle as the result of the hysteresis phenomena.

If a coil is wound on a closed ring of soft iron, which form exhibits great magnetic retentiveness, the value of L is very great if the ring is magnetized by an alternating current. But if the ring is magnetized by a rectified periodic current, that is, a periodic current which is unidirectional, the apparent value of L may be practically the same as though the iron core were not present. This behavior is due to the ring

continuously retaining the magnetization caused by the maximum current, and since the magnetism in the core therefore remains constant it does not set up a counter-voltage. By making a transverse cut in the ring, its coercive force may be reduced so much that the effects are practically the same for rectified and alternating currents.

Prob. 1. A coil with a self-inductance of .02 of a henry has a steady current of 50 amperes flowing through it. What is the energy of the magnetic field?

Prob. 2. A magnetic circuit 100 centimeters long and 200 square centimeters in cross section, the material of which has a constant permeability of 1000 units, is excited by a coil of 500 turns which has a resistance of 10 ohms. What energy has the magnetic field when 100 volts are steadily impressed at the terminals of the coil, there being no magnetic leakage?

Prob. 3. A steady current of two amperes flowing in a coil of 1000 turns creates a field which sets up 1,000,000 lines of force through each turn of the coil. What is the energy of the field?

Prob. 4. A magnetic circuit is excited by a coil of 2000 turns in which a steady current of 20 amperes flows and sets up a total of 2,000,000 lines of force. What is the energy of the field, if all the lines of force link with all the turns of the coil?

Prob. 5. In what degree does changing the reluctance of a magnetic circuit affect the energy stored therein by an exciting coil, other things remaining equal?

Prob. 6. The energy of a certain magnetic field is 20 joules. This is created by a magnetizing coil having 100 turns carrying a steady current of 10 amperes. What is the self-inductance of the circuit if all the lines of force of the field link with all the turns of the coil?

48. The Transient Transfer of Electricity in Divided Circuits. Application to a Shunted Ballistic Galvanometer. — The fact that the total quantity of electricity which passes through a wire when subjected to a transient voltage is independent of the self-inductance of the circuit when hysteresis effects are absent, as is shown above, has a bearing upon the distribution of currents in divided circuits. With no external disturbing factors, it is apparent that where a transient voltage is impressed upon

parallel circuits of different inductances, the number of coulombs which flow through each circuit would also flow were the circuits without self-inductance, but the phase of the flow in each circuit is retarded so as to lag behind that of the voltage by an amount which is proportional to the self-inductance of the circuit.

This reasoning would make it appear that shunting a ballistic galvanometer must change the constant in the ratio of the resistances of galvanometer and shunt without regard to their self-inductances, as is true when steady currents are in question. This, however, is not correct, because one of the factors in this problem has not been taken into account, *i.e.* the movement of the needle which occurs before the end of the discharge generates a counter-voltage in the galvanometer winding. This reduces the proportion of the discharge which passes through the galvanometer, but the effect is due to the needle and not to the relation of the self-inductances of the galvanometer winding and the shunt. Assuming that the number of lines of force due to the needle which link with the turns of the winding are proportional to the sine of the deflection of the needle; calling r_g and L the resistance and self-inductance of the galvanometer winding; r_s the resistance of the shunt (the inductance of the latter being assumed negligible on account of its being wound with doubled wire); and i_g and i_s being the respective instantaneous currents: then the instantaneous impressed voltage is $e = i_s r_s$. The corresponding instantaneous value of the IR drop in the galvanometer winding is $i_g r_g$, and this is equal to the impressed voltage less the corresponding instantaneous counter-voltages caused by self-induction and the swing of the needle. Therefore:

$$i_g r_g = i_s r_s - \left(\frac{L di_g}{dt} + \frac{k d(\sin \alpha)}{dt}\right),$$

where k is a constant representing the summation of linkages of magnetic lines of force from the needle with turns of the galvanometer winding when the needle stands perpendicular to the plane of the winding. Whence $i_s r_s dt - i_g r_g dt = L di_g + k d(\sin \alpha)$

and $$r_s \int_0^t i_s dt - r_g \int_0^t i_g dt = L \int_0^0 di_g + k \int_0^\alpha d(\sin \alpha).$$

This is $r_s q_s - r_g q_g = k \sin \alpha$. If the swing of the needle is small,

then $\sin\alpha$ is sensibly equal to $2\sin\frac{\alpha}{2}$, but $2K\sin\frac{\alpha}{2}=q_g$, where K is the ordinary constant of the ballistic galvanometer. Hence, $r_s q_s - r_g q_g = \frac{k q_g}{K}$. Calling Q the total discharge, which is equal to $q_s + q_g$, this becomes

$$r_s(Q-q_g)-r_g q_g=\frac{k q_g}{K}, \text{ and } q_g=\frac{Q r_s}{r_g+r_s+\frac{k}{K}}.$$

This discussion shows that the coulombs of a discharge which pass through a shunted ballistic galvanometer are less than might be predicted from the ratio of the resistances; that is, $q_g < \frac{Q r_s}{r_g+r_s}$. This deficit is caused solely by the lines of force from the needle cutting the galvanometer coils while the discharge is passing, and its value is

$$\frac{Q r_s}{(r_g+r_s)\left[1+\frac{K}{k}(r_g+r_s)\right]}.$$

The shunted ballistic galvanometer therefore gives readings which are too small, unless the duration of the discharge is very small compared with the time of vibration of the needle. But it must also be remembered that, if the needle is removed from the galvanometer or clamped in a fixed position, the division of the discharge between the galvanometer winding and the shunt is then independent of the self-inductance of the winding and shunt, and may be predicted from the ratio of resistances, or $q_g=\frac{Q r_s}{r_g+r_s}$.

Only under special conditions can a single coil with self-inductance be substituted for the coils in parallel so as to produce the same effect as the latter upon transient currents of every duration. These conditions are fulfilled when the ratio of $\frac{L}{R}$ is constant for all the coils, and an equivalent coil may then be substituted for the parallel circuits. In this case the resistance of the equivalent coil must be

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\text{etc.},$$

and its self-inductance must be

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \text{etc.}$$

These values make $\frac{L}{R}$ equal to the constant value of the ratio for the individual coils.*

49. Effect of Eddy Currents and of Hysteresis upon the Rise and Fall of Current in a Self-inductive Circuit. — If a circuit surrounds or lies near conducting material, the rise or fall of current when a voltage is applied or removed is modified by the magnetic effect of eddy currents induced in such conductors.† The eddy currents tend to quickly rise to a maximum while the main current is changing most rapidly, and then to fall gradually as the main current approaches its full value. Eddy currents are similar to the secondary currents of a transformer ‡ and are induced in the same way. As in the transformer, an additional current just sufficient to neutralize the magnetic effect of the secondary currents at the instant flows in the main or primary circuit. This extra neutralizing element of the main current causes the curve of current to rise more rapidly when the voltage is applied and fall more quickly when the voltage is withdrawn; that is, energy is expended not only in building up the magnetic field and in i^2R loss in the main circuit, but also in the eddy current circuits, as the main current rises, and while the main current is falling, the energy yielded from the magnetic field is absorbed by i^2R losses in the main circuit and also in the eddy current circuits. This transfer of energy from the main circuit to the eddy current circuits has the same effect on the rise and fall of the current in the main circuit as would be produced by increasing to an equal degree the energy changed into heat by the i^2R loss in the main circuit and excluding eddy current effects. Any transfer of energy by induction from the main circuit while the main current is changing has the same effect of hastening the change of the main current, and thus *reducing the apparent effect of the self-induction.* It is therefore possible to surround a self-inductive circuit with conducting material in such a manner as to largely mask the presence of self-inductance.

* The subject of circuits in parallel is fully treated in later pages.

† Art. 111. ‡ Art. 23 and Chap. X.

On account of the counter-magnetizing effect of eddy currents the lines of force set up by the main current tend to crowd outside of the eddy current circuits, which decreases the average cross section of their path and hence increases the reluctance. This effect of eddy currents in apparently screening the interior of an iron core from magnetic influences decreases the number of lines of force per ampere in the circuit and consequently also decreases the apparent self-inductance. It consequently, also, causes an acceleration in the rise or fall of the current. If the circuit surrounds an iron core, this effect may be quite large unless the core is finely laminated.

The phenomena of hysteresis and residual magnetism tend to produce a smaller average rate of change of the magnetic field set up in an iron core when the exciting current is withdrawn than occurs when the exciting current is introduced in the circuit ; and a current therefore dies away more quickly than it builds up in a circuit with an iron core, even though the source of voltage is switched into the circuit and later removed without altering the resistance of the circuit. The change of permeability of iron for different magnetizations, which results in a change in number of magnetic lines set up per ampere in the electric circuit, causes a distortion of the curves of rising or falling current, as already pointed out in the latter part of Art. 46. This effect becomes particularly noticeable when the magnetic circuit is completed through iron, as in the case of a closed ring core or the transformer core illustrated in Fig. 45, as any air gap in the path of the magnetic lines of force introduces a space of relatively large reluctance which is independent of the current.

50. High Voltage generated on Breaking a Self-inductive Circuit. — The condition under which the curves of rising and falling current are logarithmic and of exactly the same dimensions when voltage is applied to and withdrawn from a circuit requires that the resistance as well as the self-inductance of the circuit shall remain constant. If the circuit is quickly broken, by opening a switch or otherwise, it is a well-known fact that the *counter-voltage rises much higher than the original impressed voltage*, frequently rising to many times its value. The extreme severity of the shock which may be received upon breaking a circuit of large inductance attests the fact. This is due to

the exceedingly large increase of resistance in the circuit which is introduced by the break. This increase in resistance causes the current to fall off more quickly, and hence produces a greater rate of change of magnetism. However, as before, the energy given up by the field must be

$$W = \int_0^I L i_1 di_1 = \frac{LI^2}{2},$$

where i_1 is the current after the break, since the same number of magnetic linkages disappear from the circuit, and the energy given up must therefore be equal to the energy stored in the circuit when the full current I was flowing; and, also

$$q_1 = \int_0^t i_1 dt = \frac{LI}{R_1} < \frac{LI}{R},$$

where R_1 is the new resistance of the circuit, including the break, assumed to be constant, and q_1 is the coulombs of the discharge. The current is

$$i_1 = I\epsilon^{-\frac{R_1 t}{L}} < I\epsilon^{-\frac{Rt}{L}}$$

for finite values of t; and the induced voltage is

$$e_1 = L\frac{di_1}{dt} = IR_1\epsilon^{-\frac{R_1 t}{L}} > E\epsilon^{-\frac{Rt}{L}}$$

for small values of t. The total number of coulombs transferred is, therefore, less, and the *induced voltage greater, upon breaking a circuit than upon making it.** If hysteresis is present, the field may not be totally discharged, and the amount of energy and quantity of electricity will then not be so great as indicated by the formula, as is explained in the previous article.

Prob. 1. A certain electric circuit supports a magnetic field which is charged with an energy of 5 joules. This circuit is opened by introducing resistance increasing to infinity in .01 of a second. What is the average power exerted by the magnetic field during the time of the break, assuming that the magnetism falls to zero? Give the answer to this and the following problems in watts and neglect the effect of hysteresis and eddy currents.

* Formulas for further expressing the conditions when the resistance of a circuit is changed are given in Art. 55.

Prob. 2. If 6,000,000 lines of force linking the turns of a coil of 1000 turns are set up by a steady current of 5 amperes in the coil, what is the average power exerted by the discharge of the magnetic field if the exciting circuit is broken in .01 second by introducing resistance up to infinity, and the magnetism falls to zero?

Prob. 3. A magnetic circuit having an average permeability of 1000 units, a length of 200 centimeters, and a cross section of 50 square centimeters is excited without magnetic leakage by a coil of 1000 turns having a resistance of 25 ohms. A steady voltage of 1000 volts is impressed at the terminals of the coil. What is the average power exerted by the discharge of the magnetic field when the exciting circuit is broken through an arc maintained .01 of a second?

51. Capacity. — All insulated conductors have the property of being able to hold a charge of electricity. When an insulated conductor is connected to a source of a different potential, electricity will flow to it or from it, until its potential is the same as that of the source. The measure of the charge or quantity of electricity which is held by the conductor when at unit potential is its electrostatic **Capacity**, and the unit of capacity may be defined as the capacity of a conductor which contains a unit charge of electricity when at unit potential. *The practical unit of capacity is the capacity of an isolated conductor which would contain a charge of one coulomb when at a potential of one volt.* This is called a **Farad**, after Faraday. It is $\frac{1}{10^9}$ times as large as the unit of capacity of the C. G. S. set of units. The farad is too large a unit of capacity to be convenient in practice, and the **Microfarad**, or millionth of a farad, is commonly used as the unit of measurement. The capacity of a conductor depends upon its conformation and surroundings.

Two adjacent conducting bodies brought to different potentials and isolated from external influences assume equal and opposite charges. In this case, the capacity is measured in terms of the difference of potential between the conductors and the charge on each. The capacity is one farad when a difference of potential between the conductors of one volt produces a charge of one coulomb on each, — the charges being (relative

to each other) positive on the conductor of higher potential and negative on the conductor of lower potential.

Capacity effects caused by differences of potential between neighboring conductors or parts of the same conductor, are of constant occurrence and much importance in electrical engineering structures. In most of such instances the influences are complex because many conductors of different potentials and of different shapes and space relations relative to each other may be involved; but most of these instances may be reduced to approximate analytical processes by considering the conductors pair by pair and thus learning their influence on each other.

The term **Condenser** is applied to any pair of insulated conductors having an appreciable capacity, although it is more strictly used to designate a combination of sheets of conducting material, insulated, and laid together with the alternate layers connected in parallel. In the following discussion the term condenser will be used in its broader sense.

The **Dielectric** of a condenser is the insulating medium which surrounds the conductors; and its dimensions and quality determine the condenser capacity. The capacity is directly proportional to the area and inversely as the thickness of the dielectric and depends upon the **Specific inductive capacity**, or **Dielectric constant** of the dielectric material. The **Charge of a condenser** is the quantity of electricity comprised on either conductor or electrode of the condenser.

It should be kept clearly in mind that the capacity of a conductor or circuit is a quality of the circuit and is not dependent upon the current flowing or the voltage applied. In the same way the capacity of a vessel to hold a liquid is a quality of the vessel, dependent upon the conformation of the vessel, and irrespective of the amount of liquid that may be in the vessel or the rate at which it is running in or out.

From the foregoing definitions of capacity is at once derived the fundamental relation,

$$Q = CE,$$

where Q represents the quantity of electricity (that is, the coulombs) in the charge of a condenser, C the capacity in farads, and E the voltage between the conductors or electrodes of the condenser.

Prob. 1. A condenser is charged by bringing its electrodes to a difference of potential of 100 volts. What is its capacity if, under that voltage, its charge is .002 coulomb?

52. Conditions of Establishment and Termination of Current in a Circuit containing Capacity and Resistance in Series.—*a. Current and Rise of Charge upon Impressing Constant Voltage.*—The work done during time dt on a circuit containing resistance and capacity at instant t after a constant voltage E is impressed thereon is

$$Edq = Eidt = edq + Ri^2dt,$$

where e, i, and q are the instantaneous values at instant t of the voltage of the charge, current flowing into the condenser, and quantity of the charge. The last member contains two terms, of which the first is the work stored in the increment dq coulombs of the charge which is put in the condenser during the time dt seconds, and the second is the work converted into heat by dq coulombs flowing through the resistance R of the circuit. If this equation is divided by $idt = dq$, there results an equation of voltage

$$E = e + Ri,$$

or

$$E = \frac{q}{C} + R\frac{dq}{dt}.$$

From these equations the charge at any instant may be determined when the applied voltage E is constant during the process of charging. The equation may be put in the form

$$\frac{dq}{q - CE} = -\frac{dt}{RC};$$

whence

$$\int_0^q \frac{dq}{q - CE} = -\int_0^t \frac{dt}{RC}.$$

Integrating gives

$$\log (q - CE)\Big]_0^q = -\frac{t}{RC}\Big]_0^t,$$

or

$$\log \left\{\frac{q - CE}{-CE}\right\} = -\frac{t}{RC},$$

and

$$q - CE = -CE\epsilon^{-\frac{t}{RC}}.$$

Therefore since $CE = Q$, where Q is the final value of the charge,

$$q = Q(1 - \epsilon^{-\frac{t}{RC}}),$$

and

$$i = \frac{dq}{dt} = \frac{Q}{RC}\epsilon^{-\frac{t}{RC}}.$$

Figure 122 shows the logarithmic curves of the rise of charge in a condenser in a particular circuit and the current flowing

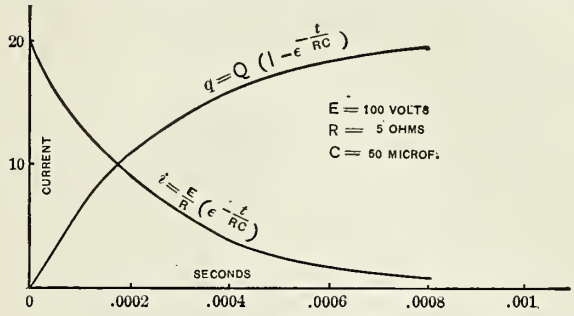


FIG. 122. — Curves of Charge of a Condenser.

into the condenser (that is, the rate of change of the charge) while the charge is reaching its full value Q after 100 volts are impressed on the circuit. Figure 122 *a* shows the rate of change of the current $\frac{di}{dt} = -\frac{Q}{R^2C^2}\epsilon^{-\frac{t}{RC}}$ during the same process. It is to be noted that the current at the instant of introducing the voltage E into the circuit (*i.e.* $t = 0$) is $i = \frac{Q}{RC} = \frac{E}{R}$ amperes, and it is therefore instantaneously equal to the current that the voltage E would cause to flow through the resistance R of the circuit if the condenser were absent.

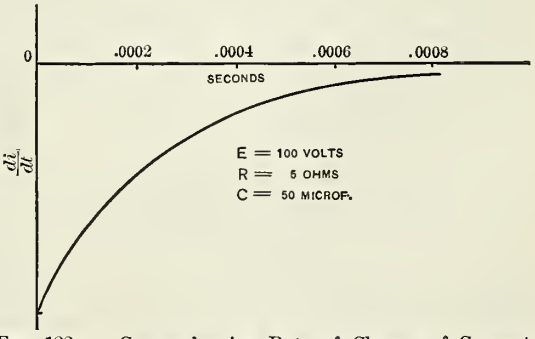


FIG. 122 *a*. — Curve showing Rate of Change of Current during the Period of Charging.

b. Current and Fall of Charge on Withdrawal of Impressed Voltage. — During the discharge, assuming that the impressed

voltage is withdrawn from the circuit without changing R or C, the impressed voltage E is zero, and therefore

$$0 = \frac{q}{C} + R\frac{dq}{dt},$$

or

$$\frac{dq}{q} = -\frac{dt}{RC},$$

and

$$\int_Q^q \frac{dq}{q} = -\int_0^t \frac{dt}{RC}.$$

Integrating gives

$$\log q\Big]_Q^q = -\frac{t}{RC}\Big]_0^t;$$

or

$$\log\frac{q}{Q} = -\frac{t}{RC},$$

whence

$$q = Q\epsilon^{-\frac{t}{RC}},$$

and

$$i = \frac{dq}{dt} = -\frac{Q}{RC}\epsilon^{-\frac{t}{RC}}.$$

Figure 123 shows the logarithmic curves of falling charge after the impressed voltage has been removed and of current flowing during the period of discharging. Figure 123 *a* shows the rate of change of current, $\frac{di}{dt}$, during the same period.

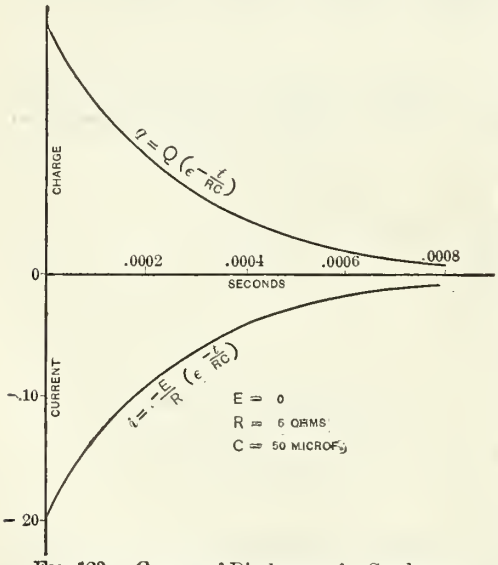


FIG. 123.—Curves of Discharge of a Condenser.

From the equations showing the charging and discharging currents, which are of forms similar to those showing the rise and fall of currents in circuits containing resistance and self-inductance, it is seen that the curves of charge and discharge in circuits containing resist-

ance and capacity in series are logarithmic when an unvarying voltage is impressed on the system. (See Figs. 122 and 123.) In many cases of practice RC is so small that the charge and discharge of a condenser are practically instantaneous, but in other cases the duration of the rise or fall of the charge may be quite appreciable. Some condensers have a quality which has some analogy to magnetic hysteresis and which is often termed dielectric hysteresis from the fact of its being due to a characteristic of the dielectric.

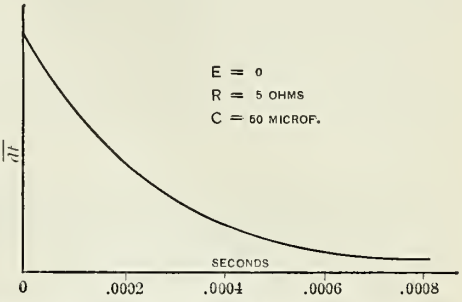


FIG. 123*a*. — Curve showing Rate of Change of Current during the Period of Discharging.

When hysteresis is present, the condenser does not at once proceed to a complete discharge when the impressed voltage is removed from its terminals, but it retains a residual charge. The amount of energy retained in this residual charge is a measure of the amount of energy absorbed by the condenser on account of hysteresis, and this energy may be converted in each cycle into heat if the condenser is subjected to alternating charges and discharges. It is probable that this effect is due to a small part of the electric charge soaking into the dielectric (especially into those layers near the condenser electrodes) and then soaking out again, because of the imperfect insulating qualities of the dielectric, rather than to molecular phenomena strictly analogous to magnetic hysteresis; and the heating of the dielectric which is sometimes observed as the result of repeated charging and discharging may be ascribed to the I^2R loss due to the current flow in the dielectric which results from the aforesaid soaking in and out of the charge. The effect is not ordinarily very important in engineering practice.

Prob. 1. A condenser of 50 microfarads capacity and negligible internal resistance is connected in series with a wire having a resistance of 10 ohms. Draw the curve of charge in this circuit when 100 volts are impressed; and of discharge

when the impressed voltage is removed without breaking the circuit.

Prob. 2. What quantity of charge has the condenser of problem 1 received at the instant when the impressed voltage has been applied for RC seconds?

Prob. 3. What quantity of charge remains in the condenser of problem 1, RC seconds after the removal of the impressed voltage?

c. Current and Charge when the Impressed Voltage is a Sine Function of the Time. — Under these conditions the voltage equation of the circuit becomes

$$e = f(t) = e_m \sin \omega t = Ri + \frac{q}{C}$$

$$= Ri + \int \frac{idt}{C}.$$

This is a linear differential equation of the first order, the solution of which is *

$$i = \frac{e_m}{\sqrt{R^2 + \frac{1}{4\pi^2 f^2 C^2}}} \sin(\alpha - \theta) + B_1 \epsilon^{-\frac{t}{RC}},$$

where $\alpha = \omega t$, $\omega = 2\pi f$, and $\theta = -\tan^{-1}\left(\frac{1}{\omega C} \div R\right)$.

In a similar manner the value of q is found to be

$$q = \int idt = -\frac{e_m}{2\pi f \sqrt{R^2 + \frac{1}{4\pi^2 f^2 C^2}}} \cos(\alpha - \theta) + B_2 \epsilon^{-\frac{t}{RC}}.$$

After a short time the exponential term of the equations becomes inappreciable and may be neglected. The effect of this exponential term will be discussed later.† The current then becomes a sine function of the time and *leads* the impressed voltage by an angle θ whose tangent is $\frac{1}{2\pi fC} \div R$, instead of

* Murray's *Differential Equations*, p. 26.
† Art. 59.

lagging behind it, as in the case of a circuit containing inductance and resistance. The current has a maximum value

$$i_m = \frac{e_m}{\sqrt{R^2 + \frac{1}{4\pi^2 f^2 C^2}}},$$

or an effective value of

$$I = \frac{E}{\sqrt{R^2 + \frac{1}{4\pi^2 f^2 C^2}}},$$

in which E is the effective value of the sinusoidal impressed voltage. The denominator of this expression is, as in the case of the inductive circuit, called the **Impedance** of the circuit. This impedance is composed of the square root of the sum of two terms, one of which is the square of the electrical resistance and the other the square of the expression $\frac{1}{2\pi fC}$. This latter term is called the **Reactance** of the circuit, and corresponds in this case to the term similarly designated in the circuit containing inductance and resistance.

53. The Energy of a Charged Condenser. —As a condenser is charged, a certain amount of work is done in raising the potential of the charge. During the time dt this is equal to

$$dw = (eidt = edq) = Cede,$$

in which e is the voltage of the charge, i the current flowing into the condenser, and q the charge in the condenser at the instant t; from which, by integration,

$$W = \frac{CE^2}{2} = \frac{EQ}{2},$$

where E and Q are respectively the final values of the voltage and the quantity composing the charge.

This represents a certain amount of work which is stored in the condenser when its charge is increased from zero to Q coulombs. When the condenser is discharged, an equal amount of work is returned to the circuit. The expression $\frac{CE^2}{2}$ is similar to that giving the work stored in a compressed spring, $W = \frac{NF^2}{2}$, where N is the compressibility of the spring meas-

ured by the distance compressed per unit of force, and F is the force applied to perform the compression. These expressions are proportional to the square of impressed electrical or physical pressure and the stored energy is truly potential. These equations thus differ from the expressions $\frac{LI^2}{2}$ and $\frac{MV^2}{2}$, which are dependent upon momentum instead of pressure and represent kinetic energy.

As we know the current and the value of the charge at each instant during the rise of the charge, it is practicable to compute the value of the energy, $w = \frac{1}{2} Ce^2 = \frac{1}{2} \frac{q^2}{C}$, stored in the condenser at each instant during the rise of charge, and the rate at which the energy is changing at each instant. These values are exhibited in the curves of Figs. 124 and 124 a for the conditions of the circuit corresponding to Fig. 122.

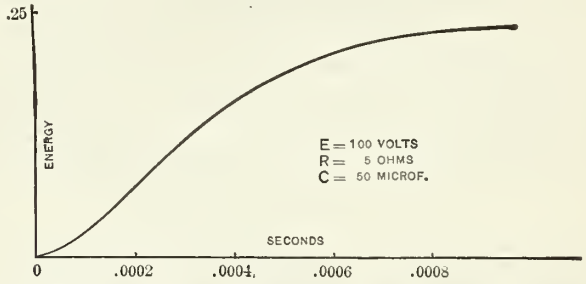


FIG. 124. — Energy stored in Condenser during Period of Charging.

The energy stored in the condenser at each instant during the fall of the charge, and the rate at which the condenser gives out energy (the rate of change of the stored energy) during the fall of the charge, for circuit conditions corresponding to those indicated in Figs. 123 and 123 a are shown in Figs. 125 and 125 a.

54. Vector Diagrams showing the Voltage and Current Relations in a Circuit containing R and C. — *Expression in Form of Complex Quantities.* — When a condenser is connected to a source of alternating voltage, as indicated in Fig. 126, a current will flow into and out of the condenser. The value of the current at any instant is proportional to the rate of change of the voltage impressed on the capacity, because the charge in the condenser at any instant is proportional to the charging

voltage then acting on the capacity, and the rate at which the charge changes must be proportional to the rate at which the voltage changes. The rate of change of the charge is equal to the number of coulombs flowing per second into or out of the condenser, and is therefore equal to the current flowing into or out of the condenser. The condenser current (i_c) at any instant is represented by $i_c = C\frac{de}{dt}$, and since, when the alternating voltage is sinusoidal,

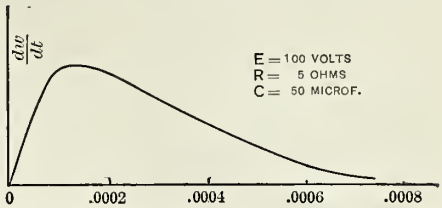


FIG. 124 *a*. — Rate of Change of Energy stored in Condenser during Period of Charging.

$$e = e_m \sin \alpha = e_m \sin \omega t,$$

where e_m is the maximum voltage acting upon the condenser during a cycle, ω is the angular velocity or advance of the cycle, and t the time measured from the beginning of a cycle,

$$\frac{de}{dt} = e_m \omega \cos \omega t.$$

But $$\omega = 2\pi f,$$

where f is the frequency of the cycles.

Therefore, $$i_c = e_m \cos \alpha \div \frac{1}{2\pi f C} = 2\pi f C e_m \cos \alpha$$

and $$e_m \cos \alpha = i_c \times \frac{1}{2\pi f C}.$$

The voltage e_c of the condenser charge, which is, of course, equal and opposite to the voltage e impressed on the plates, may be called the **Capacity voltage** or **Condenser voltage**. This voltage is purely reactive, due to the charge in the condenser. That it must be 90° in advance of the current when the impressed voltage is sinusoidal may be readily seen

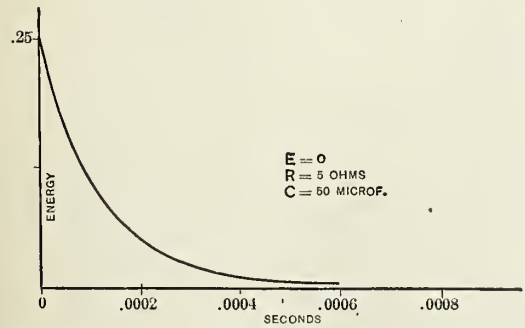


FIG. 125. — Energy stored in Condenser at Each Instant during Period of Discharging.

from the formula above and the reactions that occur in the circuit. When a sinusoidal voltage impressed at the terminals of a resistanceless condenser is rising, *a current flows into the condenser.* This current is a positive maximum at the instant the impressed voltage passes through zero in changing from negative to positive values, for the rate of change of voltage is then a positive maximum. When the impressed voltage passes through its maximum point, its rate of change is zero, and the current at that instant is therefore zero. When the voltage is falling, *a current flows out of the condenser;* that is, in the direction which discharges the condenser. Therefore, the current flowing is 90° in advance of the voltage impressed on the capacity. The capacity voltage is a counter-voltage caused by the potential of the charge, and is equal and opposite to the voltage impressed on the capacity, and is 90° in advance of the current.

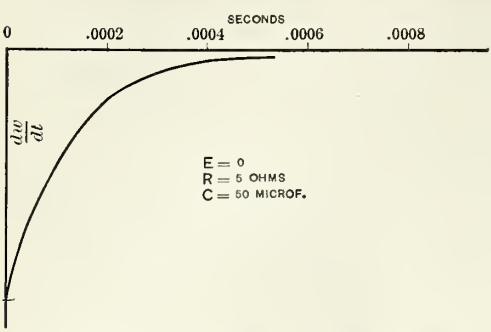


FIG. 125*a*. — Rate of Change of Energy stored in Condenser during Period of Discharging.

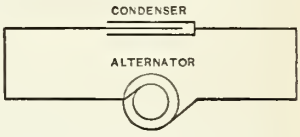


FIG. 126. — Diagram showing an Alternator connected to a Capacity Circuit.

From the formula for instantaneous current in the condenser the maximum current is seen to be

$$i_{cm} = 2\,\pi f C e_m,$$

and the effective current,

$$I_c = 2\,\pi f C E,$$

where E is the effective value of the voltage impressed on the capacity. This current I_c being 90° ahead of the phase of the voltage impressed on the capacity (which is 180° behind the capacity voltage), the phase of the active voltage which drives the current through the resistance of the circuit is also 90° ahead of the voltage impressed on the capacity. The voltage impressed upon the series circuit containing capacity and re-

sistance must therefore comprise two components, one equal to the IR drop in the circuit and the other equal and opposite to the capacity voltage. A phase diagram of these, accompanied by the corresponding sinusoidal curves, is shown in Fig. 127, where OC, CA, and OA are respectively the IR drop, the component equal but opposite to the capacity voltage, and the voltage impressed on the circuit. The triangle of voltages E_r, E_x, E, as shown in Fig. 128, is derived from this diagram, and it may be observed that

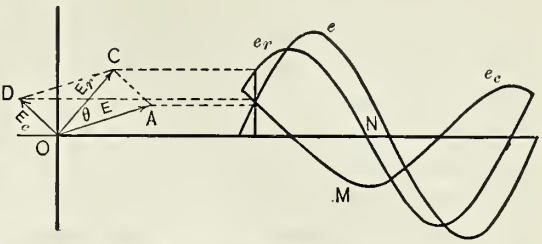


Fig. 127. — Phase Diagram of Voltages in a Circuit containing Resistance and Capacity in Series.

$$E = \sqrt{E_r^2 + E_x^2};$$

and also that the angle of lead, *i.e.* the angle by which the current and IR drop lead the voltage impressed on the circuit, is the angle θ, with

$$\tan \theta = \frac{-E_x}{E_r}.$$

The angle θ is here called negative because it relates to a leading current, and the corresponding angle for a lagging current has already been called positive.

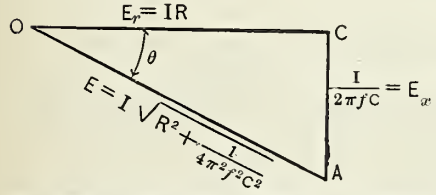


Fig. 128. — Triangle of Voltages in a Circuit containing Resistance and Capacity in Series.

The triangle of voltage (Fig. 128), taken from the phase diagram in Fig. 127, is similar to the triangle for a self-inductive circuit illustrated in Fig. 119, except that the current in the circuit and IR drop lead the impressed voltage instead of lagging behind.

It must be carefully borne in mind that in self-inductive and capacity circuits the self-inductive voltage and the capacity voltage act under the same laws as any other electric voltages. *The impressed voltage*, therefore, *must comprise two rectangular*

components, one equal and opposite to the reactive, and the other equal to the active voltage (*i.e.* the IR drop). The triangles of Figs. 119 and 128 indicate this. The vertical sides E_x of the triangles are vector components of the hypothenuses which represent the voltages impressed on the circuits, and are respectively *equal and opposite to the self-inductive or capacity* voltages in the circuit, while the active voltages (IR drop) represented by the horizontal sides of the triangles complete the respective right-angled vector triangles.

The vector representing the impressed voltage in a circuit containing resistance and capacity may be indicated by the complex quantity

$$\overline{E} = E_r - jE_x,$$

$$\overline{E} = E\,(\cos\theta - j\sin\theta).$$

E is the scalar value of $\overline{E}$. The scalar value of $\overline{E}$ and the angle θ are determined in the same manner as explained in Art. 40, as is also the sum of several voltages in series.

Since $$\theta = \tan^{-1}\frac{-E_x}{E_r},$$

the angle of lag θ is negative, which follows from the fact that the current leads the impressed voltage.

The expression

$$\overline{E} = E\,(\cos\theta - j\sin\theta)$$

obviously may be written

$$\overline{E} = E\,(\cos(-\theta) + j\sin(-\theta)).$$

55. The Effect of Introducing Resistance in a Continuous Current Circuit; Opening the Circuit. — It has already been pointed out in Art. 50 that the voltage must rise at the breaking of a direct current circuit possessing self-inductance. Breaking such a circuit has the effect of very rapidly introducing resistance until the resistance becomes infinite, with a correspondingly rapid forced decrease of the current. In case of a closed circuit of resistance R and self-inductance L upon which a steady voltage E is impressed, the equation representing the conditions, while the circuit is in process of being broken, is

$$E = L\frac{di}{dt} + [R + f(t)]\,i,$$

in which $f(t)$ is a function of time measured from the instant of starting the process and represents the rate of introducing extra resistance to open the circuit.

This may be transformed into

$$\frac{di}{dt}+\frac{[R+f(t)]\,i}{L}=\frac{E}{L}. \tag{1}$$

The rate at which extra resistance is to be introduced at breaking the circuit—namely, the form of $f(t)$—must be known before this equation can be solved. Assuming that the resistance is added as a simple function of time, then $f(t)=at$, in which a is a constant, and the equation becomes

$$\frac{di}{dt}+\frac{(R+at)\,i}{L}=\frac{E}{L}. \tag{2}$$

Putting $R+at=u\sqrt{-2aL}$ and $\frac{di}{dt}=\frac{du}{dt}\cdot\frac{di}{du}=\frac{\sqrt{a}}{\sqrt{-2L}}\,\frac{di}{du}$,

this becomes

$$\frac{di}{du}-2ui=-\frac{2E}{\sqrt{-2aL}}, \tag{3}$$

and the solution is of the form*

$$ie^{-\int 2u\,du}=-\frac{2E}{\sqrt{-2aL}}\int e^{-\int 2u\,du}\,du+K.$$

Hence, $i=-\frac{2E}{\sqrt{-2aL}}e^{u^2}\int_0^u e^{-u^2}\,du=-\frac{2E}{\sqrt{-2aL}}$

$$e^{u^2}\left\{\int_0^\infty e^{-u^2}\,du-\int_u^\infty e^{-u^2}\,du\right\}. \tag{4}$$

When this is integrated by parts and approximated by neglecting evanescent terms and terms containing the reciprocal of $(u^2)^n$, the equation reduces to

$$i=\frac{E}{R+at}. \tag{5}$$

The induced voltage caused by the process of opening the circuit is

$$e_l=-L\frac{di}{dt}. \tag{6}$$

From equation (5) is obtained the value,

$$\frac{di}{dt}=-\frac{aE}{(R+at)^2}. \tag{7}$$

Whence,

$$e_l=\frac{aLE}{(R+at)^2}=\frac{aLIR}{(R+at)^2} \tag{8}$$

* Murray's *Differential Equations*, p. 26.

in which $I = \frac{E}{R}$, that is, it is the current flowing in the circuit before the process of breaking commenced.

When the process begins, $t = 0$, and at that instant,

$$e_l = \frac{a}{R} LI = \frac{LI}{\frac{R}{a}}, \tag{9}$$

in which LI represents the number of linkages of lines of magnetic force around conductors of the circuit before the process of breaking the circuit was begun, and $\frac{a}{R}$ represents the ratio of the rate of introducing extra resistance into the circuit compared with the initial or steady resistance of the circuit. If the circuit is broken instantly, that is, $a = \infty$, the induced voltage obviously must be infinite, as the formula shows; but this fortunately cannot occur in practice, because the arc which occurs at a break prevents the resistance thereat from going instantly to infinity.

If the line voltage is not removed from the circuit upon the incipiency of the break in the circuit (as in the case of opening a switch), it must be added to the self-induced voltage e_ϵ, and the resultant becomes

$$e = e_l + E.$$

The induced voltage, e_l, is numerically equal at time $t = 0$ to as many volts as LI is a multiple of $\frac{R}{a}$. The formula (8) shows that e_l falls off rapidly with the time, especially when a is large.

If the rate of change of resistance increases with t instead of being constant, the maximum value of e_l comes when t has some finite value instead of when $t = 0$.

If a current of ten amperes is flowing through a circuit of 10 ohms resistance and .2 henry self-inductance, and the opening of a switch gives an effect at the initial instant of introducing resistance into the circuit at the rate of 10,000 ohms per second, the induced voltage at that instant is approximately 2000 volts, which is the maximum value if the rate of change of resistance does not increase. If the drawing out of the arc following the interruption of metallic contact causes the apparent rate of introduction of resistance to increase, the induced voltage may

continue to rise to a higher value, after which it drops to zero on the breaking of the arc.

By applying the same reasoning to the formula when resistance and capacity are in circuit together, it may be observed that $e_c = -E$ when $t = 0$ and there is no rise of voltage, which is to be expected from the physical characteristics of capacity. The stored energy remains stored in the condenser. Such storage unaccompanied by flow of current is manifestly impossible in the case of electromagnetic energy except through some exhibition of the phenomenon of coercive force in an iron core or the like.

56. Rate of Expenditure of Work in Circuit Containing Self-inductance and Capacity; Analogies. — The effect of self-inductance has been compared with the effect of inertia in a moving physical mass. The inertia effects of water flowing in a pipe, as suggested by Faraday, afford instructive analogies. Thus, on impressing voltage upon a circuit containing resistance and self-inductance, the current does not rise to its full value instantly, but increases as a logarithmic function, the constant of which depends upon the relation of the self-inductance to the resistance of the circuit. In the same way, if pressure is exerted upon water which fills a pipe, the water cannot begin its full flow instantly, on account of inertia. If a gate is suddenly closed in the pipe after the flow is fully under way, the momentum of the liquid tends to continue the flow, and the gate suffers a severe blow. In the same way, upon opening an electric circuit a bright spark passes on account of the so-called extra current caused by the tendency of self-inductance to uphold the flow. It must always be remembered that the analogies between the flow of electric current and moving solids or liquids are by no means exact. For instance, there is a marked difference between the effect of bends on the inertia effect in the pipe containing water and in the electric circuit. Thus, in the electric circuit, a solenoid has much more self-inductance than has the same wire straightened out. On the other hand, bends in a water pipe cause the inertia effect to be absorbed by friction. Notwithstanding the differences, the analogies are quite useful in fixing the meaning of the phenomena and worthy of further consideration.

When water in a pipe is set in motion, a part of the force

exerted upon it at any moment is utilized in overcoming the reaction, $\frac{Mdv}{dt}$, caused by the inertia of the accelerating mass, and the remainder in overcoming frictional resistances (Av). That is,

$$F = Av + \frac{Mdv}{dt},$$

where F is the pressure exerted, v the instantaneous rate of flow (velocity) of the water, M is its mass, and A is a constant. It is here assumed that the frictional resistance is proportional to the velocity, which is true only when v is small. When the velocity of the water has become so great that $Av_1 = F$, where v_1 is the final velocity, the acceleration ceases, and the water continues to flow at a uniform velocity v_1 as long as the force is applied.

In the case of the electric circuit, the impressed voltage is expended in overcoming the counter-voltage due to self-inductance $\frac{Ldi}{dt}$, and in causing the current to flow through the resistance R of the circuit, or

$$E = iR + \frac{Ldi}{dt}.$$

This is similar to the expression for the flow of a liquid as given above. The voltage exerted in overcoming the electric resistance or electric friction of the conductor is represented by iR, and $\frac{Ldi}{dt}\left(=\frac{d(Li)}{dt}\right)$ represents the voltage exerted in storing energy in the magnetic field; that is, in changing the electro-magnetic momentum of the magnetic field.* When the current reaches such a value that $iR = E$, the last term disappears and the current becomes constant. The expression $\frac{Mdv}{dt}\left(=\frac{d(Mv)}{dt}\right)$ in the formula relating to the flow of water represents, of course, the pressure or force exerted in storing energy in the water by increasing its momentum.

The power expended in the electric circuit at any instant is

$$Ei = i^2R + \frac{Lidi}{dt},$$

* Art. 47.

in which i^2R is the power expended in heating the conductor and $Li\frac{di}{dt}$ is the power expended in storing energy in the magnetic field. In this discussion, it is assumed that the electrical resistance of a conductor is a constant, and is the same for a variable current as for a constant one. This is correct within practical limits, provided the rate of variation of the current is not too great and the conductor is not too thick.

The capacity effect in a circuit has analogies with the effect of the physical capacity of a system of water pipes upon the flow of water; or, to make the comparison more simple, suppose that the pipes are so small as to have an inappreciable capacity but lead to the bottom of a large tank. The force acting upon the water must then accomplish two purposes; first, it must force the water through the resistance of the connecting pipes, and second, it must raise the level or potential of the water to the tank level. A formula may be made to express this as follows,

$$F = Av + H = Av + \frac{G}{Y},$$

where H is the head of the water in the tank, G is the volume of the water in the tank, and Y is the capacity of the tank in units of volume per unit of height or head. In case of an electric circuit the analogous equation is,

$$E = Ri + \frac{q}{C}.^*$$

The power which is used can immediately be obtained by multiplying the hydraulic equation by the rate of flow of the water and the electric equation by the current. The equation of power in the electric circuit is then

$$Ei = i^2R + \frac{iq}{C},$$

but as

$$i = \frac{dq}{dt},$$

$$iE = Ri^2 + \frac{qdq}{Cdt}.$$

In case there are both capacity and self-inductance in series in the circuit, the two equations must be combined and

$$E = Ri + \frac{Ldi}{dt} + \frac{q}{C},$$

* Art. 52 c.

and the power used is $iE = Ri^2 + \frac{Lidi}{dt} + \frac{qdq}{Cdt}$.

It must be remembered in considering this equation that energy is absorbed by the circuit on account of the second term on the right only while the current is rising; when the current falls the magnetic field gives up energy which is returned to the source. Likewise, energy is absorbed by the circuit on account of the third term of the right-hand member of the equation only while the voltage impressed on the capacity is rising and the charge in the condenser is therefore increasing; when the charge falls the condenser gives up energy which is returned to the source. These reactions are analogous to the effects in water pipes where the inertia absorbs energy while the velocity is rising, to return it as the velocity falls; while the tank stores energy while being charged and returns it on discharge. The resistance and iron and dielectric losses are analogous to the frictional resistances in the pipes and tank.

57. Effect, on the Transient State in a Circuit, of Self-inductance and Capacity Combined. — When an inductive coil of constant inductance L is included in a steady-current circuit of resistance R, and a condenser of capacity C is shunted across a portion of the circuit of resistance r, as in Fig. 129, the following conditions obtain:

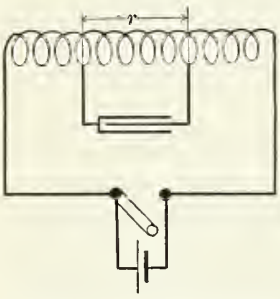


FIG. 129. — Circuit containing Resistance, Self-inductance, and Capacity.

The condenser is charged with a quantity of electricity, $Q = CIr$, where I is the steady value of the current. Now if the impressed voltage is suddenly removed by throwing the switch shown in the figure to the horizontal position, the condenser will discharge and the quantity of electricity which will pass from the condenser through the part of the circuit beyond its terminals is

$$q_c = Q\frac{r}{R} = \frac{CIr^2}{R}.$$

At the same time the self-inductance will cause a quantity of electricity to be transferred through the circuit in the opposite direction, which is equal to

$$q_l = \frac{LI}{R}.$$

Hence, the total quantity of electricity transferred through the circuit is

$$q_l - q_c = \frac{I}{R}(L - Cr^2),$$

and the effect of the condenser is to apparently reduce the self-inductance by an amount equal to the capacity of the condenser multiplied by the square of the resistance around which it is shunted. With iron in the circuit, L varies, but the general relations remain the same.

The foregoing relates to the transient state upon removing the impressed voltage which sets up the steady current I in the circuit, but it is obvious that similar conditions are produced during the transient state accompanying the establishment of the current upon introducing the impressed voltage. If Cr^2 is equal to L, it is obvious that the charging current neutralizes the extra current of self-induction and the total circuit acts like a circuit containing resistance R but lacking self-inductance or capacity. The neutralization of the effect of self-inductance cannot ordinarily be complete on account of hysteresis in the iron core of the coil (if an iron core is used) and the dielectric hysteresis of the condenser, which cause the curves of discharge from the magnetic field and condenser to vary from the logarithmic form.

58. Conditions of Establishment and Termination of Current in a Circuit containing Resistance, Inductance, and Capacity in Series. — *a. Current and Charge under Constant Impressed Voltage.* — The equation of voltage when current is established by introducing constant voltage E into the circuit is

$$\begin{aligned} E &= Ri + \frac{Ldi}{dt} + \frac{q}{C} \\ &= Ri + L\frac{di}{dt} + \frac{1}{C}\int idt \\ &= R\frac{dq}{dt} + L\frac{d^2q}{dt^2} + \frac{q}{C} \end{aligned} \qquad (1)$$

From (1) we obtain

$$0 = \frac{d^3q}{dt^3} + \frac{R}{L}\frac{d^2q}{dt^2} + \frac{1}{LC}\frac{dq}{dt}. \qquad (2)$$

Putting $q = \epsilon^{xt}$,

the equation takes the characteristic form

$$x^3 + \frac{Rx^2}{L} + \frac{x}{LC} = 0, \qquad (3)$$

the roots of which are

$$x_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

$$x_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

$$x_3 = 0.$$

The solution of such an equation as (2) is *

$$q = a_1\epsilon^{x_1 t} + a_2\epsilon^{x_2 t} + a_3, \tag{4}$$

in which x_1 and x_2 are the aforesaid roots of equation (3) and a_1, a_2, and a_3 are constants introduced by integration. Also, since

$$i = \frac{dq}{dt},$$

$$i = a_1 x_1 \epsilon^{x_1 t} + a_2 x_2 \epsilon^{x_2 t}. \tag{5}$$

These equations (4) and (5) for charge and current assume three forms according to the values of the constants of the circuit, namely: (1) when $\frac{R^2}{4L^2} > \frac{1}{LC}$; (2) when $\frac{R^2}{4L^2} < \frac{1}{LC}$; (3) when $\frac{R^2}{4L^2} = \frac{1}{LC}$, under which conditions the roots x_1 and x_2 are respectively real, imaginary, and equal.

Case (1). *When* $\frac{R^2}{4L^2} > \frac{1}{LC}$; *Non-oscillatory Effect.* — Under these conditions the values of x_1 and x_2 in equations (4) and (5) are real. The values of the constants a_1, a_2, and a_3 may be determined by solving when t is given the special values of 0 and ∞.

For the conditions here named, $q = 0$ and $i = 0$ when $t = 0$; $q = EC = Q$ and $i = 0$ when $t = \infty$. Substituting these two sets of values successively in (4) and (5) gives the following four equations:

$$a_1 + a_2 + a_3 = 0,$$

$$a_1 x_1 + a_2 x_2 = 0,$$

$$Q - a_3 = a_1\epsilon^{x_1 \infty} + a_2\epsilon^{x_2 \infty},$$

$$a_1 x_1 \epsilon^{x_1 \infty} + a_2 x_2 \epsilon^{x_2 \infty} = 0,$$

from which,

$$a_1 = \frac{Qx_2}{x_1 - x_2},$$

$$a_2 = \frac{-Qx_1}{x_1 - x_2},$$

$$a_3 = Q.$$

* Murray's *Differential Equations*, p. 64.

Substituting these values of the constants in (4) and (5), we obtain

$$q = \frac{Qx_2\epsilon^{x_1 t}}{x_1 - x_2} - \frac{Qx_1\epsilon^{x_2 t}}{x_1 - x_2} + Q$$

$$= Q - \frac{Q}{2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}\left[\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)\epsilon^{-\left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}\right.$$

$$\left. - \left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)\epsilon^{-\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}\right], \quad (6)$$

and $$i = \frac{Qx_1x_2\epsilon^{x_1 t}}{x_1 - x_2} - \frac{Qx_1x_2\epsilon^{x_2 t}}{x_1 - x_2}$$

or, since $Q = EC$,

$$i = \frac{E}{2\sqrt{\frac{R^2}{4} - \frac{L}{C}}}\left[\epsilon^{-\left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} - \epsilon^{-\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}\right]. \quad (7)$$

These equations (6) and (7) give the values of the charge and current in a series circuit containing R, L, and C, when $\frac{R^2}{4L^2} > \frac{1}{LC}$, at any time t after the introduction of a constant voltage E in the circuit, Q being the final charge corresponding to the voltage E.

Case (2). *When* $\frac{R^2}{4L^2} < \frac{1}{LC}$; *Oscillatory Effect.*—Under these conditions the roots x_1 and x_2 are imaginary. Hence, using the operator j to indicate the imaginary unit $\sqrt{-1}$, we have

$$x_1 = -\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}},$$

$$x_2 = -\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

Substituting $x_1 = a + jb$ and $x_2 = a - jb$,

where $a = -\frac{R}{2L}$ and $b = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$ and are real quantities,

equations (4) and (5) give *

$$\begin{aligned} q &= a_1\epsilon^{r_1 t} + a_2\epsilon^{r_2 t} + a_3 \\ &= a_1\epsilon^{(a+jb)t} + a_2\epsilon^{(a-jb)t} + a_3 \\ &= \epsilon^{at}(A' \cos bt + B' \sin bt) + a_3 \\ &= A'\epsilon^{at} \cos bt + B'\epsilon^{at} \sin bt + a_3, \end{aligned}$$

in which $A' = a_1 + a_2$ and $B' = j(a_1 - a_2)$.

Also $$i = (A'a + B'b)\epsilon^{at} \cos bt + (B'a - A'b)\epsilon^{at} \sin bt.$$

If p is an angle of which $\tan p = \frac{m}{n}$, it follows that

$$\sin p = \frac{m}{\sqrt{m^2 + n^2}}, \quad \cos p = \frac{n}{\sqrt{m^2 + n^2}},$$

and $$m \cos \alpha + n \sin \alpha = \sqrt{m^2 + n^2} \sin (\alpha + p).$$

Therefore,

$$q = \epsilon^{at}\sqrt{A'^2 + B'^2} \sin (bt + \theta) + a_3, \tag{8}$$

$$i = \epsilon^{at}\sqrt{(A'a + B'b)^2 + (B'a - A'b)^2} \sin (bt + \theta'), \tag{9}$$

in which $\theta = \tan^{-1}\frac{A'}{B'}$, and $\theta' = \tan^{-1}\frac{(A'a + B'b)}{(B'a - A'b)}$.

The values of the constants of the equations may be determined as in the previous case by solving when t is given the limiting values of 0 and ∞. As before, $q = 0$ and $i = 0$ when $t = 0$; $q = Q$ and $i = 0$ when $t = \infty$. Whence,

$$\begin{aligned} A' &= -Q, \\ B' &= -A'\frac{a}{b} = \frac{-QR}{2L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}, \\ a_3 &= Q. \end{aligned}$$

Substituting these values in (8) and (9), we obtain

$$q = Q - Q\epsilon^{-\frac{Rt}{2L}} \frac{\sqrt{LC}}{LC\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \times \sin\left[\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cdot t + \theta\right] \tag{10}$$

* See Art. 71 and Murray's *Differential Equations*, p. 67.

and $$i = \epsilon^{-\frac{Rt}{2L}} \frac{E}{L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cdot t \qquad (11)$$

in which $\theta = \tan^{-1} \dfrac{2L\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}}{R}$.

These equations (10) and (11) give the values of the charge and current in a series circuit containing R, L, and C, when $\dfrac{R^2}{4L^2} < \dfrac{1}{LC}$, at any time t after the introduction of the constant voltage E in the circuit. It will be noted that the expressions for charge and current each contain the product of a logarithmic term and a sine term, and their values are recurrent functions of time of an oscillatory nature. The value of the charge at any instant is equal to the difference between a constant term, $Q = CE$, and an oscillatory term which is in the lead of the current by an angle θ of which $\tan\theta = \dfrac{2L\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}}{R}$. The current consists of a sinusoid modified by a vanishing logarithmic curve, and its oscillations are of gradually vanishing amplitude.

Case (3). *When* $\dfrac{R^2}{4L^2} = \dfrac{1}{LC}$. — Under this critical condition, the roots x_1 and x_2 of the equation (3) become equal, or

$$x_1 = x_2 = x = -\frac{R}{2L} = -\frac{1}{\sqrt{LC}},$$

and the solution becomes:

$$q = a_1\epsilon^{xt} + a_2 t\epsilon^{xt} + a_3, \qquad (12)$$

$$i = a_1 x\epsilon^{xt} + a_2 xt\epsilon^{xt} + a_2\epsilon^{xt}.* \qquad (13)$$

The values of the constants may be determined by solving when t is given the special values of 0 and ∞. As before, $q = 0$ and $i = 0$ when $t = 0$; and $q = Q$ and $i = 0$ when $t = \infty$. Then, $a_1 = -Q$,

* See Murray's *Differential Equations*, p. 65.

$$a_2 = -\frac{R}{2L}Q = -\frac{Q}{\sqrt{LC}},$$

$$a_3 = Q.$$

Substituting these values in equations (12) and (13) we obtain,

$$q = Q - Q\left(1 + \frac{Rt}{2L}\right)\epsilon^{-\frac{Rt}{2L}}, \tag{14}$$

$$i = Q\frac{R^2}{4L^2}t\epsilon^{-\frac{Rt}{2L}} = \frac{E}{L}t\epsilon^{-\frac{Rt}{2L}}. \tag{15}$$

These equations (14) and (15) give the values of the charge and current in a series circuit containing R, L, and C, when $\frac{R^2}{4L^2} = \frac{1}{LC}$, at any time t after the introduction of a constant voltage E in the circuit. This condition affords the most rapid rate of charging for the circuit. A decrease of R would put the circuit in an oscillatory condition, and an increase of R would put the circuit in a condition represented by equations (6) and (7).

b. Current and Charge on Withdrawal of Impressed Voltage.— On removal of the impressed voltage from a circuit containing R, L, and C in series, the equation of the circuit is

$$E = 0 = Ri + \frac{Ldi}{dt} + \frac{q}{C}$$

$$= Ri + \frac{Ldi}{dt} + \frac{1}{C}\int idt$$

$$= R\frac{dq}{dt} + \frac{Ld^2q}{dt^2} + \frac{q}{C}. \tag{16}$$

Putting $q = \epsilon^{xt}$,

the equation takes the characteristic form

$$x^2 + \frac{R}{L}x + \frac{1}{LC} = 0, \tag{17}$$

the roots of which are

$$x_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

$$x_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}.$$

The solution takes three forms according as these roots are real, imaginary, or equal, or when

(1) $\frac{R^2}{4L^2} > \frac{1}{LC}$; (2) when $\frac{R^2}{4L^2} < \frac{1}{LC}$; (3) when $\frac{R^2}{4L^2} = \frac{1}{LC}$.

Case (1). *When* $\frac{R^2}{4L^2} > \frac{1}{LC}$; *Non-oscillatory Effect.* — Under these conditions the solution of equation (16) takes the form

$$q = a_1\epsilon^{x_1 t} + a_2\epsilon^{x_2 t}, \tag{18}$$

$$i = a_1 x_1\epsilon^{x_1 t} + a_2 x_2\epsilon^{x_2 t}. \tag{19}$$

To determine the constants we have, when $t = 0$, $q = Q$ and $i = 0$; and therefore

$$a_1 = -\frac{Qx_2}{x_1 - x_2}, \quad a_2 = \frac{Qx_1}{x_1 - x_2}.$$

Substituting these values in (18) and (19), we obtain

$$q = Q\left[\frac{R}{4L\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}} + \frac{1}{2}\right]\epsilon^{-\left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}$$

$$- Q\left[\frac{R}{4L\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}} - \frac{1}{2}\right]\epsilon^{-\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}, \tag{20}$$

and

$$i = \frac{Q}{2LC\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}\left[\epsilon^{-\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} - \epsilon^{-\left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t}\right]. \tag{21}$$

These equations (20) and (21) give the values of the charge and current in a series circuit containing R, L, and C, when $\frac{R^2}{4L^2} > \frac{1}{LC}$, any time t after the withdrawal of the impressed voltage E.

Case (2). *When* $\frac{R^2}{4L^2} < \frac{1}{LC}$; *Oscillatory Effect.* — Under these conditions the roots x_1 and x_2 of equation (17) are imaginary and proceeding as in case (*a*. 2), p. 189, the equations for charge and current become

$$q = \epsilon^{at}\sqrt{(A')^2 + (B')^2}\sin(bt + \theta), \tag{22}$$

o

$$i = \epsilon^{at}\sqrt{(A'a+B'b)^2+(B'a-A'b)^2}\,\sin\,(bt+\theta'), \qquad (23)$$

where $$a = -\frac{R}{2L},\ b = \sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}},$$

$$\tan\,\theta = \frac{A'}{B'} \text{ and } \tan\,\theta' = \frac{A'a+B'b}{B'a-A'b}.$$

To determine the values of the constants in (22) and (23) we have, when $t = 0$, $q = Q$ and $i = 0$.

Then, $$A' = Q \text{ and } B' = -\frac{a}{b}A' = -\frac{a}{b}Q.$$

Substituting these values in (22) and (23), we obtain,

$$q = Q\epsilon^{-\frac{Rt}{2L}}\frac{\sqrt{LC}}{LC\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\times$$

$$\sin\left[\sqrt{\frac{1}{LC}-\frac{R}{4L^2}}\cdot t+\tan^{-1}\frac{2L\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}{R}\right], \qquad (24)$$

and $$i = -\epsilon^{-\frac{Rt}{2L}}\frac{Q}{LC\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin\left(t\cdot\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right). \qquad (25)$$

These equations (24) and (25) give the values of the charge and current in a series circuit containing R, L, and C, when $\frac{R^2}{4L^2} < \frac{1}{LC}$, at any time t after the withdrawal of the impressed voltage E. It will be noted that the discharging process is of an oscillatory nature, the expressions for q and i being each made up of the product of a logarithmic term and a sine term, and that the charge leads the current by an angle of which

$$\tan\,\theta = \frac{2L\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}{R}$$

The period of each oscillation is

$$\frac{2\pi}{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}},$$

which is the natural oscillatory period of the circuit. If R is small compared with L, this is approximately equal to $2\pi\sqrt{LC}$.

Case (3). *When* $\frac{R^2}{4L^2} = \frac{1}{LC}$. — Under these conditions the roots of equation (17) become equal, or

$$x_1 = x_2 = x = -\frac{R}{2L} = \frac{-1}{\sqrt{LC}},$$

and the solution of equation (16) becomes

$$q = a_1\epsilon^{xt} + a_2 t\epsilon^{xt}, \tag{26}$$

and

$$i = a_1 x\epsilon^{xt} + a_2 xt\epsilon^{xt} + a_2\epsilon^{xt}. \tag{27}$$

To determine the values of the constants, we have $q = Q$ and $i = 0$ when $t = 0$.

Hence, $$a_1 = Q \text{ and } a_2 = -Qx = \frac{QR}{2L}.$$

Substituting these values in (26) and (27), we obtain

$$q = Q\left(1 + \frac{R}{2L}t\right)\epsilon^{-\frac{Rt}{2L}}, \tag{28}$$

$$i = -\frac{E}{L}t\epsilon^{-\frac{Rt}{2L}}. \tag{29}$$

These equations (28) and (29) give the values of the charge and current in a series circuit containing R, L, and C, when $\frac{R^2}{4L^2} = \frac{1}{LC}$ at any time t after the withdrawal of the impressed voltage E from the circuit. This relation of R, L, and C affords the most rapid discharge of the circuit. A decrease of R would throw the discharging process into the oscillatory state, and an increase of R would put the circuit in a condition represented by equations (20) and (21).

c. Current and Charge when the Impressed Voltage is a Sine Function of the Time. — In this case the instantaneous voltage conditions are represented by

$$e = e_m \sin \omega t = iR + \frac{Ldi}{dt} + \frac{q}{C}$$

$$= iR + \frac{Ldi}{dt} + \frac{1}{C}\int idt = R\frac{dq}{dt} + \frac{Ld^2q}{dt^2} + \frac{q}{C},$$

or

$$\frac{e_m}{L}\sin \omega t = \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} \tag{30}$$

In this case of a sine function, ω is equal to $\frac{2\pi}{T} = 2\pi f$, the angular velocity of a rotating vector generating the sinusoid.

The above is a linear differential equation of the second order which is to be solved to find the values of q and i in terms of e_m, R, L, C, f, and t. The solution of such an equation consists of the sum of two parts, the complementary function and the particular integral. The complementary function is the integral obtained by equating the second member to zero. In other words, it is the solution of case (*b*) *ante*, *i.e.* when $e = 0$, and takes the form *

$$q = a_1\epsilon^{r_1 t} + a_2\epsilon^{r_2 t},$$
$$i = a_1 x_1\epsilon^{r_1 t} + a_2 x_2\epsilon^{r_2 t}.$$

The particular integral contains no arbitrary constants and is equal to †

$$q = \frac{1}{D^2 + \frac{R}{L}D + \frac{1}{LC}} \cdots \frac{e_m}{L}\sin\omega t = \frac{1}{D - a}\frac{e_m}{R}\sin\omega t$$

$$= \frac{e_m}{R}\epsilon^{at}\int \epsilon^{-at}\sin\omega t, = \frac{-e_m}{\omega\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cos(\omega t - \theta),$$

where D is an operator $\left(\frac{d[\]}{dt}\right)$, $D^2 = -\omega^2$,

$$a = \frac{\omega^2 L}{R} - \frac{1}{CR},$$

and
$$\tan\theta = \frac{\omega^2 L - \frac{1}{C}}{\omega R} = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

The complete solution is therefore

$$q = \frac{-e_m}{\omega\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cos(\omega t - \theta) + a_1\epsilon^{r_1 t} + a_2\epsilon^{r_2 t}. \quad (31)$$

In a similar way the complete solution for the current becomes

$$i = \frac{e_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\sin(\omega t - \theta) + a_1 x_1\epsilon^{r_1 t} + a_2 x_2\epsilon^{r_2 t}. \quad (32)$$

* Murray's *Differential Equations*, Chap. VI.

† Murray's *Differential Equations*, Arts. 58 and 62.

These equations give the values of the current and charge at any time t after impressing the voltage in a series circuit containing constant R, L, and C, when such voltage is sinusoidal.

The above equation for current may be written

$$i = \frac{e_m}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}} \sin(\alpha - \theta) + a_1 x_1 \epsilon^{r_1 t} + a_2 x_2 \epsilon^{r_2 t}. \quad (33)$$

The angle θ is the angular difference of phase between the current in the circuit and the impressed voltage; that is, it is the angle of lag (or lead) of the current for frequency f.

Neglecting the exponential terms, which are rapidly evanescent where x_1 and x_2 are real, the equation for instantaneous current under a sinusoidal voltage is

$$i = \frac{e_m}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}} \sin(\alpha - \theta),$$

where e_m is the maximum voltage and the angle θ is either positive or negative according as capacity or self-inductance predominates. The current is evidently a maximum when $\sin(\alpha - \theta) = 1$, and therefore

$$i_m = \frac{e_m}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}}.$$

The relation of effective current and voltage can be obtained by dividing both sides by $\sqrt{2}$, and then

$$I = \frac{E}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}}.$$

The term $\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}$ is called the **Impedance** of the circuit and is composed of the square root of the sum of the squares of two quantities, the first of which, R, is the resistance of the circuit and the other $\left(2\pi fL - \frac{1}{2\pi fC}\right)$ is called the **Reactance** of the circuit.

If L is zero and C is infinite, the equation (33) reduces to the value

$$i = \frac{e_m}{R} \sin \alpha,$$

as already derived for the current flow when a sinusoidal voltage is impressed on a circuit containing resistance alone.

If C is infinite (which is equivalent to saying that no capacity voltage arises in the series circuit), but R and L have finite values, the equation reduces to that on page 148, or

$$i = \frac{e_m}{\sqrt{R^2 + (2\pi f L)^2}} \sin (\alpha - \theta') + A_1 \epsilon^{-\frac{Rt}{L}}.$$

The value of A_1 may be derived by giving α and t the values α_1 and t_1, corresponding to the instant of introducing the voltage in the circuit, at which instant $i = 0$ and the second term of the right-hand member must be equal to but opposite the first term;

whence $$A_1 = - \frac{e_m}{\sqrt{R^2 + (2\pi f L)^2}} \epsilon^{\frac{Rt_1}{L}} \sin (\alpha_1 - \theta'),$$

which is constant, because t_1 and α_1 and θ' have particular values,

and $$A_1 \epsilon^{-\frac{Rt}{L}} = - \frac{e_m}{\sqrt{R^2 + (2\pi f L)^2}} \epsilon^{-\frac{R(t-t_1)}{L}} \sin (\alpha_1 - \theta').$$

The value of t is measured from the instant when the sinusoidal voltage passes through the preceding zero from negative toward positive values; and t_1 is the value of t at the instant of switching the voltage into the circuit.

If L is zero, but R and C have finite values, the equation reduces to that on page 174, or

$$i = \frac{e_m}{\sqrt{R^2 + \left(\frac{1}{2\pi f C}\right)^2}} \sin (\alpha + \theta'') + B_1 \epsilon^{-\frac{t}{RC}}.$$

The value of B_1, derived from giving α and t the values α_1 and t_1, corresponding to the instant of introducing the voltage in the circuit, is

$$B_1 = - \frac{e_m}{\sqrt{R^2 + \left(\frac{1}{2\pi f C}\right)^2}} \epsilon^{\frac{t_1}{RC}} \sin (\alpha_1 + \theta'').$$

and $$B_1 \epsilon^{-\frac{t}{RC}} = - \frac{e_m}{\sqrt{R^2 + \left(\frac{1}{2\pi f C}\right)^2}} \epsilon^{-\frac{t-t_1}{RC}} \sin (\alpha_1 + \theta'').$$

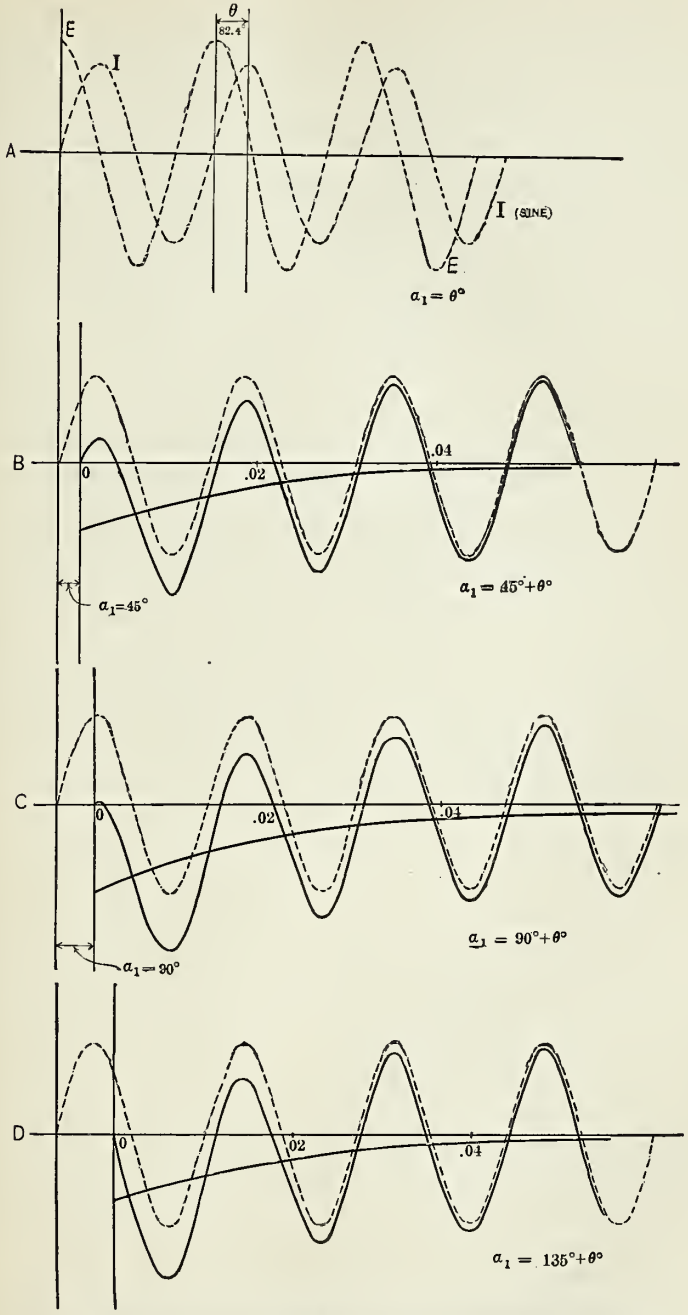


FIG. 130. —Effect of switching-in on Circuit containing R and L.

59. Effect of Exponential Terms. — The exponential members of the equations of case (*c*) show the transient effect on the form of the current wave which occurs upon first impressing sine voltage on a circuit containing resistance and inductance, resistance and capacity, or resistance, self-inductance, and capacity, in series. In circuits with resistance and inductance or resistance and capacity the exponential term quickly reduces to zero and can be then neglected. The actual influence of the exponential term in either of these two cases is to distort the current wave for the earlier periods, and the extent of the effect depends on the value of $(\alpha_1 - \theta)$ at the instant of introducing the voltage into the circuit. The value of α_1 expresses the advance of the voltage at the instant of its introduction in the circuit, and the value of $(\alpha_1 - \theta)$ expresses the advance of the corresponding normal current. Consequently, when $\alpha_1 = \theta$, the exponential value is zero and the current starts from zero, following a regular sine wave; and when $\alpha_1 = 90° + \theta$, the exponential obtains its largest value and the current wave is given the maximum of distortion. Figures 130 and 131 show the form of current wave for the first few cycles where a sinusoidal voltage of 100 volts is switched onto a circuit containing 5 ohms and .1 henry and on a circuit containing 100 ohms and 100 microfarads, the frequency being 60 periods per second in each instance. The curves are shown for $\alpha_1 = \theta$, $\alpha_1 = 45° + \theta$, $\alpha_1 = 90° + \theta$, $\alpha_1 = 135° + \theta$.

In Fig. 130 *a* the sinusoid marked *E* shows the position of the voltage; the sinusoid marked *I* shows the current. The current grows up in the circuit without deviation from the true sine form, since the switch is closed at the instant when $\alpha = \theta$; namely, at the instant when the sine current would normally pass through zero, and the exponential term of the equation is therefore zero. In Figs. 130 *b*, *c*, and *d*, the voltage curve has been omitted. The dotted curve represents the normal sine form of the current, corresponding to the curve *I* of Fig. 130 *a*, and the full line curves represent respectively the exponential term (which is equal and opposite to the value of the normal sine curve at the instant of switching in) and the actual current flow. The latter is distorted for several periods from the normal sine form by the effect of the reactance in the circuit.

Figures 131 *a*, *b*, *c*, and *d* similarly represent the relations

when the sinusoidal voltage is switched on a circuit of fixed capacity and resistance.

The effects of switching a sinusoidal voltage onto a circuit of fixed resistance and self-inductance or a circuit of fixed resistance and capacity are thus made plain. In case the self-inductance varies much during the cycles, as in the case of a coil with an iron core that becomes highly saturated in each half cycle, or the capacity varies during the cycles, as in the case of certain liquid condensers, the exponential terms become very complex on account of the variation of L or C, and the magnitude of the first rush of current may become serious. The distortion of the form of the current wave during the first few cycles may then be much greater than is indicated in Figs. 130 and 131.

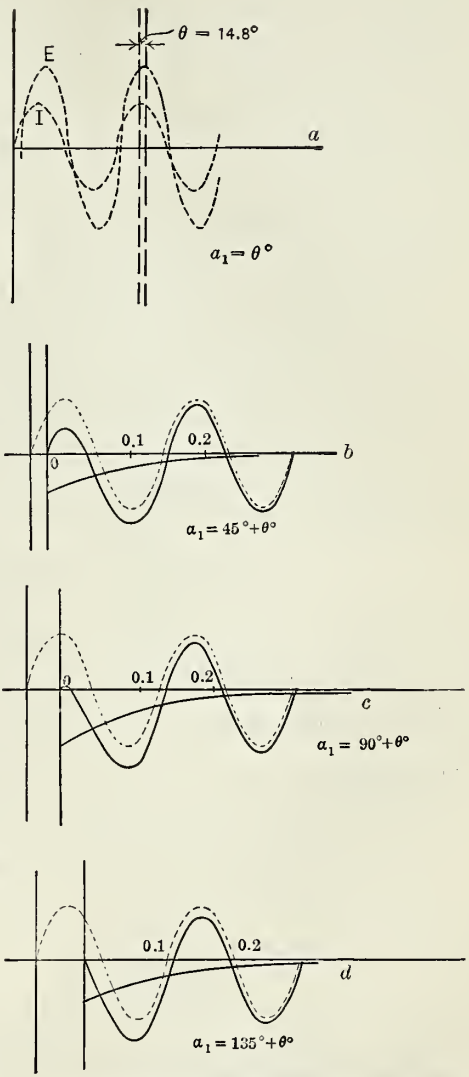


FIG. 131. — Effect of switching-in on Circuit containing R and C.

When the voltage is withdrawn from the circuit without changing the constants of the circuit, it is obvious that the first terms of the equations reduce to zero, and the current dies down along an exponential curve from its value at the instant of withdrawal of the voltage.

On the other hand, if the voltage is withdrawn in the usual manner of opening the circuit, the value of R is rapidly changed to infinity. An inspection of the two exponential terms shows that this may cause a great disturbance in an inductive circuit during the discharge of the magnetic field; but the current merely ceases in the circuit with capacity, leaving the condenser charged. In the latter case, a serious momentary current rush and distortion of the current wave may occur if the voltage, when switched on the circuit again, chances to be reversed in relation to the condenser voltage.

Turning again to the equations representing the conditions when resistance, self-inductance, and capacity are all three in series, (equations (31) and (32)), these apply to all instants of time, including the instant $t = t_1$ when the voltage is introduced in the circuit. If the capacity of the circuit is without a previous charge, the values of q and i are, of course, both zero at this instant; and, solving for a_1 and a_2 under these circumstances gives, when x_1 and x_2 are real,

$$a_1 = \frac{e_m}{Z} \cdot \frac{\epsilon^{-x_1 t_1}}{x_2 - x_1} \cdot \frac{\sin(\omega t_1 - \theta + p)}{\cos p},$$

$$a_2 = \frac{e_m}{Z} \cdot \frac{\epsilon^{-x_2 t_1}}{x_1 - x_2} \cdot \frac{\sin(\omega t_1 - \theta + p')}{\cos p'},$$

in which x_1 and x_2 have the values heretofore given them, Z is the impedance of the circuit $\left(= \sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}\right)$, t_1 is the value of t at the instant of introducing the voltage into the circuit, $\tan p = \frac{x_2}{\omega}$, and $\tan p' = \frac{x_1}{\omega}$. Therefore,

$$a_1 x_1 \epsilon^{x_1 t} + a_2 x_2 \epsilon^{x_2 t} = \frac{e_m}{2Z\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}$$

$$\left[\left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)\epsilon^{-\left(\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)(t - t_1)} \frac{\sin(\omega t_1 - \theta + p)}{\cos p}\right.$$

$$\left. - \left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)\epsilon^{-\left(\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)(t - t_1)} \frac{\sin(\omega t_1 - \theta + p')}{\cos p'}\right].$$

When the circuit constants are fixed, the right-hand member of the last equation reduces to zero rather rapidly, the precise

rate of decrease as time grows depending upon the relative values of R, L, and C; and the current will quickly come to a sinusoidal value with relatively little disturbance. However, if R, L, and C vary during the cycles of current, the disturbance may be serious on account of the instantaneous flow of an excessive current.

When $\frac{1}{LC} = \frac{R^2}{4L^2}$, equations (31) and (32) take the special forms,

$$q = -\frac{e_m}{\omega Z}\cos(\omega t - \theta) + (a_1 + a_2 t)\epsilon^{xt},$$

$$i = \frac{e_m}{Z}\sin(\omega t - \theta) + (a_1 x + a_2 x t + a_2)\epsilon^{xt}.$$

Solving for a_1 and a_2 in these equations when $q = 0$ and $i = 0$, which are the values of q and i at the instant $t = t_1$ when the sinusoidal voltage is introduced into the circuit, gives

$$a_1 = \frac{e_m}{Z}\sin\frac{(\omega t_1 - \theta + p)\epsilon^{-xt_1}}{\cos p},$$

$$a_2 = -\frac{e_m}{Z}\sin\frac{(\omega t_1 - \theta + p')}{\cos p'}\epsilon^{-xt_1},$$

in which $\tan p = \frac{xt_1 + 1}{\omega t_1}$ and $\tan p' = \frac{x}{\omega}$, where $x = -\frac{R}{2L}$.

Therefore $(a_1 x + a_2 x t + a_2)\epsilon^{xt} = (A + Bt)\epsilon^{-\frac{R(t-t_1)}{2L}}$.

When $\frac{R^2}{4L^2} < \frac{1}{LC}$ and the values of x_1 and x_2 are therefore imaginary, equations (31) and (32) take the forms

$$q = -\frac{e_m}{Z}\cos(\omega t - \theta) + M\epsilon^{at}\sin(bt + v),$$

$$i = \frac{e_m}{Z}\sin(\omega t - \theta) + N\epsilon^{at}\sin(bt + v').$$

Solving these for the values of M, N, v and v' when $t = t_1$, and therefore q and i are both equal to zero, results in expressions affording for i the equation

$$i = \frac{e_m}{Z}\sin(\omega t - \theta) + P\epsilon^{-\frac{R(t-t_1)}{2L}}\sin t \cdot \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}},$$

in which P is a real function of e_m, R, L, C, and t_1.

These equations show that the current will grow to its normal sinusoidal value without great disturbance after the voltage is switched into circuit, and will die away without great disturbance, if the values of R, L, and C give real roots; but if the roots are imaginary, oscillatory effects (instead of a plain exponential) of a frequency fixed by

$$f_1 = \frac{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}{2\pi}$$

are superimposed on the normal sinusoid at both switching in and switching off the voltage, and the disturbances may be quite great at either switching in or switching off the voltage if α_1 chances to have an appropriate value; especially if f_1 is not very different from f, as is not unusual. If these conditions are accompanied by variable values of R, L, or C, the disturbances may be still more magnified.

Prob. 1. A circuit has resistance of 20 ohms and self-inductance of .01 henry in series. What steady voltage impressed on the circuit will bring the current to a value of 20 amperes at the instant that it is increasing at the rate of 1000 amperes per second?

Prob. 2. How much power is being expended in the circuit of problem 1 at the instant in question? What proportion of this power is being converted into heat in the conductor resistance and what proportion is being stored in the magnetic field?

Prob. 3. A circuit of 10 ohms resistance has a capacity of 20 microfarads in series with the resistance. What steady voltage impressed on the circuit will bring the current to a value of 20 amperes at the instant when the charge equals $\frac{1}{2}$ coulomb?

Prob. 4. How much power is being expended in the circuit of problem 3 at the instant in question? What proportion of the power is being converted into heat in the conductor resistance and what proportion is being stored in the condenser? What value does the condenser charge finally reach, and how much energy is then stored in the charge?

Prob. 5. A circuit has a resistance of 10 ohms, a self-inductance of .02 of a henry, and a capacity of 100 microfarads, all in series relation. At a certain instant the current is 20 amperes and is increasing at the rate of 2000 amperes per second. What steady voltage is impressed on the circuit? The charge in the condenser at the instant is .001 of a coulomb.

Prob. 6. How much power is being expended in the circuit of problem 5 at the instant in question? What proportion of the power is being converted into heat in the conductor resistance, what proportion is being stored in the magnetic field, and what proportion is being stored in the condenser? What value does the condenser charge finally reach, how much energy is then stored in it, and how much energy is then stored in the magnetic field? What is the maximum value of the energy stored in the magnetic field and how long after the introduction of the impressed voltage in circuit is it reached?

Prob. 7. If the impressed voltage of problem 5 instantaneously falls to zero after a steady state of condenser charge has been reached without altering the electrical constants of the circuit, how long does it take for the condenser to fully discharge, and what is the maximum value of the discharge current?

Prob. 8. A circuit has 200 ohms resistance, .02 henry self-inductance and 5 microfarads capacity, in series. Draw the curves of rising and falling current when 100 volts are steadily impressed on the circuit until the charge rises to its full value and the impressed voltage is then withdrawn from the circuit.

Prob. 9. Draw the curves as in problem 8 when the resistance is 1 ohm, the self-inductance 10 henrys, and the capacity 10 microfarads.

Prob. 10. Draw the curves as in the preceding problems when the resistance is 1 ohm, self-inductance .2 of a henry, and the capacity 200 microfarads.

Prob. 11. Draw the curves as in the preceding problems when the resistance is .1 of an ohm, the self-inductance 5 henrys, and the capacity 500 microfarads.

Prob. 12. If the resistance, inductance, and capacity of Prob. 5 are individually segregated in the circuit, how much voltage

is impressed on the capacity, how much is impressed on the inductance, and how much is absorbed in the iR drop, at the instant considered?

60. The Time Constant of a Circuit. — The formula

$$i = \frac{E}{R}\left(1 - \epsilon^{\frac{-Rt}{L}}\right)$$

shows, as already stated, that the theoretical curve representing the rise or fall of the current in a circuit containing resistance and self-inductance, is logarithmic. The formula shows that when L is very small, the current almost immediately takes its full value $I = \frac{E}{R}$ on closing the circuit or falls to zero when the voltage is removed, since $\epsilon^{\frac{-Rt}{L}}$ quickly becomes negligible in comparison with unity. Theoretically, when inductance is present, the current can rise to its full value only after an infinite time, yet $\epsilon^{\frac{-Rt}{L}}$ becomes practically negligible after a comparatively short interval.

Since resistance has the absolute dimensions of a velocity (a length divided by a time) and inductance has the dimensions of a length, the ratio $\frac{L}{R}$ has the dimensions of a time; this ratio, in the case of any circuit, is therefore generally called the **Time constant** of the circuit, and may be represented by the Greek letter τ. In the preceding equation, $\frac{E}{R}$ represents the value which the current would instantly reach when under the constant impressed voltage, were there no inductance in the circuit. This is the same as the ultimate value when there is inductance. The equation may therefore be written

$$i = I(1 - \epsilon^{\frac{-t}{\tau}}).$$

When $t = \tau$, this becomes

$$i = I - \frac{I}{2.718},$$

or

$$i = .632\, I.$$

The current therefore reaches .632 of its ultimate or full value in a time equal to τ seconds. The value of the time con-

stant is therefore a *measure of the growth* of the current in a circuit after a fixed voltage is impressed, and it is obvious that in a circuit of great inductance and also great resistance, the current practically reaches its full value as quickly as in a circuit of small inductance and proportionally small resistance.

When the circuit is one containing resistance and capacity instead of resistance and self-inductance, the time constant τ is equal to RC, and in this case it is a measure of the growth of the charge in the circuit. When the circuit contains resistance and capacity and inductance, it has no real time constant, but the apparent time constant is less than that of the resistance and inductance or that of the resistance and capacity.

Eddy currents evidently reduce the value of the time constant by their distortion of the curves of current flow.* A variable self-inductance due to magnetic saturation and hysteresis also distorts the curves of current change, and hence changes the time constant. Thus, substituting the value of self-inductance,

$$L = \frac{n\phi}{10^8} = \frac{4\pi nn'A\mu}{10^9},$$

in the formula for the fall of current gives

$$i = \frac{E}{R}\,\epsilon^{-\left(\frac{10^9 Rt}{4\pi nn'A\mu}\right)}$$

and at any given time i is proportional to $\epsilon^{-\frac{1}{\mu}}$. A similar condition exists for rise of current. As μ and therefore L varies during the entire time of rise or fall of current in a circuit, it is evident that the curve cannot be logarithmic. Raising the value of i by the power μ gives

$$i^\mu = \left(\frac{E}{R}\right)^\mu \epsilon^{-\left(\frac{10^9 Rt}{4\pi nn'A}\right)};$$

i can therefore be obtained for any instant of time (t_1) if the constants of the energizing coil are known, provided there is no magnetic leakage.

$$i_1^\mu = \left(\frac{E}{R}\right)^\mu \epsilon^{-\left(\frac{10^9 Rt_1}{4\pi nn'A}\right)} = S,$$

where S is a constant for the instant t_1, or

$$i_1^\mu = S, \text{ and } \mu \log i_1 = \log S.$$

* Arts. 49 and 111.

If the constants of the circuit and the magnetization curve for the circuit are known, the value of μ for each value of i can be obtained. A curve can therefore be drawn in which the ordinates are equal to $\mu \log i$ — where μ is the permeability of the circuit when the current i is flowing — and the abscissas are $\log i$. $\text{Log } i_1$ is then the abscissa of a point on this curve which has an ordinate $\mu \log i_1$, and i_1 can be obtained from logarithmic tables. In this manner any number of values of the rising or falling current can be obtained with due account of changes in permeability caused by the phenomena of saturation and hysteresis. The curve of current will contain harmonics caused by the phenomena of saturation and hysteresis.

61. Examples of Time Constants. — The following are the time constants of some of the circuits for which inductances have previously been given.* A Wheatstone bridge resistance, when properly wound, generally has a time constant of a millionth of a second or less; electric bell, 4.8 millionths of a second; telephone call bell, nearly .02 of a second; modern short-core telephone bell, about .006 of a second; armature of a small magneto generator, from .005 to .013 of a second; telephone receiver, with diaphram, about .001 of a second; modern bipolar telephone receiver, .0005 of a second; mirror galvanometer for marine signaling, .0016 of a second; mirror galvanometer of 5000 ohms resistance, .0004 of a second; 2700 ohm coil of a mirror galvanometer, .001 of a second; 100,000-ohm coil of a mirror galvanometer, .0007 of a second; coil of Ayrton and Perry spring voltmeter, .0044 of a second; polarized relays, types 1, 2, 3, and 4, respectively, .0048, .0045, .0041, and .0032 of a second; Morse relays, from about .070 to .026 of a second, with about .034 of a second as an average for instruments in working adjustment; two telegraph sounders, .0095 and .0065 of a second; modern telephone induction coil primary about .006 of a second, secondary of same nearly .003 of a second; a mile of bare No. 12 B. & S. gauge copper wire on a pole line, with ground return, .00037 of a second; No. 6 wire in a similar position, with ground return, .0014 of a second; primary of large induction coil, .09 of a second; secondary of same, .065 of a second; dynamo fields, from about .01 to 10 seconds; direct-current dynamo armatures, from .005

* Art. 44.

to 5 seconds; Mordey 36-kilowatt 2000-volt alternator armature, .017 of a second; Kapp 60-kilowatt 2000-volt alternator armature, .035 of a second; primary and secondary windings of transformers, from several thousandths of a second to a number of seconds. The Du Bois electro-magnet has a time constant of 75 seconds. Finally, suppose 6 ohms is the resistance of the magnetizing coil figuring in the example of Art. 43. Then assuming the value of L to be constant, which is not exact when the core is iron, the time constant becomes in the three cases, respectively, .655, .785, and .0026 of a second.

62. Vector Relations of Current and Voltage in a Circuit containing R, L, and C; Complex Expressions. — Where resistance, self-inductance, and capacity are all in series in a circuit, the effects of the self-inductance and capacity, being opposite, tend to neutralize each other, and the diagram takes the form shown in Fig. 132. Here OC is the active voltage (IR drop), OD

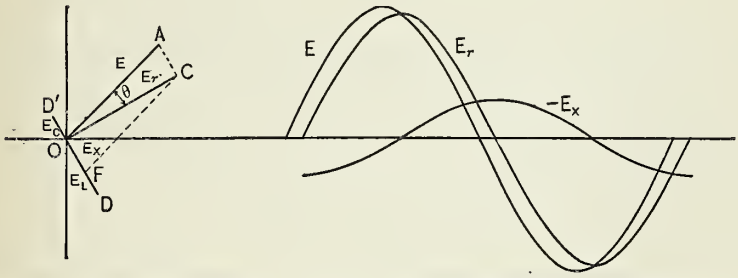


FIG. 132. — Phase Diagram of a Circuit containing Resistance, Self-inductance, and Capacity in Series.

and OD' are voltages respectively equal to the self-inductive and capacity voltages, and OA is the impressed voltage. Since OD' is 90° ahead and OD 90° behind the active voltage, they are exactly in opposition and tend to neutralize each other. If they are of equal values, the external effect is as though both were absent and the impressed and active voltages are then identical. In Fig. 132 the self-inductive voltage is greater than the capacity voltage and their difference OF is the reactive effect in the circuit. The external effect is as though a self-inductive voltage OF were present without capacity. On the other hand, if OD' should be the greater, the difference would show a capacity voltage equal to the difference between

the two reactive voltages. Figure 133 shows the polygon of the voltages shown in Fig. 132.

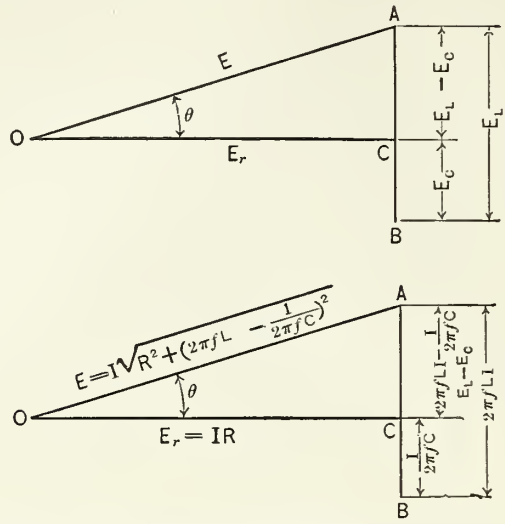


FIG. 133. — Polygon of Voltages in a Circuit containing Resistance, Self-inductance, and Capacity in Series.

The complex expression representing the voltages in such a circuit is

$$\overline{E} = E_r + jE_l - jE_c,$$

or $$\overline{E} = E_r + j(E_l - E_c)$$

and $$\overline{E} = E(\cos\theta + j\sin\theta),$$

where $$E = \sqrt{E_r^2 + (E_l - E_c)^2}$$

and $$\theta = \tan^{-1}\frac{E_l - E_c}{E_r}.$$

If $E_l > E_c$, the imaginary term of the complex expression is positive and the equation is similar to one for a circuit containing resistance and self-inductance only, and the angle θ is positive. If $E_l < E_c$, the imaginary term is negative, and the angle θ is negative.

The voltage arising from the combination of the voltages of several circuits in series is

$$\overline{E} = (E_{r_1} + E_{r_2} + \text{etc.}) + j(E_{l_1} + E_{l_2} + \text{etc.} - E_{c_1} - E_{c_2} - \text{etc.})$$
$$= \Sigma E_r + j(\Sigma E_l - \Sigma E_c).$$

and $\bar{E} = E(\cos\theta + j\sin\theta)$, where

$$E = \sqrt{(\Sigma E_r)^2 + (\Sigma E_l - \Sigma E_c)^2}, \text{ and } \theta = \tan^{-1}\frac{\Sigma E_l - \Sigma E_c}{\Sigma E_r}.$$

Either or both E_l and E_c may be zero in any of the partial circuits and E_r may be so small as to be negligible.* The angle θ is the angle of lag of the circuit. It is measured from the initial axis or axis of reals (which gives the direction of the current vector or IR drop) towards the vector of impressed voltage. When θ is positive, the circuit carries a current which lags with respect to the voltage impressed at the circuit terminals; and when θ is negative, the circuit carries a current which leads with respect to the voltage impressed at the terminals.

The current in a circuit may be represented by a vector (complex) equation in the same manner as a voltage. In this case the current must be resolved into rectangular components, preferably with one component parallel to the vector of voltage impressed on the circuit. It will be shown later that the initial axis in this case should correspond with the direction of the vector of voltage impressed on the circuit, and therefore the angle measured from the initial axis to the current vector is the angle of lag reversed; that is, it is $-(\pm\theta)$.

Knowing the rectangular components of the several currents in a number of branches or parallel circuits, the components of the main current are obtained by adding the corresponding components of the branch currents together algebraically; as

$$\bar{I} = \Sigma I_r + j(\Sigma I_x),$$

and $$\bar{I} = I(+\cos\theta - j\sin\theta).$$

In this case $$I = \sqrt{(\Sigma I_r)^2 + (\Sigma I_x)^2}$$

and $$\theta = \tan^{-1}\frac{\Sigma I_x}{\Sigma I_r}.$$

63. Impedance and Reactance and their Expression as Complex Quantities. — The quantity $\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}$ is called the **Impedance** of the circuit and $\left(2\pi fL - \frac{1}{2\pi fC}\right)$ is

* Art. 58.

called the **Reactance**, whatever may be the values of R, L, or C. The square of the impedance of a circuit is therefore equal to the sum of the squares of its resistance and reactance. Impedance and reactance are both of the dimensions of resistance and are therefore expressed in ohms. **Impedance** may be defined for circuits in general, as the total opposition in a circuit to the flow of an alternating electric current, or the ratio of the voltage to the current; and **Reactance** may be defined as the component of the impedance caused by the self-inductance and capacity of the circuit.

The reciprocal of impedance is called **Admittance.** It is equal to the current divided by the voltage. It therefore measures the tendency of a voltage to force current through a circuit. It is measured in **Mhos** or the reciprocal of ohms, as the term indicates.

The **Capacity Reactance** of a circuit is inversely proportional to the capacity and the frequency in the circuit, since it is equal to $-\frac{1}{2\pi fC}$. The **Inductive Reactance** of a circuit is directly proportional to the self-inductance and the frequency in the circuit, since it is equal to $2\pi fL$. The total reactance of a circuit containing capacity and self-inductance in series bears a complicated relation to the capacity, self-inductance, and frequency in the circuit, since it is equal to $2\pi fL - \frac{1}{2\pi fC}$. It is zero when, for any reason, $2\pi fL$ and $\frac{1}{2\pi fC}$ are both equal to zero, which is the condition of a circuit of resistance alone; or whenever $2\pi fL = \frac{1}{2\pi fC}$, that is, when $2\pi f = \frac{1}{\sqrt{LC}}$.

Polygons of impedance may be directly obtained from the polygons of voltages as are shown in Figs. 119, 128, and 133, by dividing each side of the polygon by the current I. Figure 134 shows the impedance diagram for a self-inductive circuit; Fig. 135 for a capacity circuit; and Fig. 136 for a series circuit which includes both self-induction and capacity. The capacity effect predominates over the self-inductance in the latter figure, which in this differs from the circuit represented in Fig. 133. Impedance is usually represented in formulas by the letter Z, reactance by the letter X, and admittance by the letter Y.

Impedance may be treated as a vector operator; that is, as a line having definite length and direction. It may be expressed in the complex form in this way,

$$\overline{Z} = R + jX$$

and $$\overline{Z} = Z(\cos\theta + j\sin\theta);$$

also $$Z = \sqrt{R^2 + X^2} \text{ and } \theta = \tan^{-1}\frac{X}{R}.$$

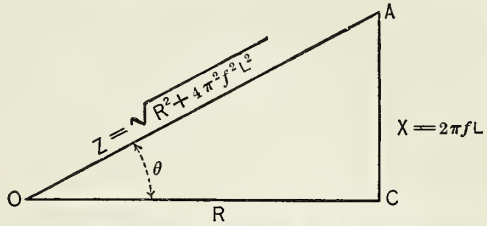


FIG. 134. — Impedance Triangle for a Self-inductive Circuit.

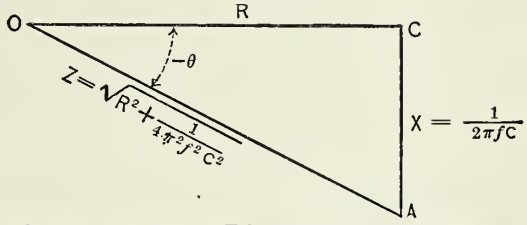


FIG. 135. — Impedance Triangle for a Capacity Circuit.

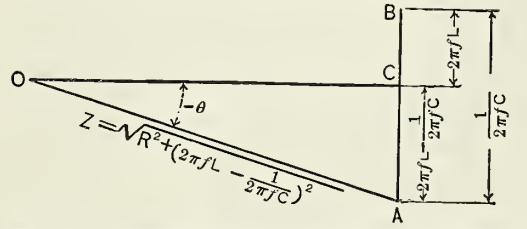


FIG. 136. — Impedance Polygon for a Circuit containing both Self-induction and Capacity, Capacity Predominating.

Since the triangles of impedance are geometrically similar to the corresponding triangles of voltage, the angle θ is the same in each, *i.e.*, it is the angle of lag in the circuit. If X and θ are negative, capacity predominates, and the imaginary terms become $-jX$ and $-j\sin\theta$.*

* Art. 81.

The combined impedance of a number of circuits in series is

$$\bar{Z} = (R_1 + R_2 + \text{etc.}) + j(X_1 + X_2 + \text{etc.}) = \Sigma R + j\Sigma X,$$

in which the reactances are to be added algebraically;

$$\bar{Z} = Z(\cos\theta + j\sin\theta)$$

and $$\tan\theta = \frac{\Sigma X}{\Sigma R}.$$

Admittance being the reciprocal of impedance,

$$\bar{Y} = \frac{1}{\bar{Z}} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}.$$

Putting g and b for the horizontal and vertical components, respectively, gives

$$\bar{Y} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2} = g - jb.$$

Thus, $g = \dfrac{R}{R^2 + X^2}$ and is called the **Conductance** of the circuit.

Also, $b = \dfrac{X}{R^2 + X^2}$ and is called the **Susceptance** of the circuit.

Then, $$\frac{b}{g} = \frac{X}{R} = \tan\theta.$$

Admittances in parallel can be combined in the same manner as impedances in series.

Assume sinusoidal voltages and currents in the following problems:

Prob. 1. A circuit has a resistance of 10 ohms and a capacity of 50 microfarads in series. What is the impedance of the circuit when the frequency is 60 periods per second? Solve by the graphical method and by means of complex quantities.

Prob. 2. A circuit has a resistance of 50 ohms and a capacity of 200 microfarads in series. What is the impedance when the frequency is 25 periods per second?

Prob. 3. If the resistance of a capacity circuit is 20 ohms and its impedance is 50 ohms, what is the capacity in microfarads, and what is the angle of lag, the frequency being 120 periods per second? Solve graphically and analytically.

Prob. 4. A circuit has an impedance of 100 ohms. The frequency is 60 periods per second and the angle of lag is 45°.

Find the resistance and inductance of the circuit by graphics and by vector equations.

Prob. 5. A circuit has an impedance of 200 ohms. The angle of lag of the circuit is $-60°$ and the frequency is 25 periods per second. Find the resistance and capacity by graphics and by vector equations.

Prob. 6. A circuit having an angle of lag of $-90°$ and a frequency of 40 periods per second has an impedance of 50 ohms. What is the resistance and what is the capacity? Solve by graphics and by vector equations.

Prob. 7. A circuit in which there is an angle of lag of 90° and a frequency of 40 periods per second has an impedance of 100 ohms. Find the resistance and the reactance by graphics and by vector equations.

Prob. 8. A circuit has a resistance of 10 ohms, a self-inductance of .01 of a henry, and a capacity of 50 microfarads in series. If the frequency is 60 periods per second, what is the value of the impedance and of the angle of lag?

Prob. 9. A circuit has a resistance of 20 ohms, a self-inductance of .01 of a henry, and a capacity of 200 microfarads in series. Find the impedance and angle of lag by graphics and vector equations when the frequency is 40 periods per second.

Prob. 10. A circuit has a resistance of 50 ohms, a self-inductance of .02 of a henry, and a capacity of 20 microfarads in series. Find the impedance and angle of lag by graphics and vector equations when the frequency is 25 periods per second.

Prob. 11. A circuit has an inappreciable resistance, a self-inductance of .01 of a henry, and a capacity of 100 microfarads in series. Find the impedance and the angle of lag when the frequency is 60 periods per second.

Prob. 12. Two circuits having self-inductances of .01 and .02 of a henry and resistances of 8 ohms and 10 ohms, respectively, are in series. Find their combined impedance and the angle of lag of the circuit when the frequency is 40 periods per second.

Prob. 13. Two circuits having capacities of 100 and 200 microfarads and resistances of 10 and 20 ohms, respectively, are in series. Find the impedance and angle of lag when the frequency is 40 periods per second.

Prob. 14. Two circuits have resistances respectively of 10 and 15 ohms, the first has a capacity of 100 microfarads and the second a self-inductance of .01 of a henry. If the circuits are connected in series, what is the combined impedance and what is the angle of lag when the frequency is 40 periods per second?

Prob. 15. If a voltage of 100 volts is impressed upon the joint circuit of problem 14, how much current will flow?

Prob. 16. If it is desired to cause a current of 10 amperes to flow through the joint circuit of problem 14, what voltage must be impressed?

Prob. 17. A circuit has a resistance of 10 ohms, a self-inductance of .01 of a henry, and a capacity of 200 microfarads all in series, and a voltage of 200 volts with a frequency of 60 periods per second is impressed on this circuit. What is the active, and what is the reactive, voltage? Find graphically and by vector equations the current flowing through the circuit.

Prob. 18. Three coils of respectively 1 ohm, 2 ohms, and 3 ohms resistance having self-inductances of .01, .02, and .03 of a henry are connected in series. If a current of 10 amperes flows through the series at a frequency of 25 periods per second, what is the voltage between the terminals of each coil, and what is the voltage between the terminals of the series ?

Prob. 19. Three condensers having internal resistances of 8, 10, and 12 ohms, respectively, and capacities of 50, 100, and 180 microfarads are connected in series. What is the total voltage impressed on the series and what is the voltage between the terminals of each condenser when the current is 20 amperes with a frequency of 60 periods per second ?

Prob. 20. A circuit of 10 ohms resistance and .01 of a henry self-inductance is in series with a circuit of 8 ohms resistance and 200 microfarads capacity and these in turn are in series with another circuit of 15 ohms resistance, .02 of a henry self-inductance, and 150 microfarads capacity. When a current of 5 amperes with a frequency of 40 periods per second is caused to flow through this series, what are the voltage and angle of lag observed at the main terminals, and what are the voltages and angles of lag observed at the terminals of each of the three above-named parts of the total circuit?

Prob. 21. The current in a circuit is 50 amperes with a period of one one-hundred-and-twentieth of a second. The circuit contains a positive (inductive) reactance of 20 ohms and a resistance of 20 ohms. What are the impressed voltage and the angle of lag?

Prob. 22. Twenty amperes flow through a circuit with an angle of lag of 30°, when the impressed voltage is 100 volts and the frequency 40 periods per second. What are the impedance, reactance, and resistance?

Prob. 23. A circuit having an impedance of 40 ohms is in series with a circuit having an impedance of 30 ohms. When a voltage of 100 volts with a frequency of 60 periods per second is applied to the two circuits, the angle of lag in the first circuit is 30° and in the second circuit is $-60°$. How much current flows through the circuits and what is the angle of lag observed at the main terminals?

Prob. 24. Three circuits are in series: the first has a resistance of 20 ohms, the second a resistance of 10 ohms and a positive (inductive) reactance of 20 ohms, the third a resistance of 5 ohms and a negative (capacity) reactance of 30 ohms, the frequency being 25 periods per second. What is the angle of lag observed at the main terminals and what voltage is impressed between the main terminals when a current of 50 amperes flows through the series?

Prob. 25. What is the angle of lag and what is the voltage observed between the terminals of each of the three parts of the main circuit of problem 24?

Prob. 26. If a voltage of 500 volts, frequency 25 periods per second, is impressed on the main circuit of problem 24, what current flows and what is the angle of lag?

Prob. 27. What is the voltage across each of the three parts of the main circuit of problem 24 under the conditions of current flow fixed by problem 26?

Prob. 28. A circuit of negligible resistance has a positive (inductive) reactance of 10 ohms when the frequency is 60 periods per second. What current flows when a voltage of 100 volts is impressed on the circuit, and what is the angle of lag?

Prob. 29. A circuit of negligible resistance has a negative (capacity) reactance of 10 ohms when the frequency is 60 periods per second. What is the current when the impressed voltage is 100 volts, and what is the angle of lag?

Prob. 30. A circuit of negligible resistance has a positive reactance of 10 ohms and a negative reactance of 10 ohms when the frequency is 60 periods per second. What current flows when 100 volts are impressed on the circuit?

Prob. 31. A circuit has a resistance of 10 ohms, a positive reactance of 10 ohms, and a negative reactance of 10 ohms, when the frequency is 60 periods per second. What current flows when a voltage of 100 volts is impressed upon the circuit, and what is the angle of lag?

Prob. 32. A circuit of negligible reactance has a resistance of 10 ohms. When a voltage of 100 volts at a frequency of 60 periods per second is impressed upon the circuit, what current flows and what is the angle of lag?

Prob. 33. Find the self-inductance and capacity in each of the circuits given in problems 28 to 32, inclusive.

64. Application. —The application to circuits in general when the currents and voltages are sinusoidal, and to alternator armatures in particular, of the preceding deductions is evident.

Thus, suppose it is desired to design an alternator which is to generate 25 amperes at an effective voltage of 1000 volts at its terminals when operating on a non-reactive load, the frequency being 100 periods per second. Take first, for example, a disk armature without iron in its core, with a resistance of 1 ohm and an average self-inductance of .01 henry. The effective value of the total voltage to be developed in this armature at full load is then

$$\bar{E} = E_r + jE_x,$$

$$E = \sqrt{E_r^2 + E_x^2} = I\sqrt{R^2 + (2\pi fL)^2},$$

$$E = \sqrt{(1000 + 25 \times 1)^2 + (2\pi \times 100 \times .01 \times 25)^2},$$

which is equal to 1037 volts. Consequently, the effect of self-inductance is to demand an increase of the total voltage equal to 12 volts. Suppose, however, the armature is of a type having an iron core and has a resistance of 1 ohm and an average

working inductance of .05 henry, the total voltage then becomes

$$E = \sqrt{(1000 + 25 \times 1)^2 + (2\pi \times 100 \times .05 \times 25)^2},$$

which is equal to 1291 volts. Hence, the total voltage must be increased by 266 volts on account of self-inductance. If the two machines were worked at full load upon resistances of absolutely no inductance or capacity, the lags of the currents with respect to the induced voltages in the circuits in the two cases would be respectively 8° 43′ and 37° 27′ (Fig. 137). These values are obtained from the relation $\theta = \tan^{-1}\frac{E_x}{E_r}$.

On the other hand, in case the load contains reactance, the figures are modified. For instance, in case the load contains 40 ohms of impedance as before, but in this instance is composed of 32 ohms resistance and 24 ohms positive (inductive) reactance, the induced voltage required to afford 1000 volts at the terminals of the generator with the iron-cored armature is

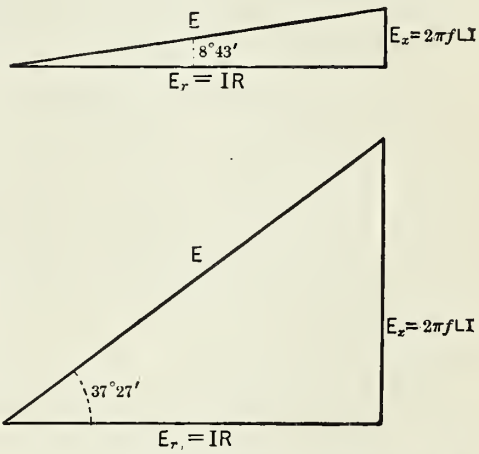


FIG. 137.—Voltage Triangle calculated for an Alternator Armature.

$$E = \sqrt{[25(1 + 32)]^2 + [25(31.4 + 24)]^2},$$

which is equal to 1612 volts; while if the reactance of the load is negative instead of positive,

$$E = \sqrt{[25(1 + 32)]^2 + [25(31.4 - 24)]^2},$$

which is equal to 845 volts. In each case the terminal voltage of the generator, that is, the voltage impressed on the load, is 1000 volts; but the voltage impressed on the entire circuit, that is, the induced voltage, is affected by the relations of the impedance of the load to the impedance of the armature winding. In case the impedance of the load contains negative reactance,

it is to be observed that the terminal voltage may be actually higher than the induced voltage on account of the interaction of the capacity of the load on the inductance of the armature winding.

Prob. 1. An alternator with terminal voltage of 10,000 volts and frequency of 60 periods per second furnishes 50 amperes to an external circuit. The armature has a resistance of 4 ohms and a self-inductance of .02 of a henry. When the external load is non-reactive, what total voltage must be generated by the machine?

NOTE. — This and the following problems are to be solved by the graphical method and by the use of the complex (vector) equations, neglecting any effects of armature reactions.

Prob. 2. An alternator generates a total voltage of 5000 volts at a frequency of 40 periods per second. The armature has 2 ohms resistance and .03 of a henry inductance. How much current will this alternator deliver to a non-reactive load of 20 ohms, and what is the terminal voltage? Also what is the value of the angle of lag between the current and the terminal voltage and the angle of lag between the current and the total generated voltage?

Prob. 3. The generator of problem 2 is connected to a load containing 10 ohms resistance and 200 microfarads capacity in series. What are the values of current, terminal voltage, and angle of lag between current and terminal voltage?

Prob. 4. The generator of problem 2 is connected to a load containing 20 ohms resistance and .02 of a henry self-inductance in series. What are the values of current, terminal voltage, and angle of lag between the current and terminal voltage?

Prob. 5. An alternator armature has a resistance of 2 ohms and a self-inductance of .02 of a henry. It produces a terminal voltage of 2000 volts at a frequency of 100 periods per second. Twenty-five amperes flow in the circuit, the angle of lag between the current and the induced voltage is 30°. What are the impedance, resistance, reactance, and self-inductance of the external circuit?

Prob. 6. In problem 5 the angle of lag changes from 30° to $-60°$ and the current to 50 amperes without changing termi-

nal voltage or frequency. What then are the impedance, resistance, reactance, and capacity of the external circuit?

Prob. 7. The external circuit in problem 5 again changes so that the tangent of the angle of lag is .4652; the current 80 amperes. What are the impedance, resistance, and reactance of the external circuit?

Prob. 8. What are the values of induced voltage corresponding respectively to the conditions in problems 5, 6, and 7; and what are the respective angles of lag between the currents and the induced voltages?

65. Equivalent Resistance and Reactance. — When there is no iron near an alternating-current circuit, the self-inductance is constant and the triangles of voltage given previously are true for all values of the current, but if iron is present, L varies with the current. In addition to this, if iron is present, hysteresis and eddy currents absorb power. Under a given impressed voltage this power reduces the reactive voltage and increases the active voltage, so that instead of leaving the triangle of voltages OAB (Fig. 138) such as would be obtained in a certain circuit when a current I is flowing and there are no **Iron losses** present, the triangle actually becomes more like OCD. The reactive voltage is reduced and the active voltage is greater in OCD than in OAB, and the angle of lag decreased. The **Equivalent** or **Working reactance** of the circuit is

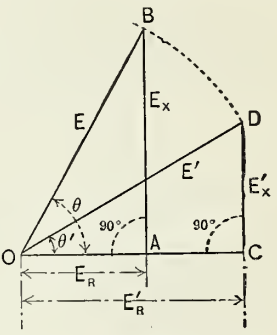


FIG. 138. — Triangles of Voltages with and without Iron Losses.

$$X' = \frac{E'_x}{I} = \frac{E \sin \theta'}{I}.$$

The **Equivalent** or **Working resistance** is

$$R' = \frac{E'_r}{I} = \frac{E' \cos \theta'}{I}.$$

These conditions are discussed further in Chapter IX.

Prob. 1. A circuit has a voltage applied to it of 100 volts and a current of 10 amperes flows with a lag of 60°. What is the apparent resistance and what is the equivalent reactance of the circuit?

66. General Equation for Current in a Circuit. — The voltage impressed upon any circuit has already been expressed * in the form

$$Ri + \frac{L di}{dt} + \frac{q}{C} = e = f(t), \tag{1}$$

in which $f(t)$ represents *any* voltage which may be impressed on the circuit. This general solution therefore includes, when properly interpreted, the various cases already discussed in Arts. 45, 52, and 58.

Since $\int i dt = q$, (1) may be written

$$\frac{Ri}{L} + \frac{di}{dt} + \frac{\int i dt}{LC} = \frac{e}{L}. \tag{2}$$

Differentiating this with respect to time to rid the equation of the integral sign, there results:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{1}{L}\frac{de}{dt} = \frac{1}{L} f'(t). \tag{3}$$

The term $\frac{de}{dt}$ can only have one finite value in an electric circuit at any specific time, and is therefore a single-valued function of time, which for convenience is called $f'(t)$. Equation (3) is a linear differential equation of the second order with its second member a function of t.†

The object is to find the value of current i, in terms of R, L, C, and the voltage.

The equation may be written

$$\left(D^2 + \frac{RD}{L} + \frac{1}{LC}\right) i = \frac{f'(t)}{L}, \tag{4}$$

where D is a symbol of operation and represents $\frac{d[\quad]}{dt}$ and D^2 means differentiation twice with respect to t. D may be used as an ordinary algebraic term.‡

The "auxiliary equation" $D^2 + \frac{RD}{L} + \frac{1}{LC} = 0$ is a quadratic equation with the two roots,

* Art. 58.

† Price's *Calculus*, Vol. II, p. 458; Forsyth's *Differential Equations*, p. 86; Perry's *Calculus for Engineers*, pp. 213–241; John Graham, *An Elementary Treatise on Calculus*, p. 221.

‡ Perry's *Calculus for Engineers*, p. 231; Forsyth's *Differential Equations*, p. 43; Murray's *Differential Equations*, Chap. VI.

$$-\frac{R}{2L}+\sqrt{\frac{1}{4}\frac{R^2}{L^2}-\frac{1}{LC}}=-\left[\frac{R}{2L}-\sqrt{\frac{1}{4}\frac{R^2}{L^2}-\frac{1}{LC}}\right], \tag{5}$$

and

$$-\frac{R}{2L}-\sqrt{\frac{1}{4}\frac{R^2}{L^2}-\frac{1}{LC}}=-\left[\frac{R}{2L}+\sqrt{\frac{1}{4}\frac{R^2}{L^2}-\frac{1}{LC}}\right]. \tag{6}$$

Letting a_1 and a_2 equal the bracket expressions respectively of the second members of equations (5) and (6), then the "complementary function" is

$$Ae^{-a_1t}+Be^{-a_2t}. \tag{7}$$

The particular integral is

$$\frac{\frac{1}{L}f'(t)}{D^2+\frac{RD}{L}+\frac{1}{LC}}=\frac{\frac{1}{L}f'(t)}{(D+a_1)(D+a_2)}$$

$$=\frac{1}{L}f'(t)\left[\frac{N_1}{D+a_1}+\frac{N_2}{D+a_2}\right], \tag{8}$$

in which the third member is obtained by resolving the second member into partial fractions. This solves into the form,

$$\frac{1}{L}\left[N_1e^{-a_1t}\int e^{a_1t}f'(t)dt+N_2e^{-a_2t}\int e^{a_2t}f'(t)dt\right]. \tag{9}$$

The values of N_1 and N_2 may be determined from the identical equation,*

$$\frac{1}{(D+a_1)(D+a_2)}=\frac{N_1}{D+a_1}+\frac{N_2}{D+a_2}; \tag{10}$$

whence

$$N_1(D+a_2)+N_2(D+a_1)=1, \tag{11}$$

$$D(N_1+N_2)=0,\ N_1a_2+N_2a_1=1, \tag{12}$$

and

$$N_1=-N_2=\frac{1}{a_2-a_1}=\frac{1}{2\sqrt{\frac{1}{4}\frac{R^2}{L^2}-\frac{1}{LC}}}. \tag{13}$$

The complete solution of equation (3), being the sum of the complementary function and the particular integral, is

$$i=\frac{1}{2L\sqrt{\frac{1}{4}\frac{R^2}{L^2}-\frac{1}{LC}}}\left[e^{-a_1t}\int e^{a_1t}f'(t)dt-e^{-a_2t}\int e^{a_2t}f'(t)dt\right]$$

$$+Ae^{-a_1t}+Be^{-a_2t}. \tag{14}$$

* Wentworth's *Complete Algebra*, p. 409.

The coefficient of the first term of the second member of the last equation may also be written

$$\frac{C}{\sqrt{R^2C^2-4LC}}.$$

The terms a_1 and a_2 may also be written

$$a_1=\frac{R}{2L}-\sqrt{\frac{1}{4}\frac{R^2}{L^2}-\frac{1}{LC}}=\frac{RC-\sqrt{R^2C^2-4LC}}{2LC},$$

$$a_2=\frac{R}{2L}+\sqrt{\frac{1}{4}\frac{R^2}{L^2}-\frac{1}{LC}}=\frac{RC+\sqrt{R^2C^2-4LC}}{2LC}.$$

A similar equation can be obtained in the same way for the charge at any instant in a condenser in the circuit.

67. Current in a Circuit when Any Periodic Voltage is Impressed. — In the case of a circuit with resistance, self-inductance, and capacity in series on which any periodic voltage is impressed, the voltage formula is

$$e=f(t)=iR+L\frac{di}{dt}+\frac{1}{C}\int_0^t i\,dt.$$

The general solution of this is equation (14), Art. 66.

This contains too many unknown terms to be of much use to the electrical engineer at present; but if $e=f(t)$ is construed to mean an alternating voltage, it may be expressed in terms of the harmonics. This formula becomes

$$e=f(t)=e_{m_1}\sin(\alpha+\theta_1)+e_{m_3}\sin(3\alpha+\theta_3)+\cdots + e_{m_n}\sin(n\alpha+\theta_n),$$

and

$$\frac{de}{dt}=f'(t)=\omega e_{m_1}\cos(\alpha+\theta_1)+3\,\omega e_{m_3}\cos(3\alpha+\theta_3)\cdots + n\omega e_{m_n}\cos(n\alpha+\theta_n).$$

Substituting this value of $f'(t)$ in the general equation for current developed in Art. 66, there results an equation in which there is a pair of similar terms for each harmonic. These may evidently be integrated by parts and the result is as follows:

$$i=\frac{e_{m_1}}{\sqrt{R^2+\left(2\pi fL-\frac{1}{2\pi fC}\right)^2}}\sin(\alpha+\theta_1)$$

$$+\frac{e_{m_3}}{\sqrt{R^2+\left(6\pi fL-\frac{1}{6\pi fC}\right)^2}}\sin(3\alpha+\theta_3)$$

$$+\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$+\frac{e_{m_n}}{\sqrt{R^2+\left(2\pi nfL-\frac{1}{2\pi nfC}\right)^2}}\sin(n\alpha+\theta_n)$$

$$+A\epsilon^{-\alpha_1 t}+B\epsilon^{-\alpha_2 t}.$$

The angle θ_n is determined by the relation

$$\tan\theta_n=\frac{2\pi fnL}{R}-\frac{1}{2\pi fnCR}=\frac{X_n}{R}.$$

Each of the terms of the right-hand member, from the first to the nth, has the form of a sine function.

The formula may be more conveniently written

$$i=\frac{e_{m_1}}{Z_1}\sin(\alpha+\theta_1)+\frac{e_{m_3}}{Z_3}\sin(3\alpha+\theta_3)\cdots$$
$$+\frac{e_{m_n}}{Z_n}\sin(n\alpha+\theta_n),$$

where Z_1, Z_3, $\cdots$ Z_n each represents the impedance which the circuit offers to the corresponding current harmonic. The exponential terms are omitted, as they ordinarily disappear quickly after the current is started; and the formula represents the conditions in the circuit after a permanent state is established. It will be noted that the impedances for the several harmonics may have very different values even though they are all affected by the same resistance, self-inductance, and capacity. This is due to the frequency of the harmonic entering into the reactance component of the impedance, as shown by the formula. Each term of the second member represents the instantaneous value at the instant t, $\left(=\frac{\alpha}{\omega}\right)$, of a sine wave current. A similar formula may obviously be derived to represent the condition when even-numbered harmonics as well as odd-numbered harmonics appear in the voltage wave.

The effective value of current is found by taking the square root of the sum of the squares of the effective values of the

harmonic currents given in the last expression, or

$$I = \sqrt{I_1^2 + I_3^2 + \cdots I_n^2}\,*$$

where $$I_1 = \frac{e_{m_1}}{\sqrt{2}\,Z_1}, \qquad I_3 = \frac{e_{m3}}{\sqrt{2}\,Z_3}, \text{ etc.}$$

Prob. 1. A certain voltage wave is composed of two harmonics, the fundamental having an effective value of 150 volts and that of three times the frequency having an effective value of 25 volts. What is the effective value of the resultant voltage?

Prob. 2. The impedance offered by a circuit to the primary harmonic of a certain voltage is 20 ohms, the effective value of the harmonic being 100 volts; the remaining harmonic of the voltage is of five times the primary frequency, has an effective value of 10 volts, and overcomes an impedance of 30 ohms. Find the effective values of the voltage and current.

Prob. 3. A current is composed of two harmonics, a primary and one of three times the frequency. The first has an effective value of 50 amperes and flows through an impedance of 50 ohms. The other has an effective value of 20 amperes and flows through an impedance of 40 ohms. What is the effective value of the voltage impressed on the circuit?

68. Irregular Current and Voltage Waves expressed as Complex Quantities. — A Fourier's series may be used to represent irregular current and voltage by means of component sine waves, as pointed out in Art. 67. The effective value of such a voltage is found from the relation,

$$E^2 = E^2_{r_1} + E^2_{r_3} + \text{etc.} + E^2_{x_1} + E^2_{x_3} + \text{etc.},*$$

and $$E = \sqrt{E^2_{r_1} + E^2_{r_3} + \text{etc.} + E^2_{x_1} + E^2_{x_3} + \text{etc.}},$$

where E_r and E_x are active and reactive effective voltages for the various harmonics. This may be written:

$$E = \sqrt{(E^2_{r_1} + E^2_{x_1}) + (E^2_{r_3} + E^2_{x_3}) + \text{etc.}}$$

and the term within each bracket is evidently the square of the scalar value of the corresponding sine voltage. A non-sinusoidal alternating curve may be empirically indicated by the expression

$$(E_{r_1}, E_{r_2}, E_{r_3}, \text{etc.}) + j(E_{x_1}, E_{x_2}, E_{x_3}, \text{etc.});$$

* Art. 16.

which is intended to convey the idea that each harmonic of the voltage curve acts individually to set up a current of the same frequency in the circuit, as though the other harmonics were not present. The square root of the sum of the squares of the effective values of the harmonic voltages, as already pointed out, gives the effective value of the total voltage impressed on the circuit. In the same way, the square root of the sum of the squares of the effective values of the individual current harmonics caused to flow in the circuit by the harmonics of voltage gives the effective value of the actual current flowing.

The irregular current relations are similar to those given for voltage.

Triangles of voltages for each harmonic may be drawn, and if the corresponding harmonic of current is divided into the value of each side, an impedance triangle results. Thus,

$$\left(\frac{E_{r_1}}{I_1} + j\frac{E_{x_1}}{I_1}\right) = R + jX_1$$

represents the impedance offered to the flow of the harmonic of primary frequency ;

$$\left(\frac{E_{r_3}}{I_3} + j\frac{E_{x_3}}{I_3}\right) = R + jX_3$$

represents the impedance offered to the flow of the harmonic of three times the frequency, etc.

In obtaining the reactances for the different harmonics, care must be taken to use the proper frequency in the formula,

$$X = 2\pi fL, \text{ or } X = \frac{1}{2\pi fC}.$$

Thus, if the resistance in a circuit is 10 ohms and the reactance for the primary harmonic is 5, there results

$$Z_1 = 10 + j5\,;\ Z_3 = 10 + j15\,;\ Z_5 = 10 + j25, \text{ etc.},$$

when the voltage and current contain harmonics of odd frequencies and the reactance is inductive. If the reactance results from capacity, the expression is

$$Z_1 = 10 - j5\,;\ Z_3 = 10 - j\tfrac{5}{3}\,;\ Z_5 = 10 - j\tfrac{5}{5}, \text{ etc.}$$

It will be noticed that the resistance is the same for all harmonics.

If the empirical expression for current

$$(I_{g_1}, I_{g_3}, \text{etc.}) + j(I_{b_1}, I_{b_3}, \text{etc.}),$$

where I_g and I_b are the rectangular components of current, has its several harmonic values divided by the respective harmonic values of the voltage, the admittances which the circuit offers to each current harmonic may be derived. Thus

$$\left(\frac{I_{g_1}}{E_1}, \frac{I_{g_3}}{E_3}, \text{etc.}\right) + j\left(\frac{I_{b_1}}{E_1}, \frac{I_{b_3}}{E_3}, \text{etc.}\right)$$
$$= (g_1, g_2, \text{etc.}) + j(b_1, b_3, \text{etc.}),$$

where g and b represent the rectangular components of the admittances.

$$\bar{Y}_1 = \frac{I_{g_1}}{E_1} - j\frac{I_{b_1}}{E_1} = g_1 - jb_1$$

is the admittance of the circuit for the current harmonic of primary frequency,

$$\bar{Y}_3 = \frac{I_{g_3}}{E_3} - j\frac{I_{b_3}}{E_3} = g_3 - jb_3$$

is the admittance for the current harmonic of three times the primary frequency, etc.

Vector voltage may obviously be multiplied by admittance or divided by impedance to give vector current; or vector current may be divided by admittance or multiplied by impedance to give vector voltage.* This process gives the relations of the equivalent sinusoids representing irregular curves of voltage and current. The total vector impedance and admittance may be written empirically

$$\bar{Z}_1 + \bar{Z}_2 + \bar{Z}_3 + \text{etc.}$$
$$= (R_1 + R_2 + R_3 + \text{etc.}) + j(X_1 + X_2 + X_3 + \text{etc.})$$

$$\bar{Y}_1 + Y_2 + \bar{Y}_3 + \text{etc.}$$
$$= (g_1 + g_2 + g_3 + \text{etc.}) - j(b_1 + b_2 + b_3 + \text{etc.})$$

$$[(R_1 + R_2 + R_3 + \text{etc.})^2 + (x_1 + x_2 + x_3 + \text{etc.})^2]^{\frac{1}{2}},$$

and $$[(g_1 + g_2 + g_3 + \text{etc.})^2 + (b_1 + b_2 + b_3 + \text{etc.})^2]^{\frac{1}{2}}.$$

The process of solving problems involving irregular curves where the harmonic components are known is similar to those

* Art. 63.

already disclosed for sinusoidal voltages or currents, but each sinusoidal component of voltage or current must be dealt with as if acting alone in the circuit, and then the results may be combined to obtain the total values according to the theorem that the effective value of any single-valued periodic curve is equal to the square root of the sum of the squares of the effective values of its harmonics. It should be noted that the impedance obtained by dividing the total voltage by the total current is *dependent upon the shapes of the curves* of voltage and current on account of the fact that fixed values of self-inductance and capacity do not have the same effect upon harmonics of different frequencies.

Prob. 1. The voltage impressed on a circuit is composed of two harmonics. The fundamental has an active component of 190 volts and a reactive component of -50 volts, while the other harmonic has three times the frequency, an active component of 25 volts, and a reactive component of 20 volts. What is the value of the voltage?

Prob. 2. A circuit contains a resistance of 10 ohms and self-inductance of .01 henry. A current with a frequency of 60 periods per second flows in this circuit. It is composed of a fundamental harmonic of 100 amperes and a third harmonic of 20 amperes. What voltage is impressed upon the circuit? (Aid: Multiply each harmonic current by the corresponding complex expression for impedance to obtain the voltage harmonics.)

Prob. 3. What impedance is offered by a circuit in which voltage $E=(100_1, 10_3)+j(50_1, 15_3)$ is impressed and current $I=(50_1, 5_3)+j(40_1, 10_3)$ flows? (Aid: First find effective voltage and effective current.)

69. Variation in the Impedance offered to Current and Voltage Harmonics of Different Frequencies. — Following out the remarks of the last paragraph of the preceding article, it is to be observed that, since the reactance of a reactive circuit is dependent on the frequency of the current flowing, it is obvious that the impedance interposed by a circuit to the different harmonics of a current may differ for the several harmonics. Inductive reactance $(2\pi fL)$ varies directly with frequency; capacity reactance $\left(\frac{1}{2\pi fC}\right)$ varies inversely with frequency; and

the reactance of a circuit containing both inductance and capacity $\left(2\pi fL - \frac{1}{2\pi fC}\right)$ is a more complex function of frequency and may increase or decrease as frequency increases, depending upon the relations of the inductive and capacity parts of the reactance.

Under these circumstances, when an irregular alternating voltage is impressed on a circuit, the harmonics of the current which flow will not bear a uniform ratio to the voltage harmonics unless the circuit is non-reactive. Assuming the self-inductance and capacity to be uniform,

$$I^2 = \frac{E^2}{Z^2} = I_1^2 + I_3^2 + \cdots + I_n^2,$$

in which $$I_1^2 = \frac{E_1^2}{Z_1^2},\ I_3^2 = \frac{E_3^2}{Z_3^2},\ \text{etc.}$$

But in a series circuit

$$Z_1^2 = R^2 + \left(2\pi f_1 L - \frac{1}{2\pi f_1 C}\right)^2,$$

$$Z_3^2 = R^2 + \left(2\pi f_3 L - \frac{1}{2\pi f_3 C}\right)^2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Z_n^2 = R^2 + \left(2\pi f_n L - \frac{1}{2\pi f_n C}\right)^2,$$

in which f_1, f_3, etc., f_n represent the frequencies of the several harmonics, the numerical value of the subscript showing the number of times to multiply the fundamental frequency to arrive at the particular frequency under consideration.

Scrutinizing these equations brings out a number of important facts.

1. Only when reactance is negligible or the voltage sinusoidal can the current flowing in a circuit have the same wave form as the impressed voltage which sets it up.

2. If the capacity reactance is negligible, as it ordinarily is in the instance of a closed metallic circuit of relatively short length, inductive reactance only is to be considered. In this instance it will be observed that the reactance opposed to each current harmonic is directly proportional to the frequency of the harmonic and the impedance is larger for the harmonics of higher frequency than for those of lower frequency. Under

these circumstances, the current produced by an irregular impressed voltage will be less irregular than the voltage producing it; that is, the inductive reactance has the effect of reducing the amplitude of the higher harmonics, and thus reducing the ripples or irregularities on the current curve. The lag angles, θ_1, θ_2, etc., of the several current harmonics with respect to the corresponding voltage harmonics also differ from each other since R is unaffected by frequency, which additionally alters the form of the current wave compared with the voltage wave. The tangent of the angle of lag between each harmonic of current and its corresponding harmonic of voltage is obviously proportional to the frequency of the particular harmonic current, in this instance.

The above conditions are illustrated in Fig. 139. The upper curves show, in a full line the voltage wave E, and in broken lines its component harmonic waves, which are the fundamental, the third, and the fifth harmonics. This voltage wave is impressed on a circuit containing resistance and inductive reactance; namely, 10 ohms resistance and .01 henry self-inductance, the fundamental frequency being 60 cycles per second. The lower curves in Fig. 139 indicate in broken lines the harmonics of current corresponding to the harmonics of voltage, while the resultant current is shown by a full line. The impedance increases with the frequency, that encountered by the first harmonic being 10.7 ohms, by the third 15.1 ohms, and by the fifth 21.4 ohms. It will also be observed that the resultant wave of current is less irregular than the voltage wave producing it, showing the effect of inductance in a circuit in smoothing out the irregularities in the current waves and dampening the effects of the higher harmonics, provided the inductance is fixed in value.

3. If the inductive reactance is negligible, capacity reactance only is to be considered. In this instance the reactance encountered by each harmonic of current is inversely proportional to the frequency of the harmonic, and the impedance encountered by each harmonic is therefore smaller as the frequency of the harmonic is larger. Consequently, the current produced in a condenser by an irregular impressed voltage is more irregular than the impressed voltage. Each harmonic of the voltage is represented by an harmonic of current, and

the higher harmonics are exaggerated in comparison with the fundamental. In the instance of a condenser without resist-

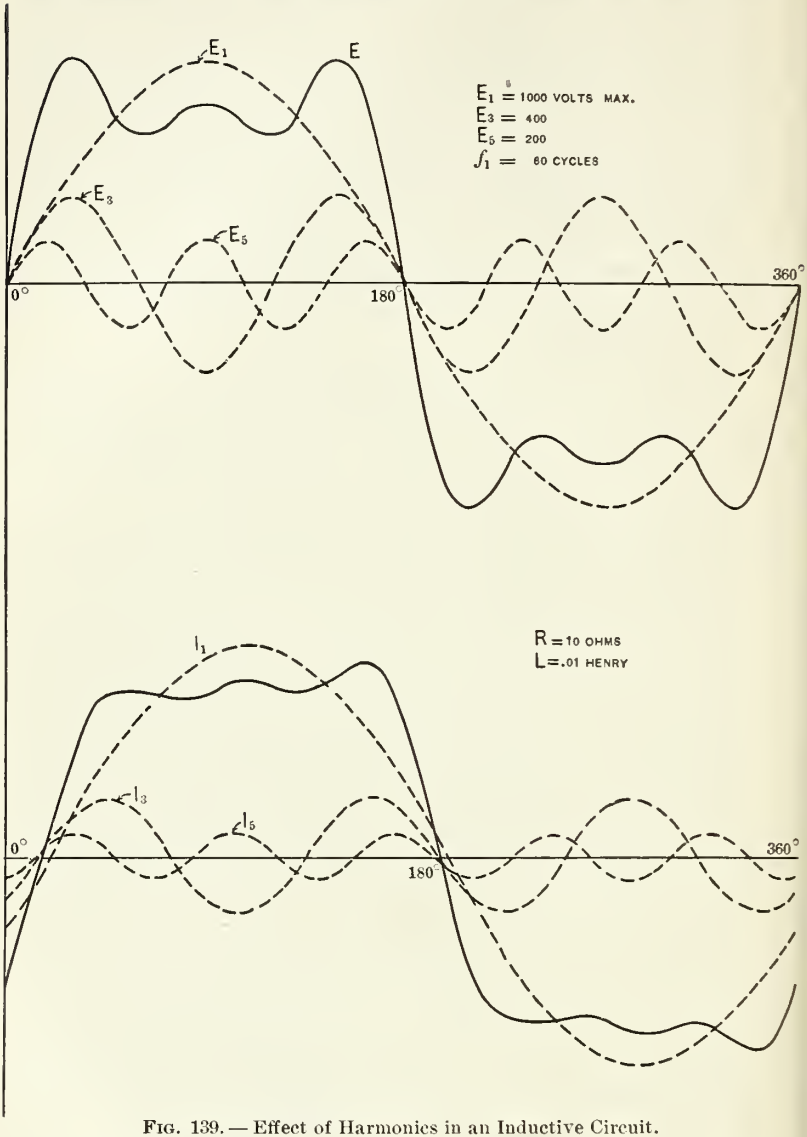


FIG. 139. — Effect of Harmonics in an Inductive Circuit.

ance, the relative amplitudes of the higher harmonics of current increase over the relative amplitudes of the harmonics of impressed voltage in direct proportion to their several frequencies.

The lag angles, θ_1, θ_2, etc., of the several current harmonics with respect to the corresponding voltage harmonics, are obviously smaller as the frequencies of the harmonics are greater. The tangent of each angle of lag is inversely proportional to the frequency of the particular harmonic.

These conditions of a circuit containing capacity reactance are illustrated in Fig. 140. In this case the same voltage as

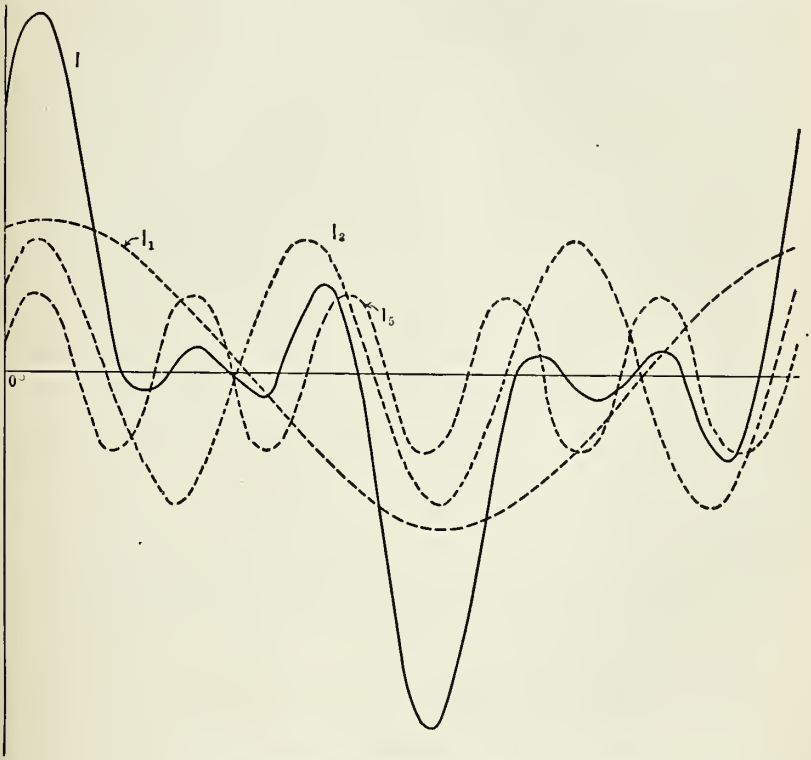


FIG. 140. — Effect of Harmonics in Capacity Circuit.

that shown in Fig. 139 is impressed on a circuit containing a resistance of 10 ohms and a capacity of 100 microfarads. The harmonics of current corresponding to the harmonics of voltage are indicated in Fig. 140 by broken lines, and the resultant current is shown by a full line. The impedance encountered by the first harmonic is 28.4 ohms, by the third 13.4 ohms, and by the fifth 11.3 ohms, while the respective angles of lag for the current harmonics are $\theta_1 = 69° 22'$; $\theta_3 = 41° 30'$; $\theta_5 = 27° 57'$.

It is observed that the resultant current is *more* irregular than the voltage producing it, showing that the effect of capacity in a circuit is to exaggerate the higher harmonics and distort the resultant wave.

4. When the circuit contains both capacity and self-inductance, these two factors of the reactance vary differently with frequency and the effects on the current harmonics are not so simply predicted. Taking the expression $2\pi fL - \frac{1}{2\pi fC}$ to represent the reactance, it will be observed: that very low frequencies, in general, make the first term negligible compared with the second; that very high frequencies make the second term negligible compared with the first; and that the two terms are equal to each other and the reactance reduces to zero at some intermediate frequency. Whatever may be the particular numerical values of L and C in any instance, some frequency will make $2\pi fL = \frac{1}{2\pi fC}$. This frequency is, plainly, $f = \frac{1}{2\pi\sqrt{LC}}$. A higher frequency makes $2\pi fL > \frac{1}{2\pi fC}$, and a lower frequency makes $2\pi fL < \frac{1}{2\pi fC}$. The condition in which $2\pi fL = \frac{1}{2\pi fC}$, in which $X = 0$ and $Z = R$, is called the condition of **Resonance**, in analogy with the tuning of a wire or a windpipe by adjusting the inertia and elasticity of the vibrating medium so as to afford a maximum natural amplitude of vibration at a particular frequency of disturbance.

Applying the above considerations to the circuit conditions when, for instance, the numerical values of L and C make $2\pi fL < \frac{1}{2\pi fC}$ for the fundamental frequency of the impressed voltage, it is to be readily seen that the impedance encountered by higher harmonics of current will decrease up to a harmonic of the frequency at which $2\pi fL = \frac{1}{2\pi fC}$, and the impedance overcome by the harmonic of that particular frequency is equal to the resistance of the circuit. Current harmonics of still higher frequency encounter impedance which is again greater than R and increases with the frequency. The angles of lag for the

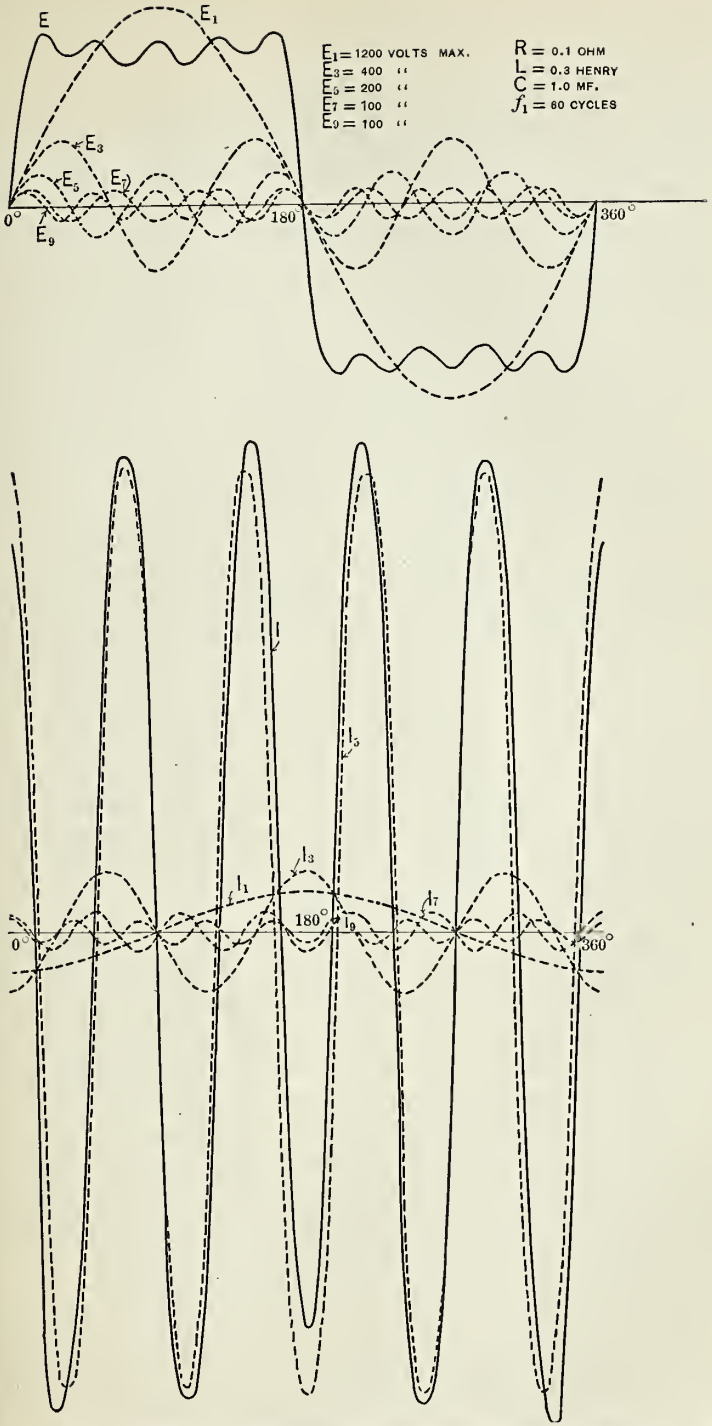


FIG. 141. — Effect of Harmonics in Circuit containing R, L, and C. (235)

various harmonics, as the frequencies increase, change from negative values through zero to positive values. In consequence of these relations, the form of the current wave in a circuit of this character may be greatly distorted from the form of the wave of impressed voltage, through the exaggeration of a certain harmonic or harmonics and the shifting of the relative positions of the higher harmonics of current with respect to corresponding voltage harmonics. These conditions are illustrated in Fig. 141. The upper curves show a voltage wave in a full line and its harmonics in broken lines. It is observed that the harmonics are the first, third, fifth, seventh, and ninth. This voltage wave is impressed on a circuit of .1 ohm resistance, .3 henry self-inductance and 1 microfarad capacity in series. The fundamental frequency is 60 periods per second. The impedance encountered by the fundamental (first) harmonic of current is 2539.4 ohms; the impedance encountered by the third harmonic is 544.9 ohms; by the fifth harmonic is 35 ohms, as at this frequency the condition of resonance is almost attained; the impedance encountered by the seventh harmonic is 412.8 ohms; and the impedance encountered by the ninth harmonic of current is 725 ohms. The respective angles of lag are each nearly 90°, being negative for the first two harmonics and positive for the others. The remarkable exaggeration of the fifth current harmonic caused by resonance illustrates clearly the circumstances here discussed.

Looking upon the foregoing considerations, it is to be observed that the form and magnitude of the voltage wave have been assumed, and the conditions of current flow have been derived. Conversely, in case a current of particular form and magnitude is imposed on a circuit, the form and amplitude of the voltage observed at the terminals of the circuit or any part thereof are dependent upon the current harmonics and the constants of the circuit. In these cases it is entirely possible to have high, and possibly injurious, voltages set up in parts of the circuit, although the effective value of the voltage at the main circuit terminals is within ordinary bounds.

Prob. 1. A certain circuit has a self-inductance of .01 of a henry and a resistance of 10 ohms, the voltage impressed upon this circuit has a frequency of 60 periods per second. The

voltage is composed of the first four harmonics having the respective effective values of 100, 75, 50, and 25 volts. What current flows under influence of each one of these harmonics, and what is the impedance opposed to the current of each harmonic? What is the effective value of the resultant current?

Prob. 2. The voltage wave of problem 1 is impressed upon a circuit of 10 ohms resistance and 200 microfarads capacity. Answer the questions asked in problem 1.

Prob. 3. The voltage wave of problem 1 is impressed on a circuit of .02 henry self-inductance and 50 microfarads capacity. Answer the questions asked in problem 1.

Prob. 4. Answer problems 1, 2, and 3 when the frequency is changed to 120 cycles per second without changing the shape of the voltage wave.

Prob. 5. Answer problems 1, 2, and 3 when the frequency is changed to 25 cycles per second without changing the shape of the voltage wave.

70. The Envelope of the Current Vector when the Conditions in Circuit Vary. — (*a*) A consideration of the effects of the variation of the constants in a series circuit gives rise to the following theorems for the locus of the current vector when the voltage and frequency are fixed.

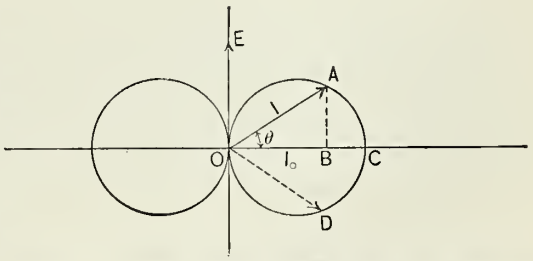


FIG. 142. — Current Locus in Series Circuit with Constant Reactance and Varied Resistance.

Case (1). *Reactance Constant and Resistance Varied.* — In Fig. 142 let OE represent the impressed voltage E and OC represent the current $I_0 = \frac{E}{X}$ when $R = 0$, in which case it is obvious that OC must lag (or lead) OE by 90°. Let OA represent the current I for some value of R between zero and infinity.

Now $$I = \frac{E}{Z} \text{ and } I_0 = \frac{E}{X}.$$

Hence, $$IZ = I_0 X \text{ and } I = I_0 \frac{X}{Z}.$$

But $$\frac{X}{Z} = \frac{OB}{OA} = \frac{OA}{OC} = \cos\theta.$$

Hence, $$I = I_0 \cos\theta,$$

which is the equation of a circle whose diameter is $I_0 = \frac{E}{X}$, and the circle is the locus of the current vector when the reactance is constant and the resistance varies.

A consideration of the limits in the variation of R shows, as above stated, that when $R = 0$, $I = \frac{E}{X}$, and when $R = \infty$, $I = 0$; and thus when R is positive, the current vector is limited to the semicircle OAC. Now suppose a negative resistance is introduced in the circuit, $-R$ varying from 0 to ∞. Negative resistance may be physically considered as the ratio of a voltage to the current flowing, where the voltage may be that of a generator, transformer or other means by which mechanical, chemical or other energy is transformed into electrical energy, instead of electrical energy being transformed into heat or mechanical energy as in ordinary resistance. The current must have a component in opposition to the circuit fall of potential and thus lies in the lower semicircle ODC.

It will be observed that the diameter $OC = I_0 = \frac{E}{X}$ will fall 90° to the right or left of the voltage vector OE according as inductive reactance or capacity reactance predominates. Thus in Fig. 142 the circle to the right of the voltage vector represents conditions when inductive reactance predominates, while the circle to the left is the locus of the current when capacity reactance prevails.

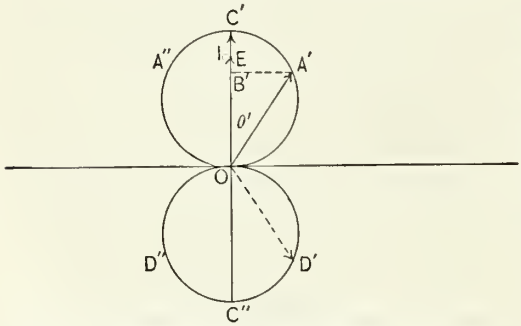


Fig. 143. — Current Locus in Series Circuit with Constant Resistance and Varied Reactance.

Case (2). *Resistance Constant and Reactance Varied.* — As before, let OE, Fig. 143, represent the impressed voltage and let OC' represent the current $I_0 = \frac{E}{R}$ when the reactance $X = 0$, in which case it will obviously be in phase with OE since there is only pure resistance in the circuit. Let OA' represent the current for some value of X between 0 and infinity. Evidently when

$$X = 0, \quad I = \frac{E}{R},$$

$$X = \infty, \quad I = 0.$$

Now $$I = \frac{E}{Z} \text{ and } I_0 = \frac{E}{R}.$$

Hence, $$IZ = I_0 R \text{ and } I = I_0 \frac{R}{Z}.$$

But $$\frac{R}{Z} = \frac{OB'}{OA'} = \frac{OA'}{OC'} = \cos \theta'.$$

Hence, $$I = I_0 \cos \theta',$$

which is the equation of a circle whose diameter is $I_0 = \frac{E}{R}$.

Following the reasoning of Case (1), when R is considered as positive and X is positive, the semicircle $OA'C'$ is the limit of the current vector, and when X is negative the semicircle $OA''C'$ is the limit of the vector. When R is considered as negative, the current becomes opposed to the voltage and comprises a component 180° from it, whence the circle $OD'C''D''$ is the limit of the current vector. As in Case (1) the current vector will fall to the right or left of the voltage OE by an angle θ' of lag or lead, according as inductive or capacity reactance predominates.

(*b*) The converse of the above cases, namely, when the resistance and reactance are constant but the voltage and frequency vary in turn, indicates the following:

Case (3). *Voltage fixed and Frequency Varied.* — Any variation in the frequency must cause a variation in the value of the reactance according to the value of

$$\left(2\pi fL - \frac{1}{2\pi fC}\right).$$

The effect of variation of the reactance, however, has already been shown in Case (2), and the deductions of Case (2) apply to this case.

Case (4). *Frequency fixed and Voltage Varied.* — Under these conditions it is obvious that the current vector is fixed in phase according to the constants of the circuit, but its magnitude increases or decreases as the voltage varies.

Case (5). *Voltage and Frequency varied proportionally together.* — In this case, if the reactance is purely inductive, the rise of voltage matches the increase in reactance, and the proportion of the total volts absorbed in the reactance per ampere of current is unaffected, but the proportion of the total volts absorbed in the resistance per ampere of current by the IR drop varies inversely with the voltage. The effect is therefore like Case (1), in which the reactance is constant and the resistance varies. When voltage is zero, frequency and reactance are zero, and $I = \frac{0}{R} = 0$, which gives the same result as $I = \frac{E}{\infty} = 0$ and corresponds to the point O of Fig. 142. When voltage increases toward the limit of infinity, frequency and reactance increase toward the same limit and R becomes of negligible influence on the impedance. This condition, therefore, gives the same result as $I_0 = \frac{E}{2\pi fL}$ and corresponds to the point C in Fig. 142. The two circles of Fig. 142 correspond to E positive and $2\pi fL$ positive for the upper half of the right-hand circle. E negative and $2\pi fL$ positive for the lower half of the right-hand circle, E positive and $2\pi fL$ negative for the upper half of the left-hand circle, and E negative and $2\pi fL$ negative for the lower half of the left-hand circle. The negative values for $2\pi fL$ may be secured by substituting the effects of mutual induction in the place of self-induction.

When capacity is associated with resistance either alone or in company with inductance, the reactance varies in a different relation with the frequency, and the relationships of the current vector are less simple to illustrate.

CHAPTER V

THE USE OF COMPLEX QUANTITIES EXTENDED

71. Vector Analysis and the Complex Quantity. — It must be continually borne in mind that as far as mathematics are useful to the engineer, or for developing the applications and extending our knowledge of discoveries in science, they must be considered as a system of logic, which may be conveniently applied to the premises which we possess for the purpose of disclosing or predetermining additional phenomena. When they are used in this manner, it is necessary for us to obtain adequate physical conceptions of the results of all mathematical transformations. Our ordinary systems of mathematics are based upon certain assumed premises called axioms, and it is clear that if, by abandoning these axioms and accepting other and different premises which we may or may not call axioms, we can thereby obtain a more convenient and satisfactory system of logic to apply to our ends, then we are justified in abandoning the old and adopting the new or of adding the new to the old. This is what we do to a certain extent when we add the processes of vector analysis to the processes of Euclidian and Cartesian geometry for the solution of alternating-current problems. We do an analogous thing when we abandon the ordinary geometry and adopt Hamilton's quaternion methods or Grasmann's powerful space analysis for treating physical problems involving vectors in space. It is just as irrational for us to confine our attention to the mathematics of a line, as is done in ordinary algebra, when we can usefully extend our consideration to an entire plane or to space, as it was absurd for the early algebraists to neglect all but positive roots in their solutions of equations. The old-time axioms might still be conclusive if viewed within the narrow horizon of Euclid's time, but in our wider horizon they extend to but a small portion of the range of our observations and we can properly extend our methods of reasoning to correspond with the borders of our knowledge.

Until the engineer has learned that the place of mathematics in his professional training is no other than that of a system of logic and therefore a tool, it is unsafe for him to use any form of mathematics higher than arithmetic and the elements of ordinary algebra; but when it is recognized that mathematics compose a system of logic and, indeed, a system of the most powerful kind which may be used in improving engineering processes, the propriety of the use of various branches of higher mathematics in engineering studies is manifest.

In Art. 6 the following forms and conditions have been given for the vector or complex quantity:

$$\bar{A} = a + jb,$$

$$\bar{A} = m(\cos\theta + j\sin\theta) = me^{j\theta},*$$

where m is the **Tensor** and θ the **Argument** of the vector quantity and a and b are the lengths of rectangular components of the vector. Also,

$$m^2 = a^2 + b^2$$

and

$$\tan\theta = \frac{b}{a} = \frac{\sin\theta}{\cos\theta}.$$

In these expressions the scalar values of $\bar{A}$ and m are equal, but the former is a vector having direction, while m may be considered as a simple number. The expression $(\cos\theta + j\sin\theta) = \epsilon^{j\theta}$ is sometimes called the **Direction coefficient** or **Versor** of the expression of which it is a part. It may be abbreviated into (cjs) θ. The tensor (also sometimes called the modulus) of the vector represents its actual numerical value and it is therefore a "scalar" quantity; that is, it is a quantity which has numerical or scale value only, without fixed direction. It is by reason of the latter fact that it is sometimes called the "absolute value" of the vector. The versor of a vector has direction fixed by the argument θ, but is of unit length only. The product of versor by tensor therefore fixes the vector in terms of both length and direction.

The operator j and the coplaner-vector quantities treated in this book are assumed to be subject to the usual algebraic laws,

* See Williamson's *Differential Calculus*, pp. 60–69.

like the law of signs, the law of indices, and, in general, the commutative, associative, and distributive laws.*

Such characteristics have been given the operator j that the imaginary quantity $\sqrt{-1}$ may be substituted for it, and the equation

$$a + jb = a + \sqrt{-1}\, b$$

holds true; for $j^2 = -1$ by definition. Hence $j = \sqrt{-1}$.† The symbol $\sqrt{-1}$ may therefore be considered to have a physical meaning like that of the operator j, and if it is met with in the roots of an equation or elsewhere, it may be replaced by j. When interpreted in this way, it is to be observed that an "imaginary" expression assumes its true significance of a vector quantity having a numerical magnitude and definite direction. The real roots of an equation or any other "real" expression may similarly be considered as vector quantities, all of which lie on the same line which is at right angles to the axis of "imaginaries."

Since $j^2 = -1$, $-j^2 = +1$, from which follows the relation $\frac{1}{j} = -j$; or, in words, the effect of applying the reciprocal of the operator to a vector quantity is the same as the effect of applying the reversed operator.

In general it must be remembered that any direction is represented equally by θ, $\theta + 2\pi$, $\theta + 4\pi$, etc., to an infinite number of angles, so that the argument of the vector has many values, except where the conditions specially fix it. We shall, as a rule, consider it for convenience as equal to θ even where it is not fixed by the conditions; but where distinct generalization of the argument is desirable, we shall write it $\theta + 2\pi n$. The complex quantity which has been developed lies in one plane and its particularly marked simplicity is a result of its uniplaner condition.

Coplaner (uniplaner) vectors, with their corresponding complex representations, are quite useful in representing the relations arising in problems which are caused by current flow in alternating-current circuits. These problems deal with periodic quantities which may be treated as though they were the result

* For a discussion of these laws in relation to vector algebra, see Hayward's *Vector Algebra and Trigonometry*, pp. 2, 14, and 39.

† Art. 6.

of uniform rotating vectors, or stationary vectors in a plane which itself rotates about a perpendicular axis. The methods used in vector algebra are analogous to those which are so largely used in the graphical solutions relating to the composition and resolution of forces in graphical statics and to the composition and resolution of velocities and moving forces in graphical dynamics. The alternating-current vectors are moving vectors, and, consequently, the methods must be similar to those used in the vector processes applied to dynamics.

72. Addition and Subtraction of Complex Expressions. — Addition and subtraction of these expressions, as has already been indicated,* is accomplished as in ordinary algebraic operations; as, for instance,

$$(a+jb)+(c-jd)=a+c+j(b-d),$$
$$(a+jb)-(c-jd)=a-c+j(b+d).$$

The rules for addition are, — reals must be added numerically to reals, and imaginaries must be added numerically to imaginaries. Whence, when addition is performed, (1) the horizontal component of the resulting vector is equal to the sum of the horizontal components of the vectors added; (2) the vertical component of the resulting vector is equal to the sum of the vertical components of the vectors added.

Subtraction being the reverse process gives: (1) the horizontal component of the resulting vector is equal to the difference between the horizontal components of the minuend and subtrahend; (2) the vertical component of the resulting vector is equal to the difference between the vertical components of the minuend and subtrahend.

In case an expression has only a horizontal or only a vertical component, then, of course, the addition or subtraction is carried on as though the lacking component were equal to zero.

73. Multiplication and Division of Complex Expressions. — The process of multiplication or division of a complex by a scalar quantity (that is, a quantity which has only a numerical value) is similar to the corresponding ordinary arithmetical process of multiplication or division. For instance,

$$(a+jb)\times 2 \text{ is } 2(a+jb)=2a+j\,2b$$
and
$$(a+jb)\div 2 \text{ is } \tfrac{1}{2}(a+jb)=\tfrac{1}{2}a+j\,\tfrac{1}{2}b.$$

* Art. 8.

This is illustrated in Fig. 144 where OA is a vector. If this vector is multiplied by 2, its direction is not changed, but its length is doubled and the vector becomes OA'. The proportions of the vector show plainly that the horizontal component of OA' is equal to $2a$ and the vertical component is equal to $j2b$. If OA is divided by 2, it becomes a vector of $\frac{1}{2}$ the length and the same direction as before, and is represented by the vector OA_1. The proportions of the vector again show clearly that the horizontal component now is equal to $\frac{1}{2}a$ and the vertical component is equal to $j\frac{1}{2}b$.

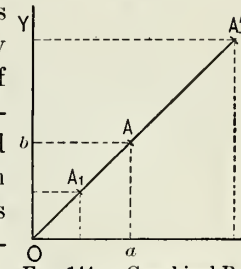


FIG. 144. — Graphical Representation of a Vector Quantity multiplied or divided by a Scalar Quantity.

Algebraical processes also immediately apply in the multiplication or division of complex quantities by complex quantities, as will be more fully illustrated. We will suppose that the two complex quantities $a+jb$ and $c+jd$ are to be multiplied together and that these quantities are equal, respectively, to $m_1(\cos\theta_1 + j\sin\theta_1) = m_1\epsilon^{j\theta_1}$ and $m_2(\cos\theta_2 + j\sin\theta_2) = m_2\epsilon^{j\theta_2}$.

Multiplying the exponential forms of the two expressions together gives as a result

$$m_1m_2\epsilon^{j(\theta_1+\theta_2)} = m_1m_2[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)].$$

If we directly multiply together the expressions

$$m_1(\cos\theta_1 + j\sin\theta_1) \text{ and } m_2(\cos\theta_2 + j\sin\theta_2),$$

we immediately come to the expression

$$m_1m_2[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)].$$

We therefore have a satisfactory demonstration that the ordinary process of multiplication in algebra applies to the multiplication of these coplaner-vector expressions. Consequently,

$(a+jb)(c+jd)$ must be equal to $ac + jad + jcb + j^2bd$,

which, again, is equal to

$$ac - bd + j(bc + ad).$$

This shows that

$ac - bd$ is equal to $m_1m_2\cos(\theta_1+\theta_2)$

and $bc + ad$ is equal to $m_1m_2\sin(\theta_1+\theta_2)$.

These equalities may also be proved by expanding the expressions $m_1 m_2 \cos(\theta_1 + \theta_2)$ and $m_1 m_2 \sin(\theta_1 + \theta_2)$, and assigning to $m_1 \sin \theta_1$, $m_1 \cos \theta_1$, $m_2 \sin \theta_2$, $m_2 \cos \theta_2$ their respective values in terms of a, b, c, and d, indicated by Fig. 145.

By an inspection of the trigonometrical form of the foregoing product, it is made evident that the product of two complex quantities is another complex quantity in which *the tensor or modulus is equal to the product of the tensors of the multiplier and multiplicand*, and in which *the argument is equal to the sum of the arguments of multiplier and multiplicand* (see Fig. 145); also that the product vector has a horizontal component which is equal to the difference between the products of the horizontal components and the vertical components respectively of multiplier and multiplicand, and a vertical component which is equal to the sum of the product of the horizontal component of the multiplier with the vertical component of the multiplicand and the product of the vertical component of the multiplier with the horizontal component of the multiplicand.

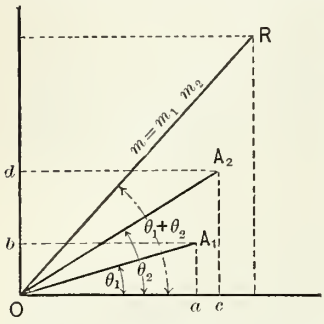


FIG. 145.—The Product of Two Vectors OA_1 and OA_2 is OR, in which OR has a length equal to the length of OA_1 multplied by the length of OA_2.

By **Complementary vectors** are meant vectors with equal horizontal components, equal vertical components, and equal but reversed arguments. Thus, if we desire to multiply $a + jb$ by $a - jb$ (*i.e.* a vector quantity by its **Complement**), we get the following result,

$$(a + jb)(a - jb) = a^2 - jab + jab - j^2 b^2 = a^2 + b^2 = m^2;$$

and again

$$(a + jb)(a - jb) = m^2 [\cos(\theta - \theta) + j \sin(\theta - \theta)] = m^2.$$

The product of a vector and its complement therefore is a vector with zero argument and may be considered as an algebraic value equal to the sum of the squares of the two components of either of the complementary vectors.

In division a similar process may be followed. We have

$$(a+jb)\div(c+jd)=m_1\epsilon^{j\theta_1}\div m_2\epsilon^{j\theta_2}=\frac{m_1}{m_2}[\epsilon^{j(\theta_1-\theta_2)}]$$
$$=\frac{m_1}{m_2}[\cos(\theta_1-\theta_2)+j\sin(\theta_1-\theta_2)].$$

The demonstration of this follows in a manner identical with the demonstration for multiplication given above. The ratio of two vectors is therefore another vector in which *the tensor is equal to the ratio of the tensors of dividend and divisor* and *the argument is equal to the difference of the arguments of dividend and divisor.*

Also multiplying both numerator and denominator by $(c-jd)$ and remembering that $-j^2=+1$, there results

$$\frac{a+jb}{c+jd}=\frac{(a+jb)(c-jd)}{c^2+d^2}.$$

The denominator in this case is a real or arithmetical quantity and the numerator is equal to the product of two vectors. Expanding shows that the above ratio is equal to

$$\frac{ac+bd+j(bc-ad)}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+j\,\frac{bc-ad}{c^2+d^2};$$

whence,
$$\frac{m_1}{m_2}\cos(\theta_1-\theta_2)=\frac{ac+bd}{c^2+d^2}$$

and
$$\frac{m_1}{m_2}\sin(\theta_1-\theta_2)=\frac{bc-ad}{c^2+d^2}.$$

Since m_2 is numerically equal to the square root of c^2+d^2, it is clear that

$$m_1\cos(\theta_1-\theta_2)=\frac{ac+bd}{(c^2+d^2)^{\frac{1}{2}}}$$

and
$$m_1\sin(\theta_1-\theta_2)=\frac{bc-ad}{(c^2+d^2)^{\frac{1}{2}}}.$$

The foregoing theorems of multiplication and division are equally applicable to stationary vectors and to rotating vectors of equal frequency. The multiplication together of rotating vectors of unequal frequencies gives a different result for the reason that θ_1 and θ_2 do not increase at the same rate. Since $\int_0^\pi \sin m\alpha \sin n\alpha d\alpha = 0$ when m and n are unequal integers, it is obvious that the product of a rotating vector by a rotating

vector having a different frequency which is any integral number of times the frequency of the first vector, is equal to zero. This also follows from the usual theorems of the products of simple harmonic motions. Also, since

$$\int \sin m\alpha \sin n\alpha d\alpha = \frac{1}{2}\left[\frac{\sin(m-n)\alpha}{m-n} - \frac{\sin(m+n)\alpha}{m+n}\right],$$

it is obvious that the product of the two rotating vectors does not reduce to zero in one-half period in case n is a mixed number instead of an integer, but the product does reduce to zero in the number of half periods equal to the reciprocal of the fractional part of n. The product of two rotating vectors of different frequencies, therefore, always reduces to zero in some definite number of half periods, although it may have a large finite value for some particular half period.

74. Reciprocal of a Vector Expression. — The reciprocal of a vector may be written this way:

$$\frac{1}{A} = \frac{1}{a+jb} = \frac{1}{m(\cos\theta + j\sin\theta)} = \frac{1}{me^{j\theta}} = \frac{1}{m}\epsilon^{-j\theta}$$

$$= \frac{1}{m}[\cos(-\theta) + j\sin(-\theta)].$$

Also since

$$a - jb = m(\cos\theta - j\sin\theta) = m[\cos(-\theta) + j\sin(-\theta)],$$

it is clear that

$$\frac{1}{a+jb} = \frac{a-jb}{m^2}.$$

This may also be proved by rationalization. Thus, multiplying the numerator and denominator by $a - jb$ gives

$$\frac{a-jb}{(a+jb)(a-jb)} = \frac{a-jb}{a^2+b^2} = \frac{a-jb}{m^2} = \frac{a}{m^2} - j\frac{b}{m^2}.$$

Hence, *the reciprocal of a vector is a vector whose tensor is equal to the reciprocal of the original tensor, and whose argument is equal to the argument of the original vector with reversed sign.*

The horizontal component of the reciprocal is equal to the horizontal component of the original vector divided by the square of the tensor, and the vertical component of the reciprocal is equal to the vertical component of the original vector with reversed sign and divided by the square of the tensor of the original vector.

75. Involution and Evolution of Complex Quantities. — The square of the vector $\bar{A}$ is

$$\bar{A}^2 = (m\epsilon^{j\theta})^2 = m^2\epsilon^{j2\theta} = m^2(\cos 2\theta + j \sin 2\theta).$$

This comes from the rules of multiplication given above; or, the *square of a vector has a tensor equal to the square of the original tensor and an argument equal to twice the original argument.*

Other powers of vectors follow the same laws, which come directly from the rules of multiplication. For instance,

$$(a + jb)^n = [m(\cos\theta + j \sin\theta)]^n = m^n\epsilon^{jn\theta} = m^n(\cos n\theta + j \sin n\theta).$$

If the exponent n is negative, its sign is reversed in the above expression as compared with the expression when n is positive. We then have the reciprocal of the tensor and the reversed angle.

Also

$$(a+jb)^{\frac{1}{n}} = [m(\cos\theta + j \sin\theta)]^{\frac{1}{n}} = m^{\frac{1}{n}}\epsilon^{j\frac{\theta}{n}} = m^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + j \sin\frac{\theta}{n}\right).$$

From this it can be seen that evolution is clearly a process of multiplication and that involution may be looked upon either as a process of multiplication or a process of division.

It has been previously shown that $j^{-1} = \frac{1}{j} = -j$; that is, j^{-1} is an indicator of rotation backwards by the amount of one quadrant. In the same way j^n indicates a rotation forward by n quadrants and $j^{-n} = \frac{1}{j^n} = -j^n$ indicates rotation backwards by n quadrants.

In case the vector $a + jb$ is squared, we have

$$(a + jb)^2 = m^2(\cos 2\theta + j \sin 2\theta),$$

and this is also equal to

$$a^2 + j\,2ab + j^2b^2 = a^2 - b^2 + j\,2ab,$$

and therefore

$$m^2 \cos 2\theta = a^2 - b^2$$

and

$$m^2 \sin 2\theta = 2ab.$$

The square of a vector therefore has a horizontal component equal to the difference of the squares of the horizontal and vertical components of the vector, and a vertical component which is equal to twice the product of the horizontal and vertical components of the vector.

In taking the root (such as the nth root) of a vector, we get the product of the given root of the tensor and $\cos\frac{\theta}{n}+j\sin\frac{\theta}{n}$. That is, the tensor of the root of a vector is equal to the real algebraic root of the tensor of the vector. But the root of a vector has direction of its own which depends on the argument of the original vector. The directions of the root vectors are given by $\frac{\theta}{n}$, which affords n different directions. For instance, let $\theta=30°$ and $n=2$; then $\frac{\theta}{2}=15°$, $\frac{\theta+360°}{2}=195°$, $\frac{\theta+720°}{2}=375°=15°$ etc., giving just two roots which have opposite directions. Again, let $\theta=30°$ and $n=5$; then,

$$\frac{\theta}{5}=6°,\ \frac{\theta+360°}{5}=78°,\ \frac{\theta+720°}{5}=150°,\ \frac{\theta+1080°}{5}=222°,$$

$$\frac{\theta+1440°}{5}=294°,\ \frac{\theta+1800°}{5}=6°,\text{ etc.,}$$

giving five different directions for the five roots.

Again, let $\theta=0$, and $n=2$, then $\frac{\theta}{2}=0°$, $\frac{\theta+360°}{2}=180°$, $\frac{\theta+720°}{2}=360°=0°$, etc., which shows that the square roots of a real quantity, that is, a quantity with a zero argument, are both real quantities (that is, they lie in the horizontal axis of reals) and are of opposite signs. Again, letting $\theta=0$, but taking $n=4$, gives $\frac{\theta}{4}=0°$, $\frac{\theta+360°}{4}=90°$, $\frac{\theta+720°}{4}=180°$, $\frac{\theta+1080°}{4}=270°$, $\frac{\theta+1440°}{4}=360°=0°$, etc.,

showing that the fourth roots of a real quantity are in opposite pairs, of which one pair are reals and the other pair pure imaginaries.

Following up in this manner our rule for evolution, it will be observed that the nth root of any vector quantity has a tensor of fixed value equal to the nth root of the original tensor, but that there are in fact n root vectors, each with a direction of its own.

The n roots of $(1)^{\frac{1}{n}}$ are

$$\sqrt[n]{1}\left(\cos\frac{2\pi k}{n}+j\sin\frac{2\pi k}{n}\right)=\sqrt[n]{1}\,\epsilon^{j\frac{2\pi k}{n}},$$

where k is to be given the values 0, 1, 2, $3 \cdots n-1$, taken in order. In this case (1) is considered to be a vector lying along the horizontal axis of reals, and therefore $\theta = 0$. Since the nth root of (1) is a vector with argument $\frac{2\pi}{n}$, it is clear that multiplying a vector by $(1)^{\frac{1}{n}}$ causes rotation by an angle $\frac{2\pi}{n}$. We have here a justification of our use of the imaginary which enables us to determine all the roots of an equation. It would be just as irrational to neglect the imaginary roots as it was in the early algebraists to also neglect the negative roots and consider only the positive roots as of any significance.

76. Differentiation and Integration of Complex Quantities. — Differentiation of a vector quantity $d(\bar{A}) = d(a + jb)$ becomes, in conformity with the distributive law of algebra, the same as $da + d(jb)$, and this becomes in conformity with the commutative law the same as $da + jdb$. The j here acts like any constant quantity in an equation which is differentiated and serves as an operator or indicator to show that db is measured vertically. We can illustrate the condition by Fig. 146 in which OA represents the vector and AA' may be considered to be its increment. AA' therefore is equal to $d(OA)$, and da and jdb are respectively the horizontal and vertical components of the increment.

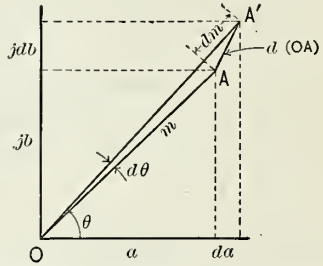


FIG. 146. — Illustration of a Vector increased by an Increment.

It is evident that an integration performed on the expression $da + jdb$ must result in $a + jb$, provided the above reasoning is correct.

Suppose the process of differentiation is applied to the expression $m(\cos\theta + j\sin\theta)$. We then have

$$d[m(\cos\theta + j\sin\theta)] = dm(\cos\theta + j\sin\theta) + md(\cos\theta + j\sin\theta)$$
$$= dm(\cos\theta + j\sin\theta) + m(j\cos\theta - \sin\theta)d\theta.$$

Now writing $(\text{cjs})\theta$ for the direction coefficient $\cos\theta + j\sin\theta$, the foregoing becomes

$$(\text{cjs})\theta dm + jm(\text{cjs})\theta d\theta = (dm + jmd\theta)(\text{cjs})\theta.$$

This, then, is the differential of the expression $a + jb$. If the

expression is in the form $m\epsilon^{j\theta}$ and we differentiate it, we have this result,

$$(\epsilon^{j\theta})dm + jm\epsilon^{j\theta}d\theta = (dm + jmd\theta)\epsilon^{j\theta}.$$

It will also be observed from this that the real part of the differential is

$$da = \cos\theta dm - m\sin\theta d\theta;$$

and the imaginary part is

$$jdb = j(\sin\theta dm + m\cos\theta d\theta).$$

We have written (cjs)θ for the expression $\cos\theta + j\sin\theta$ in the preceding paragraphs. The expression (cjs) may be pronounced as though it were spelled "siss," and it can be readily seen that it is derived by combining the initial of cos with the operator j and the initial of sin in the expression $\cos\theta + j\sin\theta$. This abbreviation is a convenient one and will be used hereafter. It should be remembered that (cjs)$\theta = \epsilon^{j\theta}$.

77. **Logarithms of Complex Quantities.** — In the preceding portion relating to the differentiation of complex quantities, we have shown that where $\bar{A}$ is a complex quantity equal to m(cjs)θ, then $d\bar{A} = (dm + jmd\theta)(cjs)\theta$; and from these we get $\frac{d\bar{A}}{\bar{A}} = \frac{dm}{m} + jd\theta$, the process performed having been to divide the vector into its differential. Since the natural logarithm of any quantity is by definition equal to the integral of the ratio of the differential of a quantity to that quantity, that is, $\log \bar{A} = \int \frac{d\bar{A}}{\bar{A}}$, it follows that $\log \bar{A} = \int \frac{dm}{m} + \int jd\theta$. Integrating these gives, $\log \bar{A} = \log m + j(\theta + 2\pi n)$, where n is 0 or an integer. The term $2\pi n$ in the final expression may be looked upon as a constant of integration, and it will be noticed that its addition to θ does not change the real direction of the vector, but shows that the angle θ is not fixed except in special cases and that the angle θ may have an infinite number of values differing from each other by $2\pi j$ or $360j°$. These expressions show that the logarithm of a complex number depends upon the logarithm of the tensor of the number and upon its argument, and that any function may have an infinite number of logarithms which arise in an arithmetical series with the difference equal to $2\pi j$.

It is easy to get from this the logarithm of the ratio $\overline{A}_1$ to $\overline{A}_2$; thus,

$$\log \frac{\overline{A}_1}{\overline{A}_2} = \log \overline{A}_1 - \log \overline{A}_2 = \log m_1 + j\theta_1 - \log m_2 - j\theta_2$$
$$= \log \frac{m_1}{m_2} + j(\theta_1 - \theta_2).$$

In the same manner $\log (\overline{A}_1 \overline{A}_2)$ may be shown to be equal to

$$\log m_1 m_2 + j(\theta_1 + \theta_2).$$

78. Combination of Admittances. — In the previous chapters, series circuits have been dealt with in most instances. The following pages of this chapter will deal more particularly with circuits in parallel and series-parallel.

If impedances are in parallel, their reciprocals must be combined, in which case the resultant is the reciprocal of the impedance of the divided circuit. It is evident that the components of the admittance (reciprocal of impedance) will not be equal to the reciprocals of the components of the impedance, but they may be found in terms of the impedance components, as pointed out in Arts. 63 and 74.

The component, g, of the admittance that lies along the horizontal axis is the component termed the **Conductance** and the vertical component, b, the **Susceptance**. The latter is positive in direction if caused by capacity, and negative if caused by self-inductance. If, then, g, b, are the admittance components (conductance and susceptance) and R, X, the impedance components (resistance and reactance) of a single circuit, we may write by the principles of geometric multiplication above stated,

$$\overline{Y} = \frac{1}{\overline{Z}} = g \mp jb = \frac{1}{R \pm jX}. \qquad (a)$$

The numerical values of the first and second terms of the right-hand member of the expression $\overline{Y} = g \mp jb$ are respectively proportional to the active current and quadrature current in a circuit. When a circuit contains inductive reactance only, X is essentially positive, but the quadrature current *lags* 90° behind the active current, so that b is essentially negative. When a circuit contains capacity reactance only, X is essentially negative, but the quadrature current *leads* the active current by 90°, so that b is essentially positive. When a circuit contains both

inductive and capacity reactance, the signs of X and b are dependent upon the relative magnitudes of the inductance and capacity.

To reduce the equation

$$g \mp jb = \frac{1}{R \pm jX} \qquad (b)$$

to a more convenient form, the numerator and denominator of the right-hand member may be multiplied by $R \mp jX$; whence

$$g \mp jb = \frac{R \mp jX}{(R \mp jX)(R \pm jX)} = \frac{R \mp jX}{R^2 + X^2},$$

since j^2 indicates the operation which is equivalent to multiplying by -1.*

Hence, $$g \mp jb = \frac{R}{R^2 + X^2} \mp j\frac{X}{R^2 + X^2};$$

but the numerical value of the impedance is

$$Z = \sqrt{R^2 + X^2}.$$

Therefore $$g \mp jb = \frac{R}{Z^2} \mp j\frac{X}{Z^2}$$

or $$g = \frac{R}{Z^2} \text{ and } b = \frac{X}{Z^2}.$$

If R, X, and Z are known or can be determined from the conditions presented, problems relating to parallel circuits can now be solved.

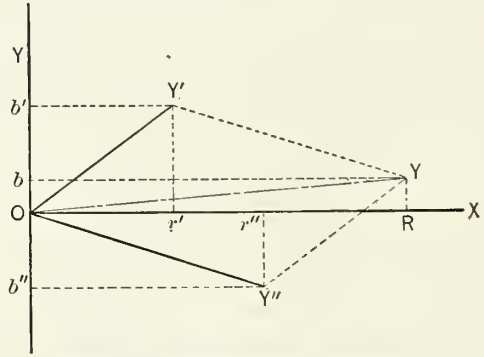


Fig. 147. — Admittances in Parallel.

If in Fig. 147 g', b' and g'', b'' are the components of the admittances of two parallel circuits having impedances Z' and Z'',

* Art. 8.

$$g' - jb' = \frac{R'}{Z'^2} - j\frac{X'}{Z'^2}$$

and

$$g'' + jb'' = \frac{R''}{Z''^2} + j\frac{X''}{Z''^2};$$

and if g and b are the components of the total admittance,

$$\bar{Y} = g + jb = g' + g'' + j(b'' - b') = \frac{R'}{Z'^2} + \frac{R''}{Z''^2} + j\frac{X''}{Z''^2} - j\frac{X'}{Z'^2}.$$

The intrinsic sign of jb depends upon the relative signs and magnitudes of $\frac{X'}{Z'^2}$ and $\frac{X''}{Z''^2}$. The impedance of the circuit is the reciprocal of the admittance thus found.

These processes which enable us to find the joint impedance and admittance of parallel or series circuits when the elements of the individual parts of the circuits are known, equally enable us to find the impedance and admittance of any combination of such circuits by computations which are almost as simple and rapid as those which would be used in dealing with a direct-current system. Also, when the impedances of any combination of circuits have been obtained, it is possible to find the voltages in any portion when a sinusoidal current is flowing and to find the current when a sinusoidal voltage is applied.

The meaning of the terms in the expression for impedance and admittance may be explained by multiplying $(R \pm jX)$ by I (current in the circuit), when it is evident that rI is the active voltage and XI the component of voltage acting against the reactance; and by multiplying $(g \mp jb)$ by E (voltage impressed on the circuit), when gE is the active component of the current and bE the quadrature current.

The following is a recapitulation of the formulas for the analytical solution by geometric processes of many problems relating to alternating-current circuits. The use of a vinculum over a letter (as $\bar{Z}$) indicates its value as a complex quantity and the letter without the vinculum indicates the tensor or numerical value. In this recapitulation the small letters r, x, y, and z represent the resistance, reactance, admittance, and impedance of parts of the circuit while the capitals R, X, Y, and Z represent the same quantities for the whole circuit. In the same manner the capitals G and B stand for circuit parts and the small letters g and b for the whole:

Geometric Equations	Algebraic Equations
$\bar{z} = r \pm jx.$	$g = \frac{R}{Z^2} = RY^2,\ b = \frac{X}{Z^2} = XY^2.$
$\bar{Z} = \Sigma\bar{z} = \Sigma r \pm j\Sigma x$ $= R \pm jX$ $= Z(\cos\theta + j\sin\theta).$	$R = gZ^2 = \frac{g}{Y^2},\ X = bZ^2 = \frac{b}{Y^2}.$
$\bar{y} = \frac{1}{\bar{z}} = G \mp jB$ $= \frac{r}{z^2} \mp j\frac{x}{z^2}.$	$Z = \sqrt{R^2 + X^2} = \frac{R}{\cos\theta} = \frac{X}{\sin\theta}.$ $Y = \sqrt{g^2 + b^2} = \frac{g}{\cos\theta} = \frac{b}{\sin\theta}.$
$\bar{Y} = \Sigma\bar{y} = \Sigma\frac{r}{z^2} \mp j\Sigma\frac{x}{z^2}$ $= \frac{R}{Z^2} \mp \frac{X}{Z^2}.$	$\tan\theta = \frac{X}{R} = \frac{b}{g}.$
$\bar{I} = \frac{E}{\bar{Z}} = E\bar{Y}.$	$I = \frac{E}{Z} = EY.$
$\bar{E} = I\bar{Z} = \frac{I}{\bar{Y}}.$	

The currents and voltages in the geometric equations are true vector quantities with magnitudes and relative directions. The impedances and admittances are of the character of vector operators.

It will be observed that when $\bar{E}$ is computed by means of the last equation in the column of geometric equations, its phase is measured with respect to the current; and when $\bar{I}$ is computed by means of the next to the last equation in the column of geometric equations, its phase is measured with respect to the phase of the voltage. The two measurements give the same angle but it is taken in opposite directions.

CHAPTER VI

SOLUTION OF CIRCUITS — APPLICATION OF GRAPHICAL AND ANALYTICAL METHODS

79. Graphical Methods. — As previously pointed out graphical methods often lend themselves satisfactorily to the solution of problems relating to circuits upon which sinusoidal voltages are impressed. This application of graphical processes was first brought to general attention by T. H. Blakesley.* This chapter is a general review of portions of chapters I and V, but it goes further, dealing extensively with circuits in parallel and with complex combinations of series and parallel circuits. It also develops a number of important general relations which have heretofore not been dealt with.

To go more into detail concerning the graphical combination of vectors than has been done previously, suppose the line OA in Fig. 148 is conceived as swinging at a uniform angular velocity ω around the point O, the angle α which it makes with the horizontal axis OX at any instant is $\alpha = \omega t$, where t is the interval of time during which the line describes the angle α. The instantaneous projection Oa upon the vertical axis OY, of the line OA, has a value $Oa = OA \sin \alpha$. If OA is proportional to the maximum value of a sinusoidal function, its instantaneous values are proportionally represented by the instantaneous projections of OA; and if OA is proportional to the effective value of a sinusoidal function, the instantaneous values of the function are proportionally represented by the product of $\sqrt{2}$ into the corresponding instantaneous projections. It is there-

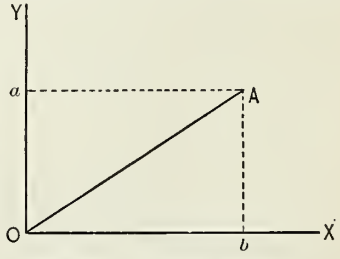


FIG. 148. — Rotating Vector.

* Blakesley's *Alternating Currents of Electricity*.

fore possible to represent all the elements of a sinusoidal function: (1) by a straight line which rotates at a uniform rate around one end; and (2) by the instantaneous projections of the line. It is evident that the motion which the projection of the end A of the rotating line makes along the axis OY is a simple harmonic motion, and that all the theorems relating to simple harmonic motion may be applied to these solutions. As is ordinarily done, the rotation of the line will always be considered to be left-handed; and angles measured from right to left will be considered positive, while those measured from left to right will be considered negative.

If two sinusoidal voltages of the same frequency, but having a phase difference θ, act in a circuit, the corresponding instantaneous values are,

$$e = \sqrt{2}\, E \sin \alpha,$$

$$e' = \sqrt{2}\, E' \sin (\alpha + \theta).$$

The total instantaneous voltage acting in the circuit is $e + e'$. In Fig. 149, the voltage E is represented by the line OA, and the voltage E' by the line OA'. Oa and Oa' are the instantaneous values of the voltage for the angular positions shown. The total instantaneous voltage in the circuit which corresponds to the angular position shown is equal to $Oa + Oa'$, or Oa''. It is readily shown that Oa'' is the projection of the diagonal of the parallelogram constructed upon OA and OA'. This is true for all angular positions, since the sum of the projections of the lines OA and OA' must be equal to the sum of the projections of the lines OA and AA'', which in turn is equal by construction to the projection of the diagonal OA''.

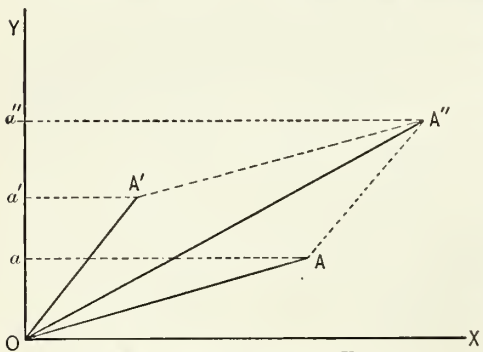


FIG. 149. — Resultant of Two Vectors.

The length of the line OA'' therefore proportionally represents the magnitude of the effective or maximum total voltage in the circuit, and its position relative to that of OA and OA'

represents the relative phase position of the total voltage. If, instead of two voltages acting in a circuit, there are three or more, as OA, OA', OA'', OA''', and OA'''', in Fig. 150, the same construction is used. Thus, completing the parallelogram for OA and OA', their resultant OA_1 is found. Completing the parallelogram for OA_1 and OA'', their resultant OA_2 is found, and again with this and OA''' the resultant OA_3 is obtained; finally OA_4, the final resultant, is obtained by combining OA_3 with OA''''. The figure shows that it is unnecessary to complete all the parallelograms. It is only neces-

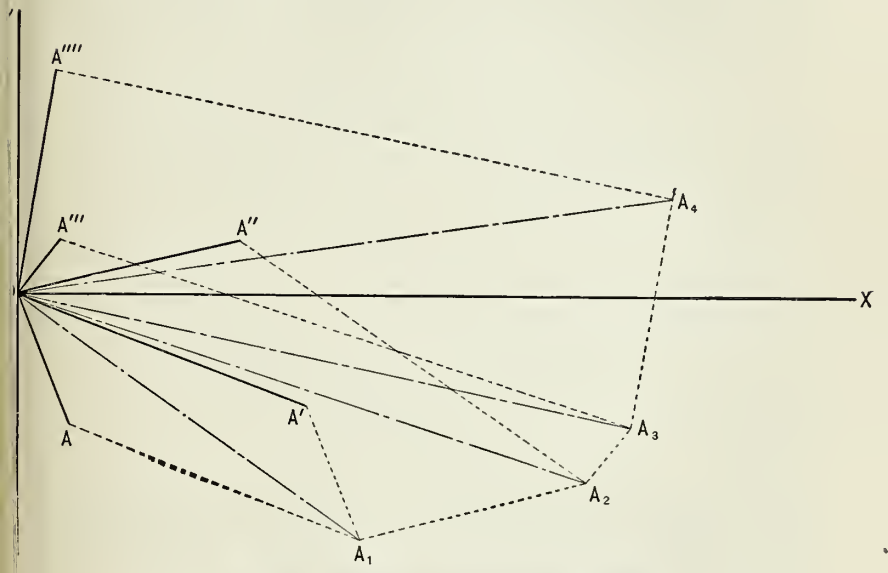


FIG. 150. — Resultant of Three or more Vectors.

sary to draw the lines OA, AA_1, A_1A_2, A_2A_3, A_3A_4 respectively parallel and equal in length to the lines OA, OA', OA'' OA''', and OA'''', and the line drawn from O to the end of the last line laid off gives the phase-position and magnitude of the total voltage in the circuit, regardless of the number of the components from which it is derived (Fig. 151). The composition of voltages is therefore exactly analogous to the composition of velocities or of forces. *As in the case of velocities or forces, the resultant of any number of voltages may be determined by this method.*

The resultant of two sinusoidal alternating currents which

flow in a divided circuit may be graphically determined in the same manner. In Fig. 149, let OA and OA' be the currents in the two reactive branches of a divided circuit. The two partial currents differ from each other in phase by an angle θ. The instantaneous values of the currents are represented by the instantaneous projections of the lines as they revolve around the point O. At each instant, the total current in the main circuit is equal to the sum of the instantaneous partial currents, or to $i + i'$. Consequently, the magnitude of the effective or maximum value of the current in the main circuit is proportionally represented by the length of the line OA'', and the angular direction of OA'' gives the angular relation of

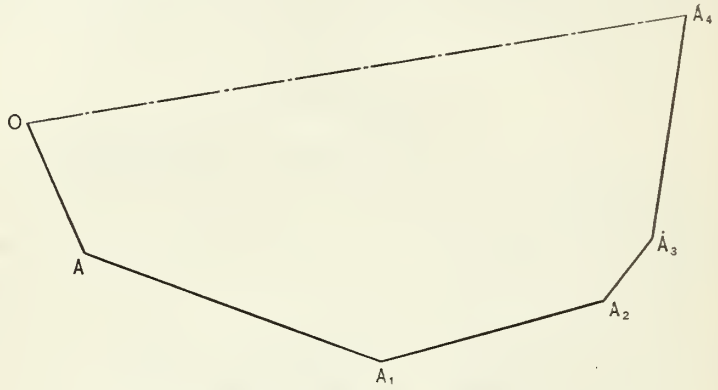


FIG. 151. — Polygon of Vectors.

the phase of the total current to the phases of the partial currents. When the divided circuit contains more than two branches, the same method may be extended as already explained for the composition of voltages.

For convenience in using the graphical methods for solving alternating-current problems, it is well to distinguish between two different diagrams. The first diagram represents the magnitude and relative phase positions of the voltages or currents by means of lines radiating from a point. This may be called the **Phase diagram**. It is analogous to the force diagram of graphical statics. The other diagram is the polygon formed by laying off lines equal and parallel to the lines in the phase diagram. This may be called the **Vector diagram** or **polygon**. It is analogous to the funicular polygon of graphical statics.

Figures 150 (full lines only) and 151 are respectively phase and vector diagrams for representing five voltages or currents. The resultant voltage is represented in magnitude and phase by the line OA_4. If the closing line OA_4 of the vector polygon is inserted in the phase diagram by drawing from O a line in the direction obtained by following round the vector diagram against the direction in which the lines were drawn (that is, from O to A_4), the line so inserted evidently represents the resultant of the component voltages or currents. If the line be drawn from O in the opposite direction, it represents a balancing voltage or current.

These simple propositions, which so evidently come from the ordinary graphical mechanics (statics and dynamics), give all the foundation that is usually necessary for the solution of problems relating to the flow of current in simple and compound circuits containing definite resistances, inductances, and capacities in their different parts. For solutions of complicated problems the analytical method using complex quantities is usually preferable to the graphical. The graphical method has the advantage of showing directly to the eye the relative phases of the voltages or currents in different parts of the circuit, but a drawing board is not always so convenient a medium as a tablet of computing paper and the sharpness of the lag angle sometimes to be dealt with may cause indefinite intersections and inaccuracy in graphical solutions. The graphical solutions have the same limitations in regard to alternating currents or voltages which are not sinusoidal as have the analytical methods; and where the wave form is not sinusoidal, only an approximation can be arrived at by judiciously correcting the results shown by the diagrams based on equivalent sine functions, unless the component sinusoids are used.

80. Classification of Circuits. — The problems relating to alternating-current circuits which can be solved by graphical methods may be divided into three classes: (1) where the current flows through all parts of the circuit in series; (2) where the same voltage is impressed upon all parts of the circuit (parallel circuits); and (3) where the first and second classes are combined. Solutions in the third class are effected by combining partial solutions of the first and second classes. Such problems can also readily be solved by the use of the com-

plex quantity. The articles immediately following give a number of illustrative problems which are solved by both methods.

In more complicated systems, it is frequently desirable to use the relations,—usually termed Kirchoff's Laws,—that *the algebraic sum of all the currents at any junction point in a circuit, or network of circuits, is zero;* and that *the algebraic sum of the voltages in a closed circuit, or any closed loop of a network of circuits, is zero.* In the first case, the currents entering the point under consideration are usually termed positive and those leaving it, negative. In the second case, it is usual to follow around the circuit in a clockwise direction, and call those voltages, in the several parts or sections of the complete circuit, positive which are considered to increase from one terminal of a section to the next, and those negative which are considered to decrease from one terminal to the next. These relations are in accordance with the simple laws of electricity and in the form above are scarcely worthy of statement as separate laws, but are rather convenient expressions of Ohm's law. The relations should be carefully borne in mind as here stated, for the purpose of affording systematic direction to the solution of complex problems.

81. Series Circuits. — *First Class.* — Suppose a circuit is given which has a certain resistance, self-inductance, and capacity, and it is desired to know what impressed voltage with a frequency f is required to pass through it a certain current I. In this case the impressed voltage is made up of two components: (1) the voltage equal to the IR drop caused by the current flowing through the resistance of the circuit, which we call the active voltage; (2) the voltage required to balance or overcome the reactive counter-voltage. The reactive voltage is 90° in phase in advance of or behind the active voltage, depending upon whether the reactance has the effect of capacity or of inductance, and the phase diagram which shows the relative phases of the voltages in the circuit is therefore like that shown in Fig. 152. The active voltage OA' is equal to IR. and the reactive voltage OA'' is equal to $2\pi fLI$, this representing a circuit with inductive reactance. The reactive component of the impressed voltage is required to balance $2\pi fLI$, and is therefore equal to and opposite to OA''. An arrowhead is therefore placed on OA'' to show that in the vector polygon its

direction must be taken from A'' to O, instead of from O outwards, as is done with the other lines. The vector polygon is therefore given by drawing O_1A_1 equal and parallel to OA', A_1A_2 equal and parallel to $A''O$, and closing the polygon by the line O_1A_2 (Fig. 152). The line O_1A_2 taken in the direction from O_1 to A_2 represents the magnitude and relative phase of the impressed voltage. When inserted in the phase diagram, it is the line OA'''. The angle θ, by which the current lags behind the impressed voltage, is the angle $A_1O_1A_2$.

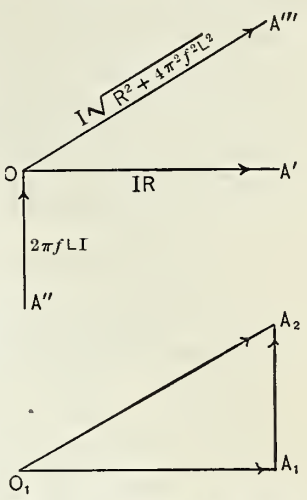


FIG. 152. — Phase and Vector Diagrams of Self-inductive Circuit.

If a number of inductive circuits are connected in series, the line in the phase diagram which represents the reactive voltage has a length equal to $2\pi fI(L_1 + L_2 + L_3 + \text{etc.})$, and the line which represents the active voltage has a length equal to $I(R_1 + R_2 + R_3 + \text{etc.})$, where L_1, L_2, L_3, R_1, R_2, R_3, etc., are the self-inductances and resistances of the different parts of the circuit. If the circuit is non-reactive, the phase diagram and vector polygon each become a single horizontal line equal in length to IR, while if the inductive circuit contains no resistance, the diagrams each become a single vertical line which is equal in length to $2\pi fLI$.

Dividing the lengths of the sides of a vector polygon of voltages by the value of I in the circuit gives the impedances of the several parts of the series circuit. A vector polygon of the voltages impressed on the parts of a series circuit and a polygon of the impedances of the corresponding parts may evidently be converted, one into the other, by a simple change of scale. It is to be remembered that in a series circuit the current has the same phase, and is equal at any given instant, in all parts of the circuit; but the phase, with reference to the current, of the voltage impressed at the terminals of different parts of the circuit depends for each part wholly upon the relation between the individual resistance and reactance of the part.

Examples. — In the following examples it is desired to find for each of the given circuits: (1) the impedance of the circuit; (2) the angle by which the current lags behind the impressed voltage; (3) the current which flows through the circuit when the impressed voltage is 100 volts; (4) the impressed voltage which is required to pass 10 amperes through the circuit. The frequency in each case is taken just under $127\frac{1}{2}$ periods per second, whence $2\pi f$ is equal to 800, and current and voltage are assumed to be sinusoidal.

Circuits containing Resistance and Self-inductance

a. The circuit is non-reactive and has a resistance of 10 ohms. The phase and vector diagrams for the first solution each consist of a horizontal line 10 units in length. The impedance of the circuit is 10 ohms, and the current which flows when a voltage of 100 volts is impressed on the circuit is 10 amperes. The diagrams for the fourth solution each consist of a horizontal line $IR(=100)$ units in length, and the impressed voltage required to pass 10 amperes through the circuit is 100 volts. The angle θ is zero. The complex expression for the impedance of the circuit is

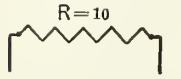


FIG. 153.— Solution *a*.

$$\bar{Z} = r + jx = 10 + j0.$$

Therefore,

(1) $Z = \sqrt{10^2 + 0^2} = 10$ ohms, and (2) $\theta = \tan^{-1}\frac{0}{10} = 0°$.

When the impressed voltage is 100,

$$\bar{I} = \frac{100}{\bar{Z}} = 100 \times \bar{Y},$$

and (3) $I = \frac{100}{Z} = 10$ amperes, in phase with the voltage.

When the current flowing through the circuit is 10 amperes,

$$\bar{E} = I\bar{Z} = Ir + jI0,$$

or $$E = 10\ I,$$

and (4) $E = IZ = 100$ volts, in phase with the current.

b. The circuit consists of an inductive coil of 10 ohms resistance and 0.01 henry self-inductance. The phase and vector

diagrams for the solutions are shown in Fig. 154. The impedance is shown by the closing line of the impedance diagram to be 12.8 ohms, whence it is seen that 7.81 amperes will flow through the circuit when the impressed voltage is 100 volts.

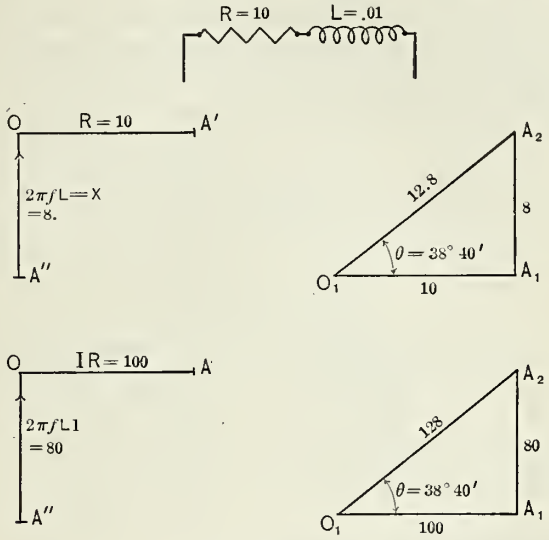


FIG. 154.—Solution *b*.

The fourth solution shows that the impressed voltage required to pass 10 amperes through the circuit is 128 volts. The angle θ is 38° 40′. The complex expression for the impedance in the circuit is

$$\bar{Z} = r + jx = 10. + j\,8$$

and (1) $Z = \sqrt{10^2 + 8^2} = 12.8$ ohms.

(2) $\theta = \tan^{-1}\frac{8}{10} = 38°\ 40'$.

When the impressed voltage is 100 volts,

$$\bar{I} = E\bar{Y} = \frac{Er}{Z^2} - j\frac{Ex}{Z^2} = \frac{1000}{164} - j\frac{800}{164},$$

and (3) $I = 100 \times \frac{1}{12.8} = 7.81$ amperes, lagging behind the voltage.

The two terms in the right-hand member of the complex expression are respectively the two rectangular components, I_g and I_b, of the current.

When the current flowing through the circuit is 10 amperes,

$$\overline{E} = I\overline{Z} = Ir + jIx = 100 + j\,80,$$

and (4) $E = IZ = 10 \times 12.8 = 128$ volts, leading the current.

The two terms in the right-hand member of the complex expression, namely Ir and Ix, are respectively the active and reactive components E_r and E_x of the impressed voltage.

c. The circuit consists of a non-inductive coil of 5 ohms in series with an inductive coil of 5 ohms and 0.01 henry. The total phase and vector diagrams for this are similar to those in example *b*. The vector diagram may be laid off as shown in Fig. 155. This shows that the voltage O_1A_3 impressed upon the circuit as a whole is not equal to the algebraic sum of the voltages O_1A_2 and A_2A_3 measured between the terminals of the parts of the circuit, but it is still equal to their vector sum.

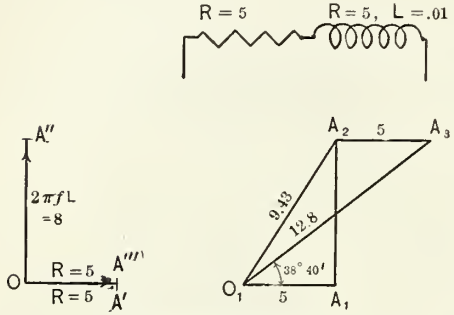


FIG. 155. — Solution *c*.

The solution by means of complex expressions is as follows:

$$\begin{aligned} \overline{Z}_1 &= 5 + j\,8 \qquad & \tan\theta &= \tfrac{8}{10} = .8,\ \theta = 38^\circ 40' \\ \overline{Z}_2 &= 5 + j\,0 \qquad & Z &= \sqrt{r^2 + x^2} = \sqrt{10^2 + 8^2} = 12.8 \text{ ohms.} \\ \overline{Z} &= 10 + j\,8 \end{aligned}$$

When 100 volts are impressed on the circuit,

$$\overline{I} = E\overline{Y} = \tfrac{1000}{164} - j\,\tfrac{800}{164},$$

and (3) $I = 7.81$ amperes, lagging behind the voltage.

When 10 amperes flow through the circuit,

$$\overline{E} = I\overline{Z} = 100 + j\,80,$$

and (4) $E = 10 \times 12.8 = 128$ volts, leading the current.

d. The circuit comprises only inductance of 0.01 henry. The phase and vector diagrams for the first solution each consist of a vertical line $2\pi fL\,(= 8)$ units in length. The impedance of the circuit is 8 ohms, and the current which flows through the

circuit under an impressed voltage of 100 volts is 12.5 amperes. The diagrams for the fourth solution each consist of a vertical line $2\pi fLI$ $(=80)$ units in length, and the impressed voltage required to pass 10 amperes through the circuit is 80 volts. The angle θ is 90°, and the current therefore lags 90° behind the impressed voltage.

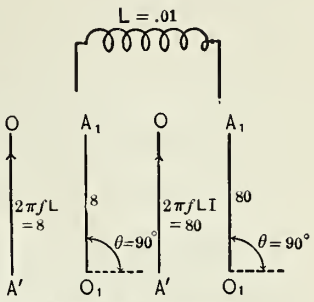


FIG. 156. — Solution *d*.

(1) $\overline{Z}=0+j\,8,\ Z=8,$

(2) $\theta=\tan^{-1}\frac{8}{0}=90°.$

When a voltage of 100 is impressed on the circuit,

$$\overline{I}=\frac{100}{\overline{Z}}=-j\frac{100}{8},$$

and (3) $I=\frac{100}{8}=12.5$ amperes, lagging behind the voltage.

When 10 amperes are flowing through the circuit,

$$\overline{E}=10\,\overline{Z}=j\,80,$$

and (4) $E=80$ volts, leading the current.

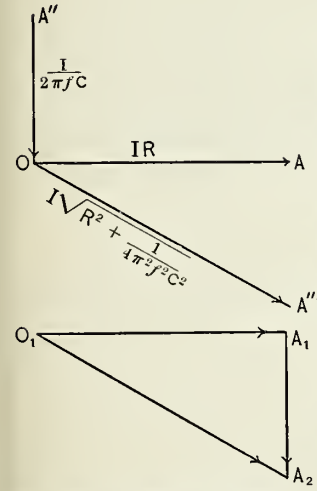


FIG. 157. — Phase and Vector Diagrams of a Capacity Circuit.

The effect of a condenser placed in series in a circuit may be shown by diagrams which are very similar to those relating to inductive circuits. The charging current of the condenser has such a phase position and magnitude that its effect on the total current flowing in the circuit is the same as the effect of a voltage which is equal to $\frac{I}{2\pi fC}$ and is 90° in advance of the circuit current. This may be called the **Condenser voltage.*** The phase diagram is like that shown in Fig. 157. The voltage impressed on the circuit when current I flows must consist of two components: (1) the voltage required to pass the current

* Art. 54.

through the resistance of the circuit; (2). the voltage required to balance the condenser voltage. The active voltage OA in Fig. 157 is equal to IR, and the condenser voltage OA'' is equal to $\frac{I}{2\pi fC}$ and is 90° in advance of the active voltage. The capacity component of the impressed voltage is required to balance $\frac{I}{2\pi fC}$, and is therefore equal and opposite to OA''. An arrowhead is therefore placed on OA'' to show that in the vector polygon its direction must be taken from A'' to O, instead of from O outwards. The vector polygon is then as shown in Fig. 157 (compare with Fig. 152).

Examples. — The following examples are to be solved for the quantities as before, the value of $2\pi f$ being taken as 800 and voltage and current being assumed to be sinusoidal.

Circuits containing Resistance and Capacity

e. The circuit contains simply a condenser having a capacity of 100 microfarads (= 0.000100 farad). The phase and vector diagrams for the first solution each consist of a vertical line $\frac{1}{2\pi fC}$ (= 12.5) units in length. The impedance of the circuit is 12.5 ohms, and the current which flows through the circuit, when 100 volts is impressed on it, is 8 amperes. The diagrams for the fourth solution each consist of a vertical line $\frac{I}{2\pi fC}$ (= 125) units in length, and the impressed voltage required to pass 10 amperes through the circuit is 125 volts. The angle θ is $-90°$; that is, the current is 90° in advance of the impressed voltage. The lines composing the diagrams for this example are drawn in a direction which is exactly opposite to that of the lines in the diagrams of the example *d*.

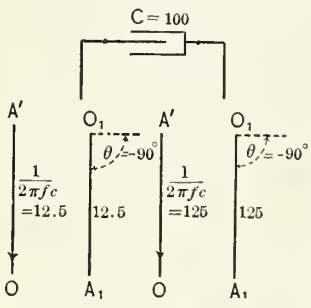


FIG. 158. — Solution *e*.

(1) $\bar{Z} = 0 - j\,12.5$, $Z = 12.5$ ohms.

(2) $\theta = \tan^{-1} - \frac{12.5}{0} = -90°$.

When the impressed voltage is 100,

$$\bar{I} = \frac{100}{\bar{Z}} = j\,\frac{100}{12.5} = j\,8,$$

and (3) $I = \dfrac{100}{12.5} = 8$ amperes, leading the voltage.

When the current flowing through the circuit is 10 amperes,

$$\bar{E} = 10\,\bar{Z} = -j\,125,$$

and (4) $E = 10 \times 12.5 = 125$ volts, lagging behind the current.

f. The circuit contains a resistance of 10 ohms and a capacity of 100 microfarads. The phase and vector diagrams are shown in Fig. 159. The impedance of the circuit is 16 ohms and the current which flows under an impressed voltage of 100 volts is 6.25 amperes. The impressed voltage required to cause 10 amperes to flow through the circuit is 160 volts. The angle θ is $-51°\ 20'$

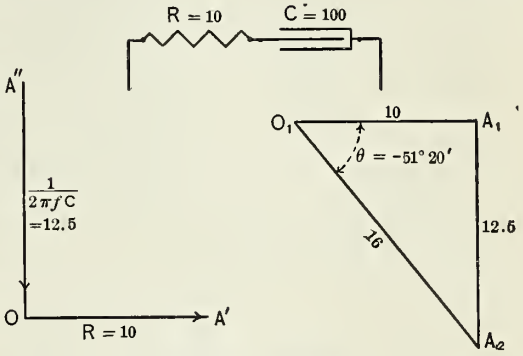


FIG. 159. — Solution *f*.

(1) $\bar{Z} = 10 - j\,12.5$, $Z = 16$ ohms,

(2) $\theta = \tan^{-1} - \dfrac{12.5}{10} = -51°\ 20'.$

For 100 volts impressed,

$$\bar{I} = 100\ \bar{Y} = \frac{1000}{256} + j\,\frac{1250}{256},$$

and (3) $I = \dfrac{100}{Z} = 6.25$ amperes, leading the voltage.

For 10 amperes flowing,

$$\bar{E} = 10\,\bar{Z} = 100 - j\,125,$$

and (4) $E = 10 \times 16 = 160$ volts, lagging behind the current.

Circuits containing Self-inductance and Capacity

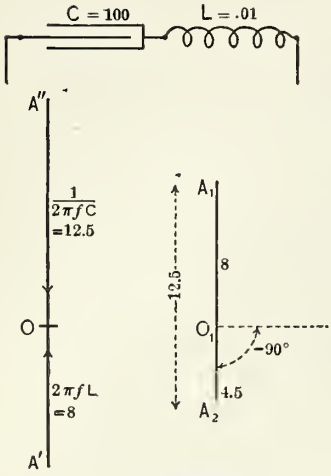


FIG. 160. — Solution *g*.

g. The circuit consists of a condenser of 100 microfarads capacity in series with a 0.01 henry inductance, the resistance being negligible. The diagrams are as shown in Fig. 160. The impedance of the circuit is 4.5 ohms. The current which flows when the impressed voltage is 100 volts is 22.2 amperes, at which time the voltage measured between the terminals of the condenser is 277.5 volts and that measured between the terminals of the inductance is 177.6 volts. The impressed voltage required to pass 10 amperes through the circuit is 45 volts. The angle θ is $-90°$.

The complex expression for impedance is

$$\begin{aligned} \bar{Z}_1 &= 0 + j\,8.0 \\ \bar{Z}_2 &= 0 - j\,12.5 \\ \hline \bar{Z} &= 0 - j\,4.5 \end{aligned}$$

and (1) $Z = 4.5$ ohms.

(2) The angle of lag is $\theta = \tan^{-1} - \frac{4.5}{0} = -90°$.

(3) When the impressed voltage is 100,

$$\bar{I} = \frac{100}{\bar{Z}} = j\frac{100}{4.5},$$

$$I = \frac{100}{Z} = 22.2 \text{ amperes, leading the voltage.}$$

(4) When 10 amperes flow, the impressed voltage is

$$\bar{E} = I\bar{Z} = -j\,10 \times 4.5 = -j\,45.$$
$$E = IZ = 45 \text{ volts, lagging behind the current.}$$

The voltage measured between the terminals of the inductive coil when 10 amperes flow is

$$\bar{E}_1 = I\bar{Z}_1 = j\,80.$$
$$E_1 = 80 \text{ volts, leading the current,}$$

and the voltage measured between the terminals of the condenser is

$$\overline{E}_2 = I\overline{Z}_2 = -j\,125,$$
$$E_2 = 125 \text{ volts, behind the current.}$$

Under a current flow of 22.2 amperes, the voltage measured between the terminals of the condenser is $22.2 \times 12.5 = 277.5$ volts, and the voltage measured between the terminals of the inductive coil is $22.2 \times 8 = 177.6$ volts as pointed out in the first paragraph.

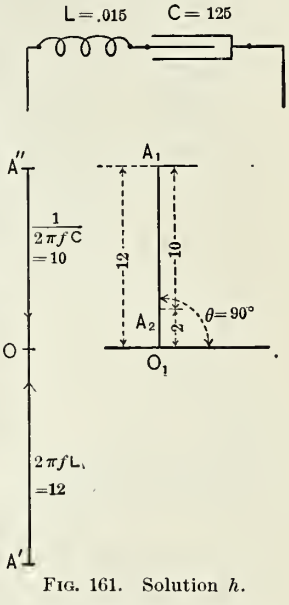


FIG. 161. Solution *h*.

h. The circuit comprises a capacity of 125 microfarads in series with an inductance of 0.015 henry, the resistance being negligible. The diagrams of Fig. 161 show that impedance of the circuit is 2 ohms and the current which flows under 100 impressed volts is therefore 50 amperes, at which time the voltage measured between the terminals of the condenser is 500 volts and between the terminals of the inductance coil is 600 volts. The impressed voltage required to pass 10 amperes through the circuit is 20 volts. With this current flowing, the voltage measured between the terminals of the condenser is 100 volts and between the terminals of the inductance is 120 volts. The angle θ is 90°.

Solving by the complex expressions gives,

$$\overline{Z}_1 = 0 + j\,12 \qquad Z = \sqrt{0^2 + 2^2} = 2 \text{ ohms}$$
$$\overline{Z}_2 = 0 - j\,10$$
$$\overline{Z} = 0 + j\,2 \qquad \tan\theta = \frac{2}{0} = \infty\,,\ \theta = 90°.$$

When 100 volts are impressed upon the circuit, the current is

$$\overline{I} = \frac{E}{\overline{Z}} = -j\,\frac{100}{2} = -j\,50$$

and $I = 50$ amperes, lagging behind the voltage.

The voltage measured between the terminals of the condenser at this time is

$$\overline{E}_2 = I\overline{Z}_2 = 50 \times -j\,10 = -j\,500,$$

and $E_2 = 500$ volts, behind the current.

The voltage measured between the terminals of the inductance coil is

$$\overline{E}_1 = I\overline{Z}_1 = 50 \times j\,12 = j\,600$$

and $E_1 = 600$ volts, ahead of the current.

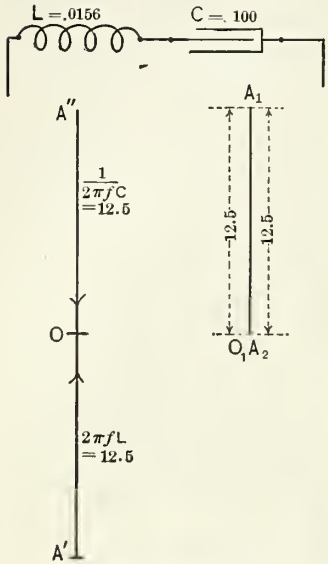


FIG. 162. — Solution *i*.

It will be observed that the 500 volts measured between the condenser terminals and the 600 volts measured between the coil terminals are in exact opposition, and the voltage measured between the terminals of the circuit made up of the condenser and coil in series is the difference of the two or 100 volts. This combination gives a large rise of voltage at the condenser and at the coil caused by the interaction of the two through their mutual transfer of energy.

When current flowing through the circuit is 10 amperes, the voltage measured between the circuit terminals is

$$\overline{E} = I\overline{Z} = j\,20$$

and $E = 20$ volts, ahead of the current.

Under these circumstances

$$\overline{E}_2 = I\overline{Z}_2 = -j\,100$$

and $E_2 = 100$ volts, behind the current.

$$\overline{E}_1 = I\overline{Z}_1 = +j\,120$$

$E_1 = 120$ volts, ahead of the current.

i. The circuit comprises a capacity of 100 microfarads in series with an inductance of 0.0156 henry, resistance being negligible. The phase and vector diagrams are shown in Fig. 162. In this case $\frac{1}{2\pi fC} = 2\pi fL$, O_1A_1 is equal and opposite to A_1A_2, and the impedance of the circuit is zero.

$$\overline{Z}_1 = 0 + j\,12.5$$

$$\underline{\overline{Z}_2 = 0 - j\,12.5}$$

$$\overline{Z} = 0$$

$Z = 0$

That is, this acts like a short circuit.

If a current of 10 amperes is caused to flow through this arrangement of capacity and inductance in series, the voltage measured between the terminals is zero; but the voltage measured between the terminals of the condenser is

$$\bar{E}_2 = I\bar{Z}_2 = -j\ 125$$

and $E_2 = 125$ volts, behind the current;

and the voltage measured between the terminals of the inductance coil is

$$\bar{E}_1 = I\bar{Z}_1 = j\ 125$$

and $E_1 = 125$ volts, ahead of the current.

Circuits containing Resistance, Self-Inductance, and Capacity

j. The circuit consists of 10 ohms plain resistance, 100 microfarads, and .01 henry in series relation. The impedance of the circuit is shown by Fig. 163, to be 10.96 ohms. The current which flows through the circuit when 100 volts are impressed at its terminals is 9.12 amperes. The voltage required to pass 10 amperes through the circuit is 109.6 volts. This is the vector sum of 125 volts and 128 volts, which are the voltages measured respectively between the condenser terminals and the remainder of the circuit. The angle θ is $-24°\ 14'$.

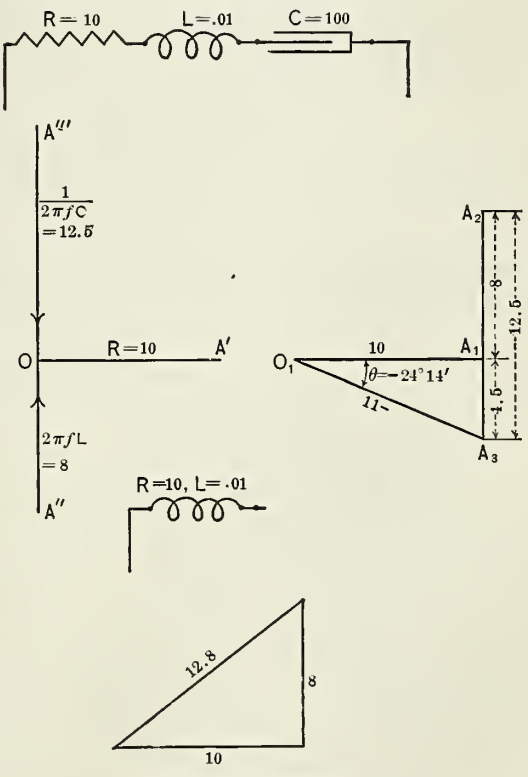


FIG. 163. — Solution *j*.

Solved by complex quantities, there results

$$\bar{Z}_1 = 10 + j\,0 \qquad Z = \sqrt{10^2 + 4.5^2} = 10.96 \text{ ohms.}$$
$$\bar{Z}_2 = 0 + j\,8 \qquad \tan\theta = -\frac{4.5}{10},\ \theta = -24^\circ\ 14'.$$
$$\bar{Z}_3 = 0 - j\,12.5$$
$$\bar{Z} = 10 - j\,4.5$$

Here the prefix of the reactance term in the expression for $\bar{Z}$ is $-j$, hence the angle θ is negative.

When 100 volts are impressed on the circuit,

$$\bar{I} = \frac{E}{\bar{Z}} = E\bar{Y} = \frac{1000}{(10.96)^2} + j\,\frac{450}{(10.96)^2}$$

and $$I = \frac{100}{10.96} = 9.12 \text{ amperes, leading the voltage.}$$

Under these conditions, the voltage measured across the terminals of the condenser is

$$\bar{E}_3 = I\bar{Z}_3 = -j\,9.12 \times 12.5$$

and $$E_3 = 114 \text{ volts, behind the current.}$$

The voltage measured across the remainder of the circuit is

$$\bar{E}_{1+2} = I(\bar{Z}_1 + \bar{Z}_2) = 9.12\,(10 + j\,8)$$

and $E_{1+2} = \sqrt{91.2^2 + 72.96^2} = 116.8$ volts, ahead of the current.

When 10 amperes is caused to flow through the circuit,

$$\bar{E} = I\bar{Z} = 100 - j\,45$$

and $$E = 109.6 \text{ volts, behind the current.}$$

The values of E_3 and E_{1+2} under these circumstances may be computed in the same manner.

k. The circuit comprises 10 ohms plain resistance, 150 microfarads, and 01042 henry in series. The diagrams in Fig 164 show that the impedance of the circuit is 10 ohms. One hundred volts is therefore the impressed voltage that gives a current of 10 amperes. When 10 amperes flow in the circuit, the voltage measured between the terminals of the condenser is 83.3 volts, and that measured between the terminals of the remainder of the circuit is 130 volts. The angle θ is zero.

$$\bar{Z}_1 = 10 + j\,0 \qquad Z = 10 \text{ ohms.}$$
$$\bar{Z}_2 = 0 + j\,8.33 \qquad \tan\theta = \frac{0}{10},\ \theta = 0^\circ.$$
$$\bar{Z}_3 = 0 - j\,8.33$$
$$\bar{Z} = 10 + j\,0$$

When 100 volts are impressed on this circuit,

$$\bar{I} = \frac{\bar{E}}{\bar{Z}} = \frac{100}{10} = 10$$

and $I = 10$ amperes, in phase with the voltage.

The voltage measured between the terminals of the condenser (assuming it to be resistanceless) is

$$\bar{E}_3 = -j\,10 \times 8.33 = -j\,83.3$$

and $E_3 = 83.3$ volts, behind the current.

The voltage measured between the terminals of the inductance coil (assuming it to be resistanceless) is

$$\bar{E}_2 = +j\,10 \times 8.33 = +j\,83.3$$

and $E_2 = 83.3$ volts, ahead of the current.

These two reactive components of voltage are equal and opposite to each other, and the voltage measured across the circuit is the same as that measured across the plain resistance coil, namely, 100 volts in phase with the current.

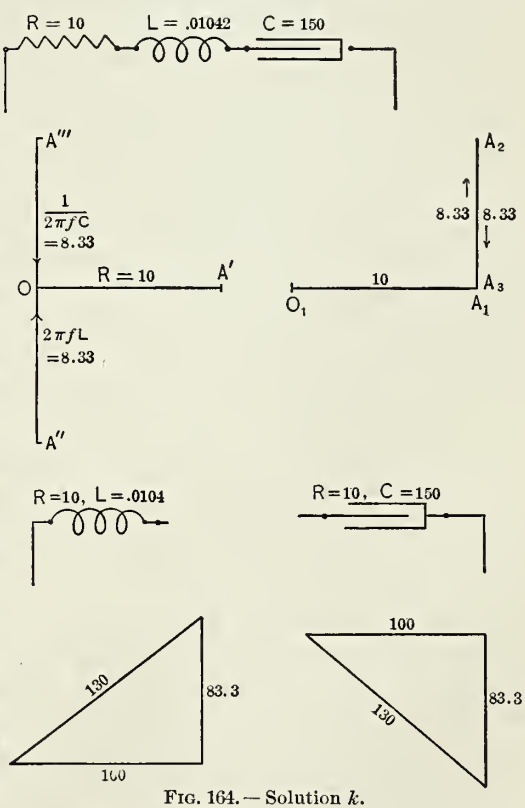


Fig. 164. — Solution *k*.

82. Conclusions in Regard to Series Circuits. — The eleven examples thus given cover every fundamental arrangement of series circuits which may occur. An examination of the diagrams makes the following statements evident for circuits in which are sinusoidal voltages and currents:

1. When *non-inductive circuits* are connected in series, the total impressed voltage equals the sum of the voltages measured between the terminals of the individual parts, and the total resistance of the circuit is equal to the sum of the resistances of the individual parts.

2. When *inductive circuits of equal time constants* are connected in series, the total impressed voltage equals the sum of the voltages measured between the terminals of the individual parts, and the total impedance of the circuit is equal to the sum of the individual impedances.

3. When *inductive and non-inductive circuits* are connected in series with each other, or when *inductive circuits of unequal time constants* are connected in series, the total impressed voltage equals the vector sum, which is always less than the algebraic sum, of the voltages measured between the terminals of the individual parts, and the individual voltages are each less than the total impressed voltage. The total impedance of the circuit is equal to the vector sum of the individual impedances, each of which is less than the total.

4. When *condensers* are connected in series *by conductors of negligible resistance*, the total impressed voltage equals the sum of the voltages measured across the individual condensers, and the total impedance of the circuit is equal to the sum of the impedances of the individual condensers.

5. When *condensers* are connected in series with *non-inductive circuits*, the total impressed voltage equals the vector sum, which is always less than the algebraic sum, of the voltages measured between the terminals of the individual parts of the series, and the individual voltages are each less than the total impressed voltage. The total impedance of the circuit is equal to the vector sum of the individual impedances, each of which is less than the total.

6. When *condensers* are connected in series *with inductive circuits*, the total impressed voltage equals the vector sum, which is always less than the algebraic sum, of the voltages measured between the terminals of the individual parts of the series. Since the effects of capacity and self-inductance respectively cause the lag angle to become negative and positive, *the individual voltages may be* EITHER GREATER OR LESS *than the total impressed voltage*, depending upon the relation between the

various resistances, capacities, and inductances in the circuit. The total impedance of the circuit is equal to the vector sum of the individual impedances, each of which may be either greater or less than the total impedance.

The third, fifth, and sixth paragraphs above make the following proposition evident: When in series circuits the angles measured between the phase of the current and the phases of the individual voltages measured between terminals of parts of the series, are all either positive or negative, the total impressed voltage is always greater than any of the individual or partial voltages. When the angles measured between the phase of the current and the phases of the partial voltages are in part positive and in part negative, *some or all of the partial voltages may be greater than the total impressed voltage.*

83. Parallel Circuits. — *Second Class.* — The graphical and analytical treatment of problems relating to parallel circuits is entirely analogous to that given for series circuits. As the simplest cases of parallel circuits are those in which the same voltage is impressed upon all the parts of the circuit, these will be treated first. In this class the same general operations are used in solving problems as in the first class (series circuits), but alternating currents and admittances are dealt with instead of alternating voltages and impedances. Suppose a circuit is made up of two branches in parallel, each with a known resistance and reactance, and it is desired to know what impressed voltage with a frequency f is required to cause a sinusoidal current I to flow through the circuit. In this case the total current is made up of two components, each of which flows through one of the branches and is inversely proportional to the impedance of the branch, and the phase of which has an angular retardation with respect to the impressed voltage which depends upon the time constant of the branch. The total current, which is inversely proportional to the equivalent impedance of the parallel circuit, is equal in magnitude and position to the resultant of the branch currents. The condition is represented by Fig. 165, in which OA' and OA'' are the currents in the two branches respectively, θ' and θ'' being their respective lag angles. The relative phase position of the impressed voltage is taken on the horizontal line. Then the resultant or total current in the circuit is represented in mag-

nitude and phase by OA. OA' is equal to $\frac{E}{Z_1}$, OA'' is equal to $\frac{E}{Z_2}$, and OA is equal to $\frac{E}{Z}$ where E is the voltage impressed on the branches and the denominators of the fractions are the respective impedances of the branches and of the total circuit. It is therefore evident that the reciprocal of the joint impedance Z (= the joint admittance) of a parallel circuit may be at once derived from the admittances of the branches, by taking their vector sum, as is shown in Fig. 166. The joint admittance or the joint impedance of a branched circuit being known for a particular frequency, the voltage of that frequency required to pass a given current through it, or the current flowing under a given impressed voltage of the same frequency, may be at once derived. In dealing with series circuits, the phase of the current (or active voltage) has been assumed to be along the horizontal line, or line of reference. In dealing with parallel circuits, it is more convenient to assume the phase of the impressed voltage (or conductance) for the reference phase. It must always be remembered, however, that angles of lag are measured from lines representing current to lines representing voltage. Thus, in Figs. 165 and 166 the angle θ is negative because the current leads the voltage.

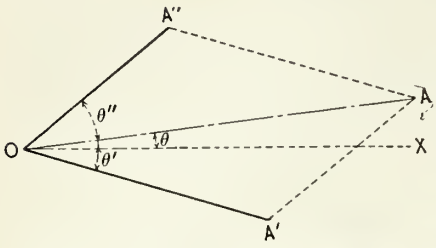


FIG. 165. — Currents in Parallel Branches.

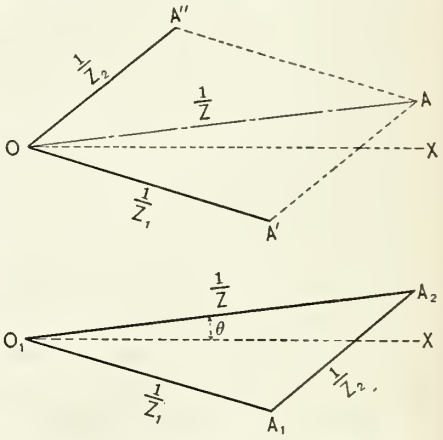


FIG. — 166. Vector Diagram of Admittances.

The resultant current is also obtained by adding the complex expressions representing the branch currents. Thus the resultant of two currents in parallel circuits is

$$\bar{I} = \bar{I}_1 + \bar{I}_2 = E\bar{Y}_1 + E\bar{Y}_2 = E(g_1 + g_2) - jE(b_1 + b_2).$$

In this equation $Eg_1 + Eg_2$ is the active component I_g of the particular current, and $Eb_1 + Eb_2$ the component I_b in quadrature. In the same way the combined admittance of parallel circuits is obtained. The addition is performed graphically in Figs. 165 and 166.

Examples. — In the following examples it is desired to find for each of the given circuits: (1) the joint impedance of the circuit at the given frequency, (2) the angle by which the total current lags behind the impressed voltage, (3) the current which flows through the circuit when the impressed voltage is 100 volts, (4) the impressed voltage which is required to pass 10 amperes through the circuit. The frequency is taken as in examples of the series circuit to be just under $127\frac{1}{2}$ periods per second, whence $2\pi f$ is equal to 800; and current and voltage are supposed to be sinusoidal.

Circuits containing Resistance and Self-inductance

a. The circuit consists of two non-reactive* branches in parallel, one having 20 ohms resistance and the other 10 ohms resistance. The phase diagram for the solutions, using the admittances as a basis of work, is two horizontal lines superposed, of lengths respectively .05 and .10 unit. The vector diagram is obtained by drawing these consecutively, and the equivalent admittance of the circuit is OA_2 in Fig. 167 and is equal to .15 mho. The joint impedance of the circuit is therefore 6.67 ohm. The current flowing under 100 volts impressed is therefore 15 amperes, and the voltage required to pass 10 amperes through the circuit is 66.7 volts. The complex expressions for these circuits reduce to the ordi-

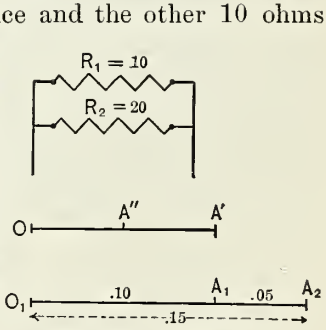


FIG. 167. — Solution *a*.

* The terms *inductive*, *capacity*, and *reactive* circuit are used in this book with the following significations: an *inductive circuit* is one containing inductance, but not capacity; a *capacity* or *condenser circuit* is one containing capacity, but not inductance; a *reactive circuit* is one containing either inductance or capacity, or both inductance and capacity. A *non-reactive circuit* is, therefore, one which contains neither inductance nor capacity, that is, one which contains plain resistance only.

nary forms derived from Ohm's Law and are, therefore, omitted.

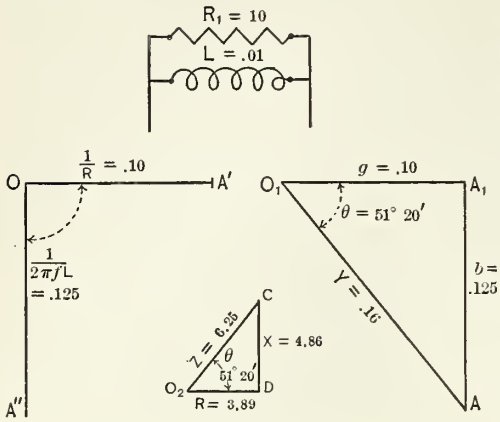


Fig. 168. Solution *b*.

b. The circuit consists of a non-reactive branch of 10 ohms and an inductive branch of .01 henry in parallel. The phase diagram consists of two lines at right angles (one being horizontal), since the current in the non-reactive branch is in phase with the impressed voltage and that in the inductive branch lags 90° behind. The lengths of the lines are respectively $\frac{1}{R}(= .10)$ units and $\frac{1}{2\pi fL}(= .125)$ units. The vector polygon is as shown. The admittance of the circuit is .16 mho and the impedance is 6.25 ohms. The current flowing under an impressed voltage of 100 volts is therefore 16 amperes, and it requires 62.5 volts to cause 10 amperes to flow. The angle θ is 51° 20′. The small triangle of Fig. 168 is an impedance triangle drawn on a different scale from the larger admittance triangle. Its sides are derived from the relation $\overline{Z} = \frac{1}{\overline{Y}}$ and $Z = \frac{1}{Y}$, whence $R = \frac{g}{Y^2}$ and $X = \frac{b}{Y^2}$.

In the complex expression we have $g = \frac{R}{Z^2}$, $b = \frac{X}{Z^2}$, and $\overline{Y} = g \mp jb$, therefore

$$\overline{Y}_1 = \frac{10}{100 + 0} - j\,0 \qquad Y = \sqrt{.1^2 + .125^2} = .16 \text{ mho.}$$

$$\overline{Y}_2 = 0 \qquad -j\frac{8}{64 + 0} \qquad \tan\theta = \frac{.125}{.1},\ \theta = 51° 20'.$$

$$\overline{Y} = .1 - j\,.125 \qquad Z = \frac{1}{Y} = 6.25 \text{ ohms.}$$

When 100 volts are impressed on this branched circuit,

$$\bar{I} = E\bar{Y} = 10 - j\,12.5$$

and $I = EY = 16$ amperes, lagging behind the voltage.

When 10 amperes flow through the branched circuit,

$$\bar{E} = I\bar{Z} = \frac{10}{.1 - j\,.125} = \frac{1}{.0256} + j\frac{1.25}{.0256}$$

and $E = IZ = 62.5$ volts, leading the current.

The current in each branch under either set of conditions is equal to the admittance of the branch multiplied by the impressed voltage corresponding to the conditions.

c. The circuit consists of a non-reactive branch of 10 ohms in parallel with an inductive branch having a resistance of 10 ohms and an inductance of .01 henry. The impedance and the angle of lag for the inductive branch are found by the method given under Series Circuits: First Class, and the admittance of the branch is laid off in the phase diagram on a line making with the horizontal axis an angle equal to the angle of lag taken backwards. This is line OA'' in the diagram, Fig. 169. The line OA' represents the admittance of, and the relative phase of current in, the non-reactive branch. The length and direction of the line O_1A_2 in the vector polygon shows the value of the equivalent or joint admittance of the circuit and the angle by which the phase of the main current lags behind the phase of the impressed voltage. The joint admittance of the circuit is .168 mho and the joint impedance 5.95 ohms. The current flowing under an impressed voltage of 100 volts is 16.8 amperes, and the voltage required to pass 10 amperes through the circuit is 59.5 volts. The angle θ is 16° 52′. The triangle $OA'A$ is equal to the impedance triangle of the self-inductive branch rotated on its base until its apex points downward. This reversed construction of the

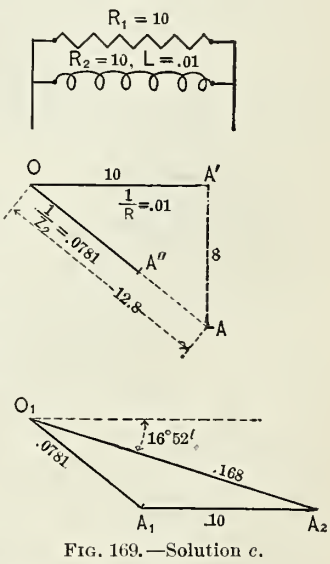


FIG. 169.—Solution *c*.

impedance line (used also in following problems) is made for convenience in obtaining the correct position of the admittance line, since admittance is the reciprocal of impedance and taking the reciprocal of an operator reverses its angle. If the position of the impedance and its components are desired, an impedance triangle should be laid out as in the case of any series circuit.

The problem may be worked out by the use of complex quantities as follows:

$$\overline{Y}_1 = \frac{10}{100} - j\,0 \qquad Y = \sqrt{.161^2 + .049^2} = .168 \text{ mho.}$$

$$\overline{Y}_2 = \frac{10}{100+64} - j\frac{8}{100+64} \qquad Z = \frac{1}{Y} = 5.95 \text{ ohms.}$$

$$\overline{Y} = .161 - j\,.049 \qquad \tan\theta = \frac{.049}{.161},\ \theta = 16^\circ\,52'.$$

When 100 volts are impressed on the circuit,

$$\overline{I} = E\overline{Y} = 16.1 - j\,4.9$$

and $I = EY = 16.8$ amperes, lagging behind the voltage.

When 10 amperes flow through the circuit,

$$\overline{E} = I\overline{Z} = \frac{I}{\overline{Y}} = \frac{1.61}{.0282} + j\frac{0.49}{.0282}$$

and $E = IZ = 59.5$ volts, ahead of the current.

The complex expressions for currents in the branches when 100 volts are impressed are

$$\overline{I}_1 = E\overline{Y}_1 = 100\,(.1 - j\,0),$$

$I_1 = 10$ amperes, in phase with the voltage,

$$\theta_1 = \tan^{-1}\frac{0}{10} = 0^\circ;$$

and $\overline{I}_2 = E\overline{Y}_2 = 100\,(.061 - j.049)$,

$I_2 = 7.81$ lagging behind the voltage,

$$\theta_2 = \tan^{-1}\frac{.049}{.061} = 38^\circ\,40'.$$

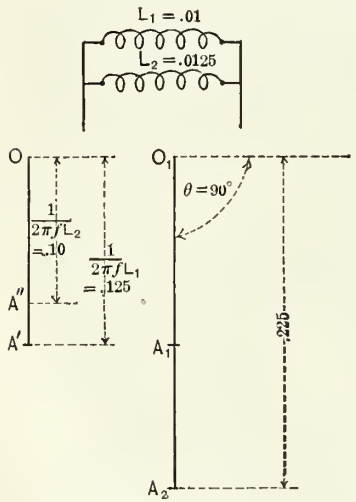


FIG. 170.—Solution *d*.

It will be observed that the arithmetical sum of the currents, 10 amperes and 7.81 amperes, in the two branches is greater than the current, 16.8 amperes, in the main line.

d. The circuit consists of two inductive branches of respectively .01 and .0125 henry in parallel. The diagrams consist of vertical lines, as shown in Fig. 170. The admittance is .225 mho and the impedance is 4.44 ohms. The current flowing when 100 volts is impressed on the circuit is 22.5 amperes, and it requires 44.4 volts to cause 10 amperes to flow. The angle of lag is 90°.

The complex expressions for admittance are

$$\overline{Y}_1 = -j.1 \qquad Y = .225 \text{ mho.}$$
$$\overline{Y}_2 = -j.125 \qquad Z = 4.44 \text{ ohms.}$$
$$\overline{Y} = -j.225 \qquad \theta = \tan^{-1}\frac{.225}{0} = 90°.$$

The currents in the branches when 100 volts are impressed on the circuit are

$$I_1 = EY_1 = .1 \times 100 = 10 \text{ amperes, lagging } 90°,$$

and $$I_2 = EY_2 = .125 \times 100 = 12.5 \text{ amperes, also lagging } 90°.$$

The main current is

$$\overline{I} = E\overline{Y} = -j\,22.5$$

and $$I = 22.5 \text{ amperes, lagging } 90°.$$

When 10 amperes are caused to flow through the circuit,

$$\overline{E} = \frac{I}{\overline{Y}} = j\,\frac{10}{.225},$$

$$E = IZ = 44.4 \text{ volts, } 90° \text{ ahead of the current.}$$

e. The circuit consists of two reactive branches of respectively .005 henry and 10 ohms, and .0125 henry and 8 ohms in parallel. The diagrams are as shown in Fig. 171. The admittance of the circuit is shown to be .165 mho, and the impedance 6.06 ohms. The current flowing under a voltage of 100 volts is therefore 16.5 amperes, and the voltage required to pass 10 amperes through the circuit is

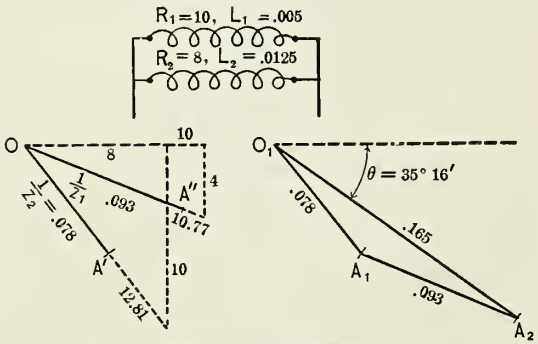


FIG. 171. — Solution *e*.

60.6 volts. The angle θ is 35° 16′. The problem may be solved by the use of complex quantities as follows:

$$\overline{Y}_1 = \frac{10}{100+16} - j\frac{4}{100+16}$$

$$\overline{Y}_2 = \frac{8}{64+100} - j\frac{10}{64+100}$$

$$\overline{Y} = .1349 - j.0954$$

$$Y = \sqrt{.1349^2 + .0954^2} = .165 \text{ mho.}$$

$$Z = \frac{1}{.165} = 6.06 \text{ ohms.}$$

$$\tan\theta = \frac{.0954}{.1349} = .7072, \quad \theta = 35° 16'.$$

The currents in the branches when 100 volts are impressed on the circuit are

$I_1 = EY_1 = 9.3$ amperes, lagging behind the voltage,

and $I_2 = EY_2 = 7.8$ amperes, also lagging behind the voltage;

$$\theta_1 = 21° 48'$$

and

$$\theta_2 = 51° 20'.$$

The total current is

$$\overline{I} = E\overline{Y} = 13.5 - j\,9.54$$

and $I = EY = 16.5$ amperes, lagging behind the voltage.

The five preceding examples cover all the fundamental combinations of resistance and inductance in parallel circuits. The following four in like manner cover the combinations of resistance and capacity. The solutions in the two cases are similar, but the lag angles become negative in the latter on account of the influence of the capacities.

Circuits containing Resistance and Capacity

f. When two or more condensers of negligible internal resistance are connected in parallel by wires of negligible resistance, they evidently act upon the circuit exactly as though it contained one condenser with a capacity equal to the combined capacity of those in parallel. The impedance of a condenser is equal to $\frac{1}{2\pi fC}$, and its admittance to $2\pi fC$. The admittance of several condensers in parallel is therefore evidently

$$2\pi f(C_1 + C_2 + \text{etc.}).$$

g. The circuit consists of a non-reactive branch of 10 ohms in parallel with a capacity branch of 100 microfarads and negligible resistance. The diagrams are as shown in Fig. 172. The admittance of the circuit is .128 mho and its impedance is 7.81 ohms. The current flowing under a voltage of 100 volts is 12.8 amperes, and the voltage required to pass 10 amperes through the circuit is 78.1 volts. The angle θ is $-38° 40'$.

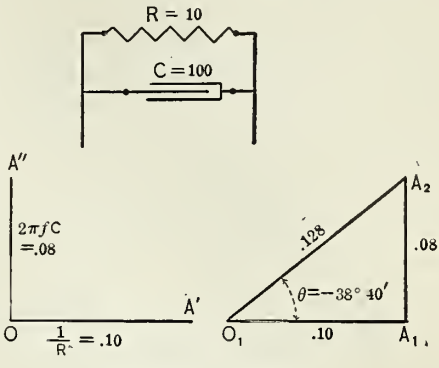


FIG. 172. — Solution *g*.

The complex expressions for this case are of the same form as for Solution *b* except that the direction of the vertical component is reversed.

$$\bar{Y}_1 = .1 - j\,0 \qquad Y = \sqrt{.1^2 + .08^2} = .128 \text{ mho.}$$

$$\bar{Y}_2 = 0 + j\,.08 \qquad Z = \frac{1}{\bar{Y}} = 7.81 \text{ ohms.}$$

$$\bar{Y} = .1 + j\,.08 \qquad \theta = -\tan^{-1}\frac{.08}{.1} = -38° 40'.$$

When 100 volts are impressed on the circuit,

$$\bar{I} = E\bar{Y} = 10 + j\,8$$

and $$I = EY = 12.8 \text{ amperes;}$$

$$\bar{I}_1 = E\bar{Y}_1 = 10 - j\,0$$

and $$I_1 = EY_1 = 10 \text{ amperes;}$$

$$\bar{I}_2 = E\bar{Y}_2 = j\,8$$

and $$I_2 = EY_2 = 8 \text{ amperes;}$$

$$\theta_1 = 0°,\ \theta_2 = -90°.$$

When a current of 10 amperes is caused to flow through the circuit,

$$\bar{E} = \frac{I}{\bar{Y}} = \frac{1}{.0164} - j\,\frac{0.8}{.0164}$$

and $$E = IZ = 78.1 \text{ volts.}$$

h. The circuit consists of a non-reactive branch of 10 ohms in parallel with a reactive branch of 10 ohms and 100 microfarads. The admittance of the circuit is shown by the diagrams of Fig. 173 to be .147 mho and the impedance 6.8 ohms. The current flowing under 100 volts is 14.7 amperes, and the voltage required to pass 10 amperes through the circuit is 68 volts. The angle of lag is $-19°\ 20'$. The solution by means of complex quantities is

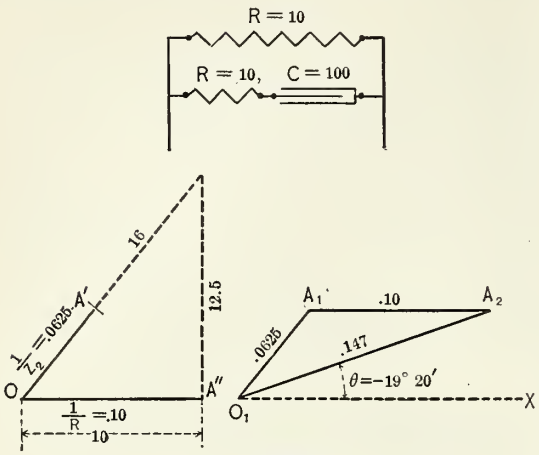


FIG. 173.—Solution *h*.

$$\overline{Y}_1 = \frac{10}{100+0} - j\frac{0}{100+0}$$

$$\overline{Y}_2 = \frac{10}{100+156} + j\frac{12.5}{100+156}$$

$$\overline{Y} = .1390 + j.0488$$

$$Y = \sqrt{.1390^2 + .0488^2} = .147 \text{ mho.}$$

$$Z = \frac{1}{.147} = 6.8 \text{ ohms.}$$

$$\tan\theta = -\frac{.0488}{.1390} = -.3509,\ \theta = -19°\ 20'.$$

When 100 volts are impressed on the circuit, the currents are

$$\overline{I} = E\overline{Y} = 13.9 + j\,4.88$$

and $$I = EY = 14.7 \text{ amperes;}$$

$$\overline{I}_1 = E\overline{Y}_1 = 10 - j\,0$$

and $$I_1 = EY_1 = 10 \text{ amperes;}$$

$$\overline{I}_2 = E\overline{Y}_2 = \frac{1000}{256} + j\frac{1250}{256}$$

and $$I_2 = EY_2 = 6.25 \text{ amperes};$$

$$\theta_1 = 0°,\ \theta_2 = -\tan^{-1}\frac{12.5}{10} = -51° 20'.$$

When 10 amperes are caused to flow through the circuit,

$$\bar{E} = \frac{I}{\bar{Y}} = \frac{1.39}{.0218} - j\frac{.488}{.0218}$$

and $E = IZ = 68$ volts.

i. The circuit consists of two reactive branches in parallel, respectively, of 10 ohms and 100 microfarads, and of 20 ohms and 250 microfarads. The admittance of the circuit is shown by the diagram in Fig. 174 to be .105 mho and the impedance 9.5 ohms. The current flowing under a voltage of 100 volts is 10.5 amperes, and the voltage required to pass 10 amperes through the circuit is 95 volts. The angle θ is $-35° 10'$.

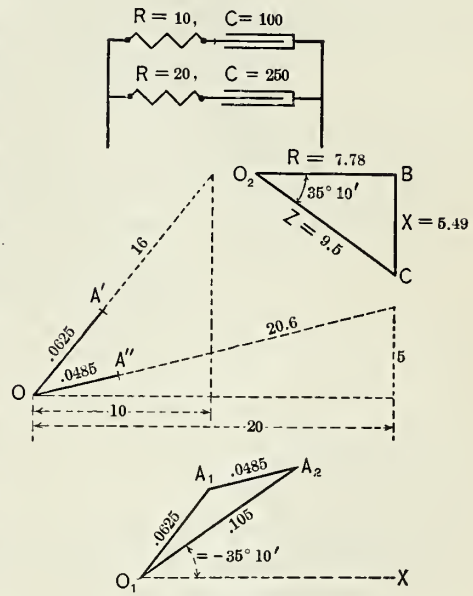


Fig. 174. Solution *i*.

The solution with complex quantities is

$$\bar{Y}_1 = \frac{10}{256} + j\frac{12.5}{256} \qquad Y = \sqrt{.0862^2 + .0605^2} = .105 \text{ mho}.$$

$$\bar{Y}_2 = \frac{20}{425} + j\frac{5}{425} \qquad Z = \frac{1}{Y} = 9.52 \text{ ohms}.$$

$$\bar{Y} = .0862 + j.0605 \qquad \theta = -\tan^{-1}\frac{.0605}{.0862} = -35° 10'.$$

When 100 volts are impressed on the circuit, the currents are

$$\bar{I} = E\bar{Y} = 8.62 + j6.05$$

and $$I = EY = 10.5 \text{ amperes};$$

$$\bar{I}_1 = E\bar{Y}_1 = \frac{1000}{256} + j\frac{1250}{256}$$

and $I_1 = EY_1 = 6.25$ amperes;

$$\bar{I}_2 = E\bar{Y}_2 = \frac{2000}{425} + j\frac{500}{425}$$

and $I_2 = EY_2 = 4.85$ amperes;

$$\theta_1 = -\tan^{-1}\frac{12.5}{10} = -51^\circ 20', \quad \theta_2 = -\tan^{-1}\frac{5}{20} = -14^\circ 5'.$$

When 10 amperes are caused to flow through the circuit,

$$\bar{E} = \frac{\bar{I}}{\bar{Y}} = \frac{.862}{.0111} - j\frac{.605}{.0111},$$

$$E = IZ = 95.2 \text{ volts}.$$

The impedance components of the combined circuit may be found as in Problem *b* or from the formula

$$\bar{Z} = \frac{1}{\bar{Y}}(\cos\theta + j\sin\theta)$$

or

$$\bar{Z} = \frac{1}{.105}(.817 - j.576),$$

$$\bar{Z} = 7.78 - j5.49.$$

The impedance diagram is shown in the triangle O_2BC, Fig. 174.

The following examples cover the fundamental combinations of circuits containing resistance, capacity, and self-inductance.

j. The circuit consists of two reactive branches in parallel, respectively of 5 ohms and .005 henry, and of 10 ohms and 100 microfarads. The admittance of the circuit is shown by the diagrams of Fig. 175 to be .168 mho and the impedance 5.95 ohms. The current flowing under a voltage of 100 volts is 16.8 amperes, and the voltage required to cause a current of 10 amperes to flow is 59.5 volts. The angle θ is 16° 50′.

The solution by means of the complex quantities is as follows:

$$\bar{Y}_1 = \frac{5}{25+16} - j\frac{4}{25+16} \qquad Y = \sqrt{.161^2 + .0487^2} = .168 \text{ mho}.$$

$$\bar{Y}_2 = \frac{10}{100+156} + j\frac{12.5}{100+156} \qquad Z = \frac{1}{.168} = 5.95 \text{ ohms}.$$

$$\bar{Y} = .1610 - j.0487 \qquad \tan\theta = \frac{.0487}{.1610} = .3025, \ \theta = 16^\circ 50'.$$

When 100 volts are impressed on the circuit,

$$\overline{I} = E\overline{Y} = 16.1 - j\,4.87$$

and $$I = EY = 16.8 \text{ amperes};$$

$$\overline{I}_1 = E\overline{Y}_1 = \frac{500}{41} - j\frac{400}{41}.$$

and $$I_1 = EY_1 = 15.6 \text{ amperes};$$

$$I_2 = E\overline{Y}_2 = \frac{1000}{256} + j\frac{1250}{256}$$

and $$I_2 = EY_2 = 6.25 \text{ amperes};$$

$$\theta_1 = \tan^{-1}\frac{400}{500} = 38^\circ\,40', \quad \theta_2 = -\tan^{-1}\frac{1250}{1000} = -51^\circ\,20'.$$

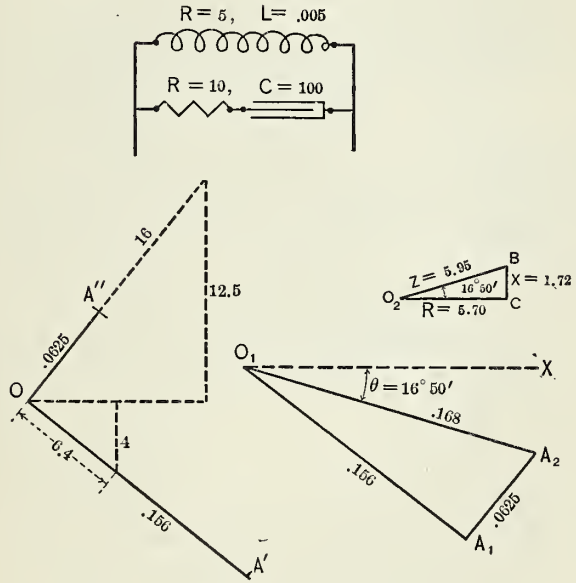


FIG. 175. — Solution *j*.

When 10 amperes are caused to flow through the circuit,

$$\overline{E} = I\overline{Z} = \frac{1.61}{.0284} + j\,\frac{.487}{.0284},$$

and $$E = IZ = 59.5 \text{ volts}.$$

The impedance diagram for the combined circuit is shown in the triangle O_2BC, Fig. 175, and is obtained as in the preceding cases, thus:

$$\overline{Z} = R + jX,$$

or $$\overline{Z} = \frac{1}{Y}\ (\cos\theta + j\sin\theta),$$

or $$\overline{Z} = \frac{1}{.168}\ (\cos 16^\circ\ 50' + j\sin 16^\circ\ 50'),$$

or $$\overline{Z} = Z\ (\cos\theta + j\sin\theta),$$

and from either of the last two,

$$\overline{Z} = 5.70 + j\,1.72.$$

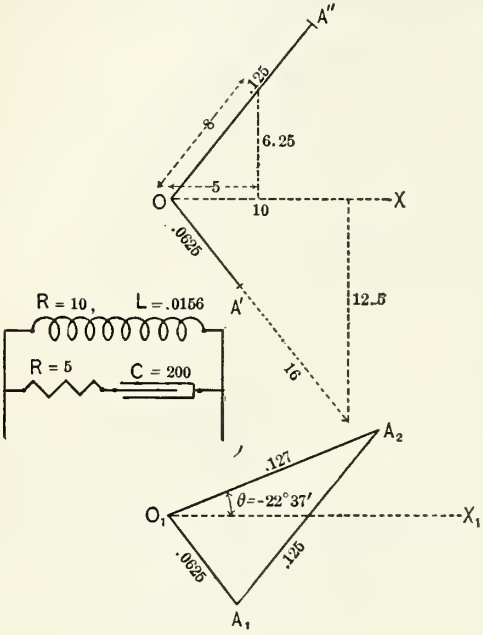


FIG. 176. — Solution *k*.

k. The circuit consists of two reactive branches in parallel, respectively, of 10 ohms and .0156 henry, and of 5 ohms and 200 microfarads. The admittance is shown by the diagrams of Fig. 176 to be .127 mho, and the impedance of the circuit is 7.87 ohms. The current flowing under a voltage of 100 volts is 12.7 amperes, and 78.7 volts are required to pass 10 amperes through the circuit. The angle θ is $-22^\circ\ 37'$.

The solution using complex quantities is as follows:

$$\overline{Y}_1 = \frac{10}{100 + 156.3} - j\,\frac{12.5}{100 + 156.3}$$

$$\overline{Y}_2 = \frac{5}{25 + 39} + j\,\frac{6.25}{25 + 39}$$

$$\overline{Y} = .1173 + j\,.0489$$

$$Y = \sqrt{.1173^2 + .0489^2} = .127 \text{ mho.}$$

$$Z = \frac{1}{.127} = 7.87 \text{ ohms.}$$

$$\tan\theta = -\frac{.0489}{.1173},\quad \theta = -22^\circ\ 37'.$$

When 100 volts are impressed on the circuit,

$$\bar{I} = E\bar{Y} = 11.73 + j\,4.89$$

and $$I = EY = 12.7 \text{ amperes};$$

$$\bar{I}_1 = E\bar{Y}_1 = \frac{1000}{256.3} - j\,\frac{1250}{256.3}$$

and $$I_1 = EY_1 = 6.25 \text{ amperes};$$

$$\bar{I}_2 = E\bar{Y}_2 = \frac{500}{64} + j\,\frac{625}{64}$$

and $$I_2 = EY_2 = 12.5 \text{ amperes};$$

$$\theta_1 = \tan^{-1}\frac{12.50}{10} = 51^\circ\,20', \quad \theta_2 = -\tan^{-1}\frac{6.25}{5} = -51^\circ\,20'.$$

When 10 amperes are caused to flow through the circuit,

$$\bar{E} = I\bar{Z} = \frac{1.173}{.0162} - j\,\frac{.489}{.0162}$$

and $$E = IZ = 78.7 \text{ volts}.$$

l. The circuit consists of two reactive branches in parallel, respectively, of 10 ohms and .01042 henry, and 10 ohms and

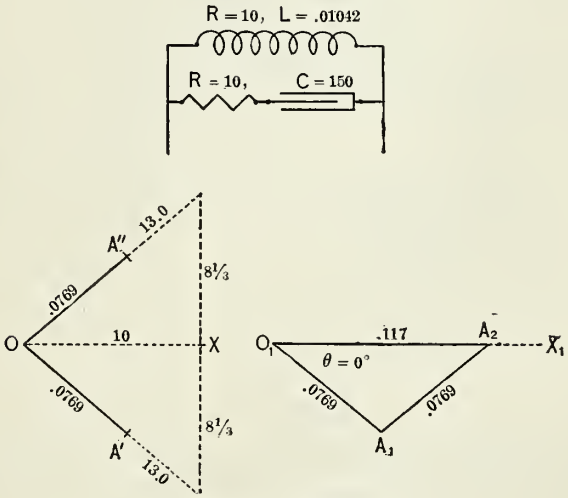


FIG. 177.—Solution *l.*

150 microfarads. The diagrams, Fig. 177, show that the admittance of the circuit is .118 mho and the angle θ is equal to zero.

The vector expression for admittance is as follows:

$$\bar{Y}_1 = \frac{10}{100+69.4} - j\,\frac{8.33}{100+69.4} \qquad Y = .1181 \text{ mho.}$$

$$\bar{Y}_2 = \frac{10}{100+69.4} + j\,\frac{8.33}{100+69.4} \qquad Z = \frac{1}{.1181} = 8.47 \text{ ohms.}$$

$$\bar{Y} = .1181 \quad - \quad j\,0 \qquad \tan\theta = \frac{0}{.1181},\ \theta = 0.$$

The effect of the reactances in this circuit is noteworthy. It will be observed that the self-inductance and capacity neutralize each other's effects so that the angle of lag is zero, but the admittance is not much more than one half as great as would be the case if the two branches each contained 10 ohms of resistance and no reactance.

When 100 volts are impressed on this circuit,

$$\bar{I} = E\bar{Y} = 11.81 - j\,0$$

and $$I = EY = 11.81 \text{ amperes;}$$

$$\bar{I}_1 = E\bar{Y}_1 = \frac{1000}{169.4} - j\,\frac{833}{169.4}$$

and $$I_1 = EY_1 = 7.7 \text{ amperes;}$$

$$\bar{I}_2 = E\bar{Y}_2 = \frac{1000}{169.4} + j\,\frac{833}{169.4}$$

and $$I_2 = EY_2 = 7.7 \text{ amperes;}$$

$$\theta_1 = \tan^{-1}\frac{833}{1000} = 39^\circ\,48',\ \theta_2 = \tan^{-1} - \frac{833}{1000} = -39^\circ\,48'.$$

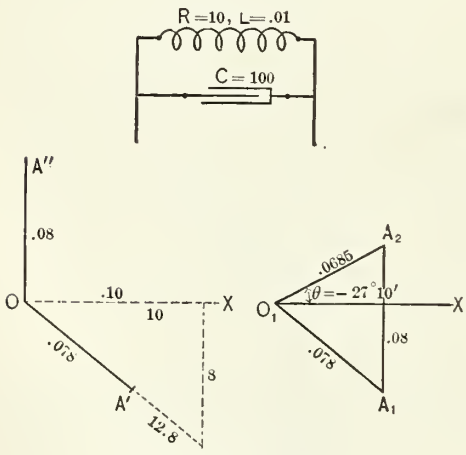


FIG. 178. — Solution *m*.

When 10 amperes are caused to flow through the circuit,

$$\bar{E} = I\bar{Z} = \frac{10}{.1181} + j\,0$$

and

$$E = IZ = 84.7 \text{ volts.}$$

Under the latter conditions the currents in the two branches are, respectively, 6.52 amperes and 6.52 amperes.

m. The circuit consists of two reactive branches, respectively,

of 10 ohms and .01 henry, and of 100 microfarads. The diagrams in Fig. 178 show the joint admittance to be .0685 mho. The joint impedance is 14.6 ohms the impedances of the branches are, respectively, 12.8 and 12.5 ohms, so that when the impressed voltage is 100 volts, 6.85 amperes flow in the main circuit, while 7.8 and 8 amperes, respectively, flow in the branches. The angle θ is $-27° 10'$.

It will be noticed in this case that not only is the arithmetical sum of the branch currents greater than the main current, but *each branch current is itself greater than the main current.*

The solution by complex quantities is as follows:

$$\overline{Y}_1 = \frac{10}{100+64} - j\frac{8}{100+64} \qquad Y = \sqrt{.0610^2 + .0312^2} = .0685 \text{ mho.}$$

$$Z = 14.6 \text{ ohms.}$$

$$\overline{Y}_2 = 0 \qquad +j.08$$

$$\overline{Y} = .0610 \qquad +j.0312$$

$$\tan\theta = -\frac{.0312}{.0610} = -.511,$$

$$\theta = -27° 10'.$$

When 100 volts are impressed on this circuit,

$$\overline{I} = E\overline{Y} = 6.1 + j\,3.12$$

and $$I = EY = 6.85 \text{ amperes;}$$

$$\overline{I}_1 = E\overline{Y}_1 = \frac{1000}{164} - j\frac{800}{164}$$

and $$I_1 = EY_1 = 7.8 \text{ amperes;}$$

$$\overline{I}_2 = E\overline{Y}_2 = j\,8$$

and $$I_2 = EY_2 = 8 \text{ amperes;}$$

$$\theta_1 = \tan^{-1}.8 = 38° 40', \quad \theta_2 = -\tan^{-1}\infty = -90°.$$

When 10 amperes are caused to flow through the circuit,

$$\overline{E} = I\overline{Z} = \frac{.610}{.00469} - j\frac{.312}{.00469}$$

and $$E = IZ = 146 \text{ volts.}$$

n. The circuit consists of two reactive branches in parallel, respectively, of .01 henry, and of 10 ohms and 100 microfarads. The diagrams, Fig. 179, show the admittance to be .0856 mho. and the impedance to be 11.7 ohms. The impedances of the branches are, respectively, 8 and 16 ohms, so that when 100 volts are impressed upon the circuit, 8.56 amperes flow in the

main leads, while 12.5 and 6.25 amperes flow, respectively, in the two branches. The angle θ is 62° 53′.

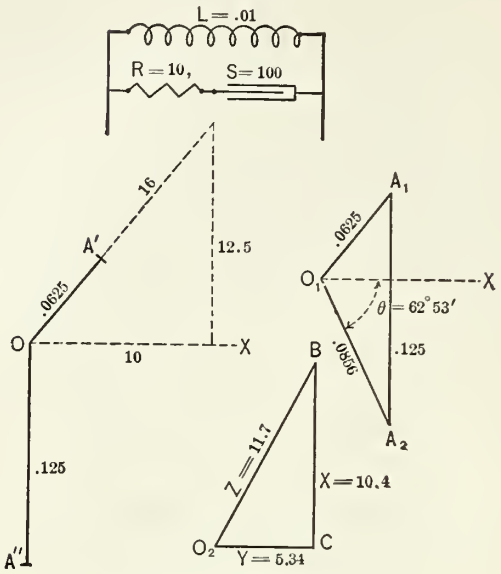


Fig. 179. — Solution *n*.

Here one branch current is much larger than the main current.

The solution by complex quantities is,

$$\overline{Y}_1 = \quad 0 \quad - j\frac{8}{0 + 64}$$

$$\overline{Y}_2 = \frac{10}{100 + 156.25} + j\frac{12.5}{100 + 156.25}$$

$$\overline{Y} = .0391 - j\,.0762.$$

$$Y = \sqrt{.0391^2 + .0762^2} = .0856 \text{ mho.}$$

$$Z = \frac{1}{.0856} = 11.7 \text{ ohms.}$$

$$\tan\theta = \frac{.0762}{.0391}, \quad \theta = 62° 53'.$$

When 100 volts are impressed on the circuit,

$$\overline{I} = E\overline{Y} = 3.91 - j\,7.62$$

and $$I = EY = 8.56 \text{ amperes;}$$

$$\bar{I}_1 = E\bar{Y}_1 = -j\,12.5$$

and $$I_1 = EY_1 = 12.5 \text{ amperes};$$

$$\bar{I}_2 = E\bar{Y}_2 = \frac{1000}{256.3} + j\frac{1250}{256.3}$$

and $$I_2 = EY_2 = 6.25 \text{ amperes};$$

$$\theta_1 = \tan^{-1}\infty = 90°, \quad \theta_2 = -\tan^{-1}\frac{1250}{1000} = -51° 20'.$$

When 10 amperes are caused to pass through the circuit,

$$\bar{E} = I\bar{Z} = \frac{.391}{.00733} + j\frac{.762}{.00733}$$

and $$E = IZ = 117 \text{ volts}.$$

The impedance diagram of the combined circuit O_2BC is given in the figure (Fig. 179). The sides of the triangle are shown from the expression

$$\bar{Z} = Z(\cos\theta + j\sin\theta)$$

or $$\bar{Z} = 5.34 + j\,10.4.$$

o. The circuit consists of two reactive branches in parallel, respectively, of .01 henry and of 100 microfarads. The diagrams, Fig. 180, show that the admittance of the circuit is .045 mho, and the admittance of the branches are, respectively, .125 and .08 mho. When the impressed voltage is 100 volts, the main current is 4.5 amperes, and those in the branches are 12.5 and 8 amperes, respectively. The angle θ is 90°.

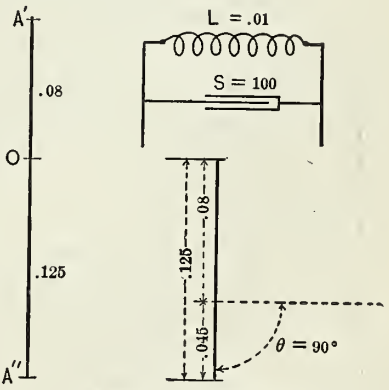


FIG. 180. — Solution *o*.

The corresponding complex quantities are:

$$\bar{Y}_1 = \frac{0}{0+64} - j\frac{8}{0+64} \qquad Y = .045 \text{ mho}.$$

$$\bar{Y}_2 = \frac{0}{0+156} + j\frac{12.5}{0+156} \qquad Z = 22.2 \text{ ohms}.$$

$$\bar{Y} = 0 - j.045 \qquad \tan\theta = \frac{.045}{0} = \infty, \ \theta = 90°.$$

In this problem the active (horizontal) component of current is zero, which is a condition that can only be approximated in practice. The vertical component of current composes the total current. The branch currents are in exactly opposite phases, and the main current is equal to their arithmetical difference.

When 100 volts are impressed on this circuit,

$$\bar{I} = E\bar{Y} = -j\,4.5$$

and $$I = EY = 4.5 \text{ amperes};$$

$$\bar{I}_1 = E\bar{Y}_1 = -j\,12.5$$

and $$I_1 = EY_1 = 12.5 \text{ amperes};$$

$$\bar{I}_2 = E\bar{Y}_2 = +j\,8$$

and $$I_2 = EY_2 = 8 \text{ amperes};$$

$$\theta_1 = \tan^{-1}\infty = 90°,\ \theta_2 = -\tan^{-1}\infty = -90°.$$

When 10 amperes are caused to pass through the circuit,

$$\bar{E} = I\bar{Z} = j\,222,$$

$$E = IZ = 222 \text{ volts}.$$

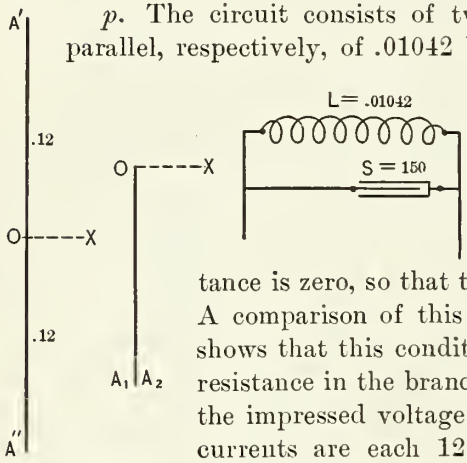


FIG. 181.—Solution *p*.

p. The circuit consists of two reactive branches in parallel, respectively, of .01042 henry and 150 microfarads. The diagrams, Fig. 181, show that the two branch currents are in opposition and of equal value, and that the admittance is zero, so that the main current is zero. A comparison of this problem with problem *l* shows that this condition can only arise when resistance in the branches is negligible. When the impressed voltage is 100 volts, the branch currents are each 12 amperes, and when 10 amperes flow in each branch, the voltage is 83.3 volts. No current can be caused to flow in the main circuit leading to these branches. The complex quantities representing the admittances are:

$$\bar{Y}_1 = 0 - j\,.12$$

$$\underline{\bar{Y}_2 = 0 + j\,.12}$$

$$\bar{Y} = 0 \mp j\,.0$$

$$Y = 0.$$

$$Z = \infty \text{ ohms}.$$

$$\theta = \tan^{-1}\frac{0}{0} = \text{indeterminate}.$$

If there is any appreciable resistance, however little, in either of these branches, Y and Z have finite values and θ is zero. In practical cases all resistance cannot be eliminated, and hence the angle may be considered zero.

If circuits approximating those given in this problem are constructed with very small resistance, excessively large local currents may flow when the generator current is almost negligible. Such cases are not unknown in practical operation.

84. Conclusions in Regard to Parallel Circuits. — *Second Class.*— The sixteen examples just presented cover every fundamental arrangement of simple parallel circuits. An examination of the diagrams and the principles involved in their construction makes evident the following statements, applicable to circuits in which the voltages and currents are sinusoidal, which are in many respects analogous to those heretofore given as applying to series circuits :

1. When *non-reactive circuits* are connected in parallel, the total current equals the arithmetical sum of the currents in the branches, and the joint admittance of the circuit is equal to the arithmetical sum of the branch admittances.

2. When *inductive circuits of equal time constants* are connected in parallel, the total current equals the arithmetical sum of the currents in the branches, and the joint admittance of the circuit is equal to the arithmetical sum of the branch admittances.

3. When *inductive and non-reactive circuits* are connected in parallel with each other, or when inductive *circuits of unequal time constants* are connected in parallel, the total current is equal to the vector sum, which is always less than the arithmetical sum, of the branch currents, and the individual branch currents are each smaller than the total current. The joint admittance of the circuit is equal to the vector sum of the branch admittances, each of which is less than the joint total.

4. When *condensers* are connected in parallel *by wires of negligible resistance*, the total current equals the arithmetical sum of the branch currents, and the joint admittance equals the arithmetical sum of the branch admittances.

5. When *condensers* are connected in parallel *with non-reactive resistances*, the total current equals the vector sum, which is always less than the arithmetical sum, of the branch

currents, and the individual branch currents are each smaller than the total current. The joint admittance of the circuit equals the vector sum of the branch admittances, each of which is smaller than the joint total.

6. When *condensers* are connected in parallel *with inductive circuits*, the total current equals the vector sum, which is always less than the arithmetical sum, of the currents in the branches. Since the effects of capacity and of self-inductance, respectively, cause the angle θ to become negative and positive, *the individual branch currents may be* EITHER GREATER OR LESS *than the main or total current*, depending upon the relation between the various capacities and inductances in the circuit. The joint admittance of the circuit equals the vector sum of the branch admittances, each of which may be either greater or less than the joint admittance.

The third, fifth, and sixth paragraphs make evident this proposition, which is similar to that given for series circuits : * When in parallel circuits the currents in the branches are all either lagging or leading with respect to the impressed voltage, the total or main current is always greater than the current in any one of the branches. When the currents in part of the branches lead the impressed voltage and in other branches lag behind the voltage, *some or all of the branch currents may be greater than the total or main current.* It is even theoretically possible for the angles to have such a relation that *a large current may circulate in the branches, while the main current is zero.*

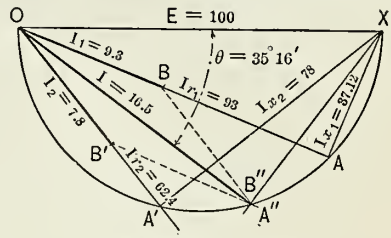
This is obviously the result of the opposing relations of inductive susceptance and capacity susceptance. It may be conceived that a local current is caused to circulate between the condenser and the inductance, under the stimulus of the alternating impressed voltage, acting to transfer energy back and forth between the capacity of the condenser and the magnetic field of the inductance.

When the current is not sinusoidal, the deductions given above do not strictly apply, but equivalent sinusoids † may frequently be substituted, when the approximation of the deductions is usually satisfactory. The deductions are general when applied to the harmonics of current and voltage when the latter are not sinusoidal.

* Art. 82. † Art. 90.

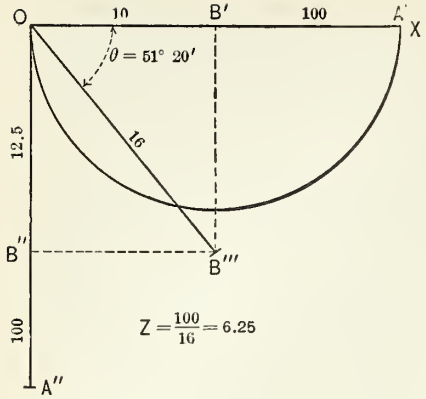
In every case referred to in these problems, it is assumed that the parts of the circuits have no appreciable mutual magnetic effect. If the parts are mutually inductive, the mutual effects must be added to those thus far treated, as will be seen in a later chapter.

85. Solution of Parallel Circuits by the Impedance Methods. — The solutions of parallel circuits may be made by another method in which voltages and impedances are principally involved instead of currents and admittance. This method may be readily exemplified by illustrations. Suppose, for instance, it is desired to find the joint impedance of the branched circuit in example *e* (Art. 83). It may be assumed that an impressed voltage of 100 volts acts on the circuit. Upon a line, *OX*, representing this voltage (Fig. 182) is drawn a semicircle. From *O* draw the line *OA* making a lag angle of θ_1 with *OX*, where $\tan\theta_1 = \frac{X_1}{R_1} = \frac{2\pi fL_1}{R_1} = .4$. Then *OA* is equal to I_1R_1 and *XA* is equal to I_1X_1, since the angle at *A* is a right angle. The current in this branch, when the impressed voltage is equal to *OX*, is $\frac{OA}{R_1}$, and this may be laid off from *O* to *B*. The current in the second branch is given by laying off the direction of the line *OA'* so that it makes a lag angle of θ_2 with *OX*, where $\tan\theta_2 = \frac{X_2}{R_2} = \frac{2\pi fL_2}{R_2} = 1.25$. The current in the second branch is equal to *OA'* divided by R_2, and when laid off from *O* gives *OB'*. The total current in the circuit is the resultant of *OB* and *OB'*, or *OB''*. Its value in amperes is 16.5. The impedance of the circuit is then $\frac{E}{I} = \frac{OX}{OB''} = \frac{100}{16.5} = 6.06$ ohms. The angle of lag is $\theta = \tan^{-1}\left(\frac{XA''}{OA''}\right) = 35°\ 16'$. Figures 183, 184, 185, 186, 187, and 188 give the solutions by the same method for examples



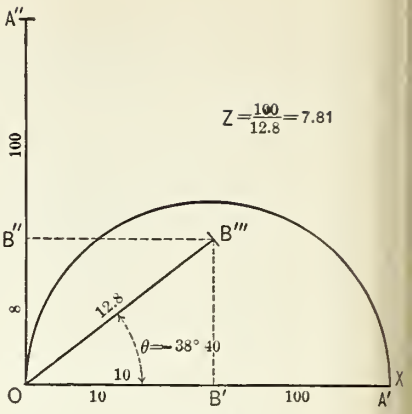
$Z = \frac{100}{16.5} = 6.06$ OHMS

FIG. 182. — Resultant Current in a Parallel Circuit.



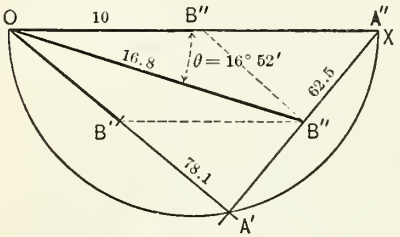
$$Z = \frac{100}{16} = 6.25$$

FIG. 183.



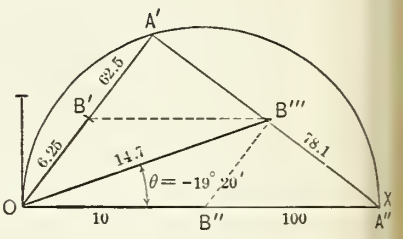
$$Z = \frac{100}{12.8} = 7.81$$

FIG. 186.



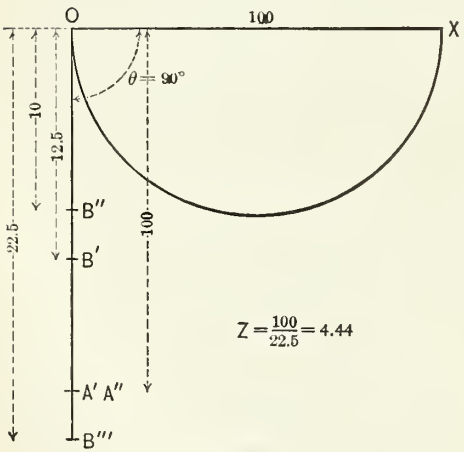
$$Z = \frac{100}{16.8} = 5.95 \text{ OHMS}$$

FIG. 184.



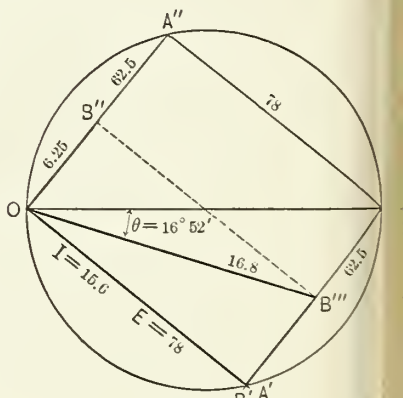
$$Z = \frac{100}{14.7} = 6.8$$

FIG. 187.



$$Z = \frac{100}{22.5} = 4.44$$

FIG. 185.



$$Z = \frac{100}{16.8} = 5.95$$

FIG. 188.

b, *c*, *d*, *g*, *h*, *j* of Art. 83. These show fully the application of the method.*

It is seen that the triangles *OXA*, *OXA'*, and *OXA''* are voltage diagrams with the apeces turned down for convenience; and that they can be converted into impedance diagrams by dividing the sides by the respective currents in the branches and combined circuit. Therefore, it is possible to use the complex quantities expressing the impedances for solving the problem. Thus,

$$\bar{Z}_1 = 10 + j\,4.$$

$$\bar{Z}_2 = 8 + j\,10.$$

$$Z_1 = \sqrt{10^2 + 4^2} = 10.77 \text{ ohms.}$$

$$Z_2 = \sqrt{8^2 + 10^2} = 12.81 \text{ ohms.}$$

$$I_1 = \frac{E}{Z_1} = \frac{100}{10.77} = 9.3 \text{ amperes.}$$

$$I_2 = \frac{E}{Z_2} = \frac{100}{12.81} = 7.8 \text{ amperes.}$$

$$\theta_1 = \tan^{-1} .4 = 21^\circ\, 48'.$$

$$\theta_2 = \tan^{-1} 1.25 = 51^\circ\, 20'.$$

$$\bar{I}_1 = I_1(\cos\theta_1 - j \sin\theta_1).$$

$$\bar{I}_1 = 8.62 - j\,3.45.$$

Likewise, $\bar{I}_2 = 4.87 - j\,6.09.$

The two current vectors added give

$$\bar{I} = 13.49 - j\,9.54.$$

$$I = \sqrt{13.49^2 + 9.54^2} = 16.5 \text{ amperes.}$$

$$Z = \frac{E}{I} = \frac{100}{16.5} = 6.06 \text{ ohms.}$$

The angle of lag of the main current is

$$\theta = \tan^{-1}\frac{9.54}{13.49} = 35^\circ\, 16'.$$

This is evidently more cumbersome than using the branch admittances directly, as in the process described in the preceding article.

* Compare Bedell and Crehore, *Alternating Currents*, p. 292; Loppé et Bouquet, *Courants Alternatifs Industriels*, p. 111.

86. Series and Parallel Circuits Combined. — *Third Class.* — Where series and parallel circuits are combined, the fundamental solutions already given apply directly, and it simply requires experience to acquire facility in the solutions relating to the most complicated circuits. Several examples are given below to indicate the general procedure. In these examples it is desired to determine as before: (1) the total impedance of the circuit, (2) the lag angle between the total current and the impressed voltage, (3) the total current flowing under a voltage of 100 volts, and (4) the voltage required to cause 10 amperes to flow. The frequency is taken nearly $127\frac{1}{2}$ cycles per second so that $2\pi f = 800$.

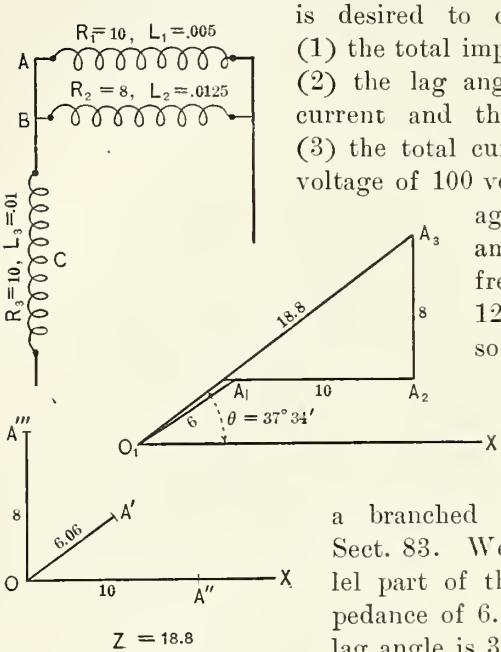


Fig. 189. — Solution *a*.

a. The circuit consists of an inductive coil of 10 ohms and .01 henry in series with a branched circuit similar to *e*, Sect. 83. We know that the parallel part of the circuit has an impedance of 6.06 ohms, and that the lag angle is 35° 16′. Therefore OA' (Fig. 189) is laid off in the phase diagram 6.06 units in length, and making the proper angle with the horizontal. The line OA'' is then laid off horizontally $R_3(=10)$ units long, and OA''' is then laid off vertically $2\pi f L_3(=8)$ units long. In the vector diagram O_1A_1 is equal and parallel to OA', A_1A_2 to OA'', and A_2A_3 to OA'''. The length of the line O_1A_3 gives the impedance of the circuit, which is equal to 18.8 ohms. The current which flows under a voltage of 100 volts is 5.32 amperes, and it requires 188 volts to cause 10 amperes to flow through the circuit. The angle of lag, θ, is the angle A_3O_1X, and is equal to 37° 34′.

By means of complex quantities we take first the admittance of the parallel branches:

$$\bar{Y}_A = .0862 - j\,.0345$$
$$\bar{Y}_B = .0487 - j\,.0609$$
$$\bar{Y}_{AB} = .1349 - j\,.0954$$
$$\tan\theta_{AB} = \frac{.0954}{.1349},\quad \theta_{AB} = 35°\,16'.$$
$$Y_{AB} = .165 \text{ mho.}$$
$$Z_{AB} = \frac{1}{.165} = 6.06 \text{ ohms.}$$

The impedance Z_{AB} can be divided into its components, thus (note the change of sign caused by changing from admittances to impedances):

$$\bar{Z}_{AB} = \frac{1}{\bar{Y}_{AB}} = \frac{.1349}{.0272} + j\frac{.0945}{.0272},$$

or

$$\bar{Z}_{AB} = 6.06(\cos\theta_{AB} + j\sin\theta_{AB})$$
$$= 6.06(.816 + j\,.577)$$
$$= 4.95 + j\,3.50.$$

This impedance is to be added to the impedance of the remaining portion of the circuit, giving:

$$\bar{Z}_{AB} = 4.95 + j\ 3.50$$
$$\bar{Z}_C = 10 \quad + j\ 8$$
$$\bar{Z} = 14.95 + j\,11.50$$
$$\tan\theta = \frac{11.50}{14.95},\quad \theta = 37°\ 34'.$$
$$Z = 18.8 \text{ ohms.}$$
$$I = \frac{100}{18.8} = 5.32 \text{ amperes when 100 volts are impressed.}$$
$$E = 10 \times 18.8 = 188 \text{ volts when 10 amperes flow.}$$

The components of current when 100 volts are impressed on the circuit are

$$\bar{I} = \frac{E}{\bar{Z}} = \frac{100}{14.95 + j\,11.50}$$
$$= 4.2 - j\,3.23;$$

that is, the active current is 4.2 amperes and the quadrature current is 3.23 amperes.

When 100 volts are impressed on the circuit, the current flowing in the main circuit is 5.32 amperes, and the currents in

the branches A and B are found from the condition that the voltage across the branched part of the circuit is

$$E_{AB} = Z_{AB}I = 6.06 \times 5.32 = 32.3 \text{ volts.}$$

The currents in the branches are, therefore,

$$\bar{I}_A = 32.3\ \bar{Y}_A = 2.78 - j\,1.11$$

and $$I_A = 2.99 \text{ amperes;}$$

$$\bar{I}_B = 32.3\ \bar{Y}_B = 1.57 - j\,1.97,$$

or $$I_B = 2.52 \text{ amperes.}$$

When 10 amperes are caused to flow in the main circuit, the currents in the branches are in the same ratio as 2.99 to 2.52.

b. The circuit consists of an inductance of .01 henry in series with a branched circuit having two parallel branches containing, respectively, 40 ohms and 100 microfarads. The joint impedance of the branched part of the circuit is first found in the usual manner. This is 11.9 ohms, and the lag is $-72°\,40'$. In the phase diagram, Fig. 190, OA' is therefore laid off equal to 11.9 and making a lag angle of $-72°\,40'$, and OA'' is laid off $2\pi fL(=8)$ units in length and making a lag angle of 90°. Laying off the vector diagram gives O_1A_2 equal to 4.9 and making a lag angle of $-43°\,40'$. The current flowing under a voltage of 100 volts is 20.4 amperes, and the voltage required to cause 10 amperes to flow is 49 volts. When 10 amperes flow in the main circuit, the voltage at the terminals of the branched circuit is 119 volts, and the currents which flow through the resistance and the condenser are, respectively,

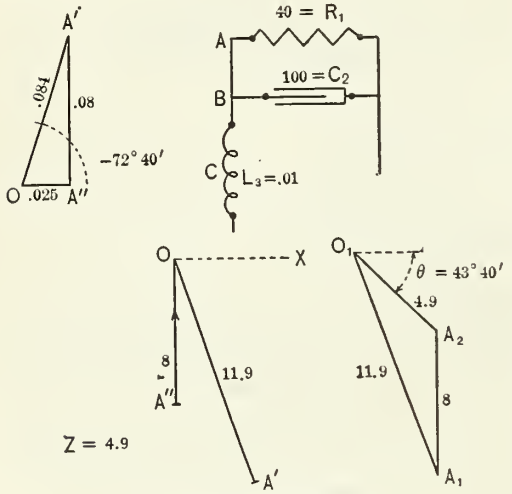


FIG. 190. — Solution *b*.

3 amperes and 9.5 amperes. The voltage across the inductance L_3 is then 80 volts.

By complex quantities the joint admittance of the parallel branches is:

$$\begin{aligned}\overline{Y}_A &= .025 + j\ 0\\ \overline{Y}_B &= 0 \quad + j\ .08\\ \overline{Y}_{AB} &= .025 + j\ .08\end{aligned}$$

$$\theta_{AB} = -\tan^{-1}\frac{.08}{.025} = -72^\circ\ 40'.$$

The components of the impedance of the branched circuit are:

$$R_{AB} = \frac{g}{Y_{AB}^2} = \frac{.025}{(.084)^2} = 3.55 \text{ ohms},$$

$$X_{AB} = \frac{b}{Y_{AB}^2} = \frac{.08}{(.084)^2} = 11.38 \text{ ohms}.$$

Therefore the impedance of the total circuit is:

$$\begin{aligned}\overline{Z}_{AB} &= 3.55 - j\ 11.38\\ \overline{Z}_C &= 0 \quad + j\ 8.0\\ \overline{Z} &= 3.55 - j\ 3.38\end{aligned}$$

$$Z = \sqrt{3.55^2 + 3.38^2} = 4.9 \text{ ohms}.$$

$$\theta = -\tan^{-1}\frac{3.38}{3.55} = -43^\circ\ 40'.$$

The components of the branched circuit may be obtained from the trigonometric expression as indicated in Solution *a*, or, if more convenient, from the logarithmic formula.

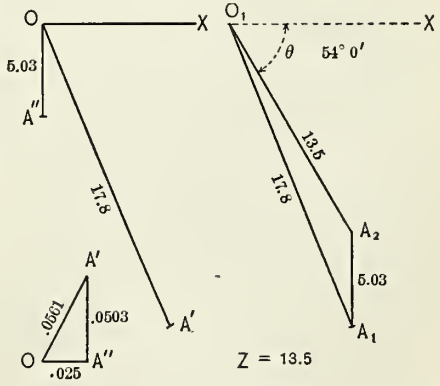


FIG. 191. — Solution b_1.

b_1. If the frequency in the preceding example is cut down to 80, the relations are materially changed. The impedance of the branched circuit becomes 17.8 ohms, and the lag angle in it is $-63^\circ\ 32'$. The phase diagram, therefore, is as

shown in Fig. 191. From the vector diagram it is seen that the joint impedance of the whole circuit is 13.5 ohms, and the total current is 54° ahead of the impressed voltage. The total current flowing when the impressed voltage is 100 volts is 7.4 amperes, and it requires 135 volts to cause 10 amperes to flow. When 10 amperes are flowing, the voltage at the terminals of the branched circuit is 178 volts, and the currents which flow through the resistance and the condenser are 4.45 and 8.9 amperes, respectively, while the voltage across the inductance L_3 is 50 volts. To maintain a voltage of 100 volts at the terminals of the divided circuit requires a voltage of 76 volts impressed on the total circuit. With this voltage, 5.6 amperes flow through the circuit.

$$\begin{aligned}
\bar{Y}_A &= .025 - j\,0 \\
\bar{Y}_B &= 0 \quad + j\,.0503 \\
\hline
\bar{Y}_{AB} &= .025 + j\,.0503 \\
\bar{Z}_{AB} &= 17.8\,(\cos\theta_{AB} - j\sin\theta_{AB}) \\
&= 17.8\,(.4456 - j\,.8960) \\
\bar{Z}_{AB} &= 7.94 - j\,15.95 \\
\bar{Z}_C &= 0 \quad + j\;5.03 \\
\hline
\bar{Z} &= 7.94 - j\;10.92
\end{aligned}$$

$$\tan\theta_{AB} = -\frac{.0503}{.025},\ \theta_{AB} = -63°32'.$$

$$Y_{AB} = .0561 \text{ mho.}$$

$$Z_{AB} = \frac{1}{.0561} = 17.8 \text{ ohms.}$$

$$\tan\theta = -\frac{10.92}{7.94},\ \theta = -.54°\,0'.$$

$$Z = 13.5 \text{ ohms.}$$

$$I = \frac{100}{13.5} = 7.4 \text{ amperes when 100 volts are impressed.}$$

$$E = 10 \times 13.5 = 135 \text{ volts when 10 amperes flow.}$$

When 100 volts are maintained on the divided part of the circuit,

$$\bar{I}_{AB} = E\bar{Y}_{AB} = .025 \times 100 + j\,.0503 \times 100 = 2.5 + j\;5.03$$

and $$I_{AB} = \sqrt{2.5^2 + 5.03^2} = 5.61 \text{ amperes.}$$

The voltage measured between the terminals of the part C under these conditions is

$$\bar{E}_C = \bar{Z}_C \times 5.61 = (0 + j\;5.03) \times 5.61,$$

$$E_C = 28.2 \text{ volts,}$$

and $$\theta_C = 90°.$$

The voltage impressed on the total circuit under these conditions is

$$\bar{E} = 5.61\,\bar{Z} = 44.5 - j\;61.3,$$

and $$E = 5.61\,Z = 76 \text{ volts.}$$

It will be observed that the current, voltage, and angle of lag in any part of the circuit under any conditions may be computed by these processes when the conditions are sufficiently known.

c. The circuit consists of a combination as shown in Fig. 192. The resistance of the branches of the circuit are $R_1 = 8$ ohms, $R_2 = 10$ ohms, $R_3 = 5$ ohms; the inductances are $L_1 = .0125$ henry, $L_2 = .01$ henry, $L_3 = .005$ henry; and the capacities are

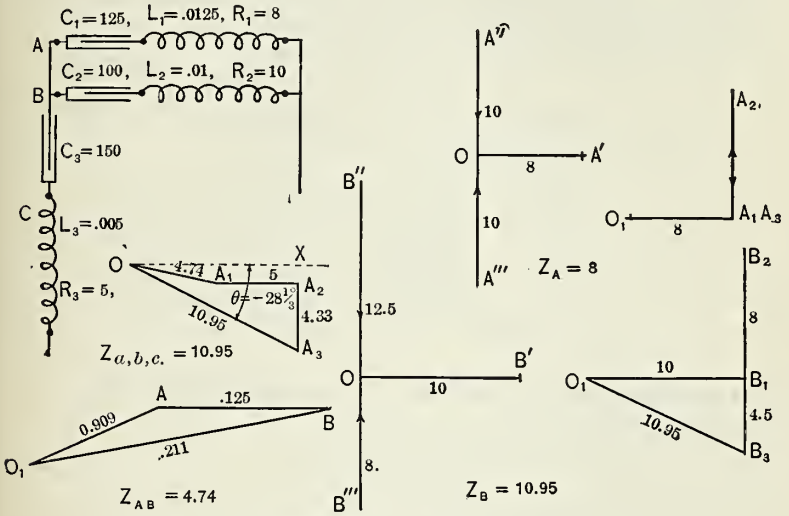


FIG. 192 —Solution *c*.

$C_1 = 125$ microfarads, $C_2 = 100$ microfarads, $C_3 = 150$ microfarads. The diagrams show the impedance of the branched part of the circuit to be 4.74 ohms, and its lag angle is $-10° 12'$. From the complete diagrams it is seen that the joint impedance of the whole circuit is 10.95 ohms, and the total current is $28° 10'$ ahead of the impressed voltage. The total current flowing when the impressed voltage is 100 volts is 9.12 amperes, and it requires 109.5 volts to cause 10 amperes to flow. When 10 amperes are flowing, the voltage at the terminals of the branched circuit is 47.4 volts, and the currents which flow through the branches are 5.9 and 4.3 amperes, respectively. The voltage across the first part of the circuit is then 66 volts. To maintain a voltage of 100 volts on the branched part of the circuit requires an impressed voltage of

231.1 volts on the whole. With this voltage 21.1 amperes flow through the circuit.

The calculation by complex quantities may be made as follows:

$$\overline{Z}_A = 8 + j\,10 - j\,10 = 8 + j\,0.$$

$$\overline{Z}_B = 10 + j\,8 - j\,12.5 = 10 - j\,4.5.$$

Hence,

$$\overline{Y}_A = .125 - j\,0 \qquad Y_{AB} = \sqrt{b^2 + g^2} = .211 \text{ mho.}$$

$$\overline{Y}_B = .083 + j\,.0374 \qquad Z_{AB} = \frac{1}{.211} = 4.74 \text{ ohms.}$$

$$\overline{Y}_{AB} = .208 + j\,.0374$$

$$\tan \theta_{AB} = -\frac{b}{g}, \quad \theta_{AB} = -10^\circ\,12'.$$

$$\overline{Z}_{AB} = \frac{1}{\overline{Y}_{AB}} = 4.66 - j\,0.84$$

$$\overline{Z}_C = 5 - j\,4.33$$

$$\overline{Z} = 9.66 - j\,5.17$$

$$Z = \sqrt{R^2 + X^2} = 10.95 \text{ ohms.}$$

$$\theta = \tan^{-1}\frac{X}{R} = -28^\circ\,10'.$$

The main current with 100 volts impressed between the circuit terminals is

$$\bar{I} = \frac{E}{\overline{Z}} = \frac{966}{120} + j\,\frac{517}{120},$$

$$I = \frac{E}{Z} = 9.12 \text{ amperes.}$$

While, with 10 amperes flowing in the main circuit,

$$E = IZ = 109.5 \text{ volts,}$$

and the voltage at the terminals of the branched circuit is

$$E_{AB} = IZ_{AB} = 47.4 \text{ volts.}$$

The currents flowing in the parallel branches are then

$$I_A = E_{AB} Y_A = 5.9 \text{ amperes}$$

and

$$I_B = E_{AB} Y_B = 4.3 \text{ amperes.}$$

Also, the voltage across C is

$$E_C = IZ_C = 66 \text{ volts.}$$

To obtain a voltage of 100 volts on A and B requires that

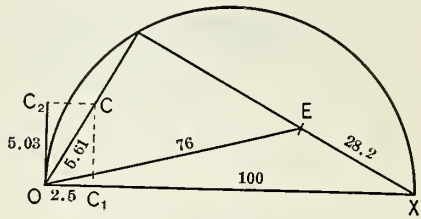
$$I = \frac{100}{Z_{AB}} = 21.1 \text{ amperes.}$$

Hence the voltage impressed on the terminals is

$$E = IZ = 231.1 \text{ volts.}$$

87. Solution of Series-Parallel Problems by the Impedance or Impressed Voltage Method. — The second method of solution for parallel circuits may be applied to circuits like those included in the above article. Figure 193 shows the solution for example b_1 made by that method. In this it is assumed that 100 volts are impressed upon the branched part of the circuit. Then lay off a length OX on the horizontal axis representing 100 volts and mark

$$OC_1 = \frac{100}{40} = 2.5,$$



$$Z = \frac{76}{5.61} = 13.5$$

FIG. 193. — Solution of Problem b_1 by Voltage Diagram.

which is the current in the first branch. The current in the second branch is 90° in advance of the voltage, and is represented by OC_2, which is vertical and $2\pi f C_2 E\,(= 5.03)$ units in length. The resultant of these currents is OC, which is 5.61 units in length. The impressed voltage measured across the terminals of the entire circuit is the resultant of the 100 volts at the terminals of the branched part of the circuit, and the voltage required to pass 5.61 amperes through the inductance $L_1 = .01$ henry. The line representing the latter voltage is perpendicular to the line representing the current in the circuit. Drawing a semicircle on OX, and from the intersection of OC with the semicircle drawing a line to X gives the direction of this voltage. The magnitude of the voltage is $2\pi f L_1 I = 28.2$ volts. This must be laid off from X to E, and the total impressed voltage is represented by OE. This shows that when 100 volts are maintained at the terminals of the branched circuit, 76 volts must be impressed on the total circuit. The impedances of the circuit may be calculated from the data thus

found, as also can the voltage required to maintain a certain current through the circuit.

Figure 193*a* shows the solution of example *c* by this method. As before, the voltage at the terminals of the divided circuit is assumed to be 100 volts for the purpose of the solution. This is laid down as OX, and a semicircle is drawn upon the line as a diameter. Tan θ_A is equal to zero, so that the current in the first branch is laid off on OX to B, a distance of 12.5 units.

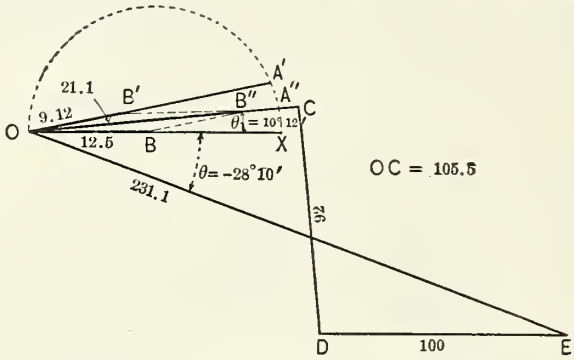


FIG. 194. — Solution of Problem *c* by Voltage Diagram.

Tan $\theta_A = .45$, as shown by calculation, and the line OA' is laid off at that angle from OX. From O on this line, OB' is laid off equal to the current in the second branch, or $\frac{1}{Z_B} OX$. The resultant of the lines OB and OB' is OB'', which represents the total current in the circuit. The total voltage impressed on the circuit is the resultant of the voltage impressed on the divided circuit, the active voltage due to resistance R_3, and the reactive voltage due to L_3 and C_3. The active voltage required to pass current OB'' through R_3 is represented by OC, which is equal to IR_3. The reactive pressure is perpendicular to this and is equal to $2\pi f L_3 I - \frac{I}{2\pi f C_3}$; it is represented by the line CD. The voltage impressed on the parallel circuit is represented by the line DE, which is equal and parallel to OX. The closing line, OE, represents the impressed voltage E on the circuit when the current is I, and therefore the impedance of the circuit is $\frac{E}{I} = 10.95$. The angle of lag is the angle $COE = -28° 10'$.

87a. Caution. — The foregoing deductions of Arts. 81 to 88 relate to the impedances and admittances in circuits within which the currents and voltages are sinusoidal, and to the effective values of such voltages and currents. They also relate to the impedances and admittances encountered by single harmonics of distorted voltages and currents. That is, impedances, admittances, effective voltages, and effective currents combine vectorially. But the student must constantly remember that *instantaneous* values of voltages or of currents at a junction point must be combined algebraically since the instantaneous values are simple algebraic values and the vector relations heretofore considered do not extend to them. It is frequently possible also, for approximate solutions, to substitute equivalent sinusoids * for the irregular curves, though on account of the extreme activity which is sometimes exhibited by the higher harmonics † — especially where a condition of resonance is approached — each problem should be carefully studied before such a substitution is made.

* Art. 90. † Art. 69.

CHAPTER VII

POWER, POWER FACTOR

88. The Power expended in a Circuit on which a Sinusoidal Voltage is Impressed. — *a. Non-reactive Circuit.* In a circuit without inductance or capacity, the current agrees in phase with the voltage which sets it up. The rate of expenditure of energy in the circuit at any instant is equal to the product of the corresponding instantaneous current and voltage. The average rate of expenditure of energy, or the average value of the power expended in the circuit, during a complete period is equal to the average of all the instantaneous products. Or,

$P = \frac{2}{T}\sum_0^{\frac{T}{2}} ie\ \Delta t$, but with the voltage and current sinusoidal

$$e = e_m \sin\ a \text{ and } i = \frac{e}{R},$$

where T is the time of a period and e_m is the maximum voltage;

hence, $$P = \frac{2}{T}\sum_0^{\frac{T}{2}} ie\ \Delta t = \frac{e_m^2}{\pi R}\int_0^{\pi} \sin^2 a da = \frac{e_m^2}{2R},$$

but $$E = \frac{e_m}{\sqrt{2}} \text{ and } I = \frac{E}{R} = \frac{e_m}{\sqrt{2}\,R}.$$

Hence, $$P = \frac{e_m^2}{2R} = IE,$$

I and E being the effective values of the current and voltage.

b. Reactive Circuit. If the circuit under consideration is reactive, the current is caused to lag behind or lead the voltage by the angle θ. The rate of expenditure of energy in the circuit at any instant is evidently still equal to the product of the corresponding instantaneous values of the current and voltage.

The expression for the average power expended in the circuit is, therefore, as before,

$$P = \frac{2}{T}\sum_0^{\frac{T}{2}} ie\ \Delta t.$$

In this case, however, with the voltage and current sinusoidal

$$e = e_m \sin \alpha, \text{ and } i = \frac{e_m}{Z} \sin (\alpha - \theta).*$$

$$\text{Hence, } P = \frac{e_m^2}{\pi Z}\int_0^{\pi} \sin \alpha \sin (\alpha - \theta) d\alpha = \frac{e_m^2 \cos \theta}{\pi Z}\int_0^{\pi} \sin^2 \alpha d\alpha$$

$$- \frac{e_m^2 \sin \theta}{\pi Z}\int_0^{\pi} \sin \alpha \cos \alpha d\alpha = \frac{e_m^2 \cos \theta}{2Z},$$

and since $e_m = E\sqrt{2}$, and $\frac{e_m}{Z} = i_m = I\sqrt{2}$, there follows

$$P = IE \cos \theta.$$

Also
$$\cos \theta = \frac{P}{IE}.$$

Since sinusoidal current and voltage curves have equal positive and negative loops, the expression thus derived for the power expended in a circuit during one half period applies to every half period, and therefore to continuous operation. In the ordinary measurements of current and voltage the effective values of the quantities are determined. Consequently, *the product of amperes and volts*, thus determined, *does not represent the power expended in a reactive circuit, but the product must be multiplied by the cosine of the angle of lag.* On the other hand, a **Wattmeter**, that is, an electrodynamometer with one coil of low resistance connected in series with the circuit and another coil of high resistance connected in shunt with the circuit, averages the instantaneous products, and therefore gives readings that are directly proportional to the power absorbed.

The preceding expressions show that the power expended in a reactive circuit is equal to the voltage E, multiplied by a component of the current which is in phase with the voltage and is equal to $I \cos \theta$. This component may be called the **Active** or **Energy current**. The remaining rectangular component of current, $I \sin \theta$, is in quadrature (that is, at 90° difference of phase) with the voltage and does not contribute to the

* Arts. 58 *c* and 62.

power expenditure when taken through a complete period. This quadrature component which does no work during a full period may be called the **Wattless** or **Quadrature current**. For illustration, suppose in Fig. 195 that OE is the vector voltage impressed on a circuit, OI is the current, and θ the angle of lag. Resolving OI into its components in consonance and in quadrature with the voltage, gives $OI_a = OI \cos\theta$ and $OI_x = OI \sin\theta$. Multiplying the former by E gives the power expended, $EI \cos\theta$, and the component OI_x is therefore inactive when averaged over a whole period.

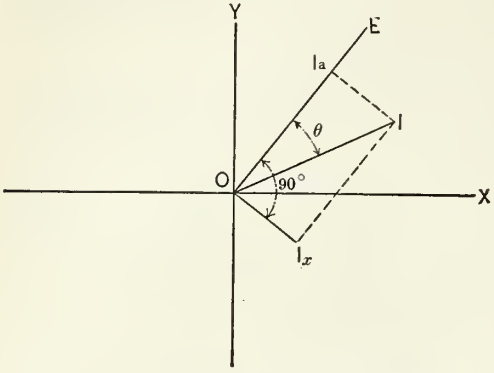


FIG. 195. — Vector Diagram of Current Components and Voltage.

Since instantaneous power is equal to ei, it is obvious that quadrature current must represent work done during some part of the period, and that the wattless character must therefore be due to alternate absorption and return of power by the circuit. Taking the summation of $ei\Delta t$ for successive quarters of a period proves this. Thus,

$$\frac{2\,e_m^2}{\pi Z}\int_0^{\frac{\pi}{2}} \sin\alpha \sin(\alpha \mp 90°)d\alpha = -\frac{2\,e_m^2}{\pi Z}\int_{\frac{\pi}{2}}^{\pi} \sin\alpha \sin(\alpha \mp 90°)d\alpha,$$

and

$$\frac{e_m^2}{\pi Z}\int_0^{\pi} \sin\alpha \sin(\alpha \mp 90°)d\alpha = 0.$$

This shows that the quadrature current is inactive only when considered for any consecutive length of time equal to a full half period. It in fact is a vehicle of energy which swashes back and forth in the circuit, in one quarter period charging up the magnetic or electrostatic fields and in the next quarter period discharging the fields and returning the energy to the source, the next quarter period charging the fields in the opposite direction, followed again by the return of the energy

to the source in the succeeding quarter period, thus balancing off the ebb and flow of energy in each half period and calling for no continuous expenditure of power.

If θ were 90°, the total current would be in quadrature with the voltage and therefore inactive. This would be possible only in a circuit having no electrical resistance or other grounds for conversion of electrical energy into heat, light, or mechanical power; otherwise some power would necessarily be expended in heating the conductors, etc., and the current would have to include an energy component. It is possible, however, to make the ratio of inductive reactance $2\pi fL$ so great in comparison with the resistance R, by using special inductive coils, that the angle θ is very nearly 90°. It is also possible to make the capacity of a circuit so great in comparison with the resistance that the angle θ is nearly $-90°$; and thereby these deductions in regard to wattless current may be tested in the laboratory.

Prob. 1. The current flowing in a circuit is 65 amperes and the voltage is 200 volts; the angle of lag between the current and voltage is 60°. What is the power expended in the circuit? Would a change of frequency change the power, the voltage, current and angle of lag remaining the same?

Prob. 2. Three reactive coils respectively comprising resistance and self-inductance as follows: 4 ohms and .01 henry, 8 ohms and .02 henry, and 10 ohms and .05 henry, are connected in series. Fifty amperes flow in the circuit. If the frequency is 60 periods per second, how much power is expended in the total circuit, and how much in each part?

Prob. 3. An overhead transmission line has a resistance of 2 ohms and a self-inductance of .0002 of a henry. This circuit supplies fully loaded transformers which in effect give a non-inductive load which has an equivalent resistance of 10 ohms. If the voltage at the receiving end where the transformers are placed is 1000 volts when the frequency is 60 periods per second, what is the voltage at the generator, how much power is lost in the line, and how much is utilized by the transformers?

Prob. 4. The three reactive coils of Prob. 2 are placed in parallel, and 100 volts with a frequency of 60 periods per second are impressed between their terminals. What is the total power expended in the circuit?

89. Power in a Circuit. *An Exercise.* — Blakesley has given a graphical proof of the formula $P = IE \cos \theta$ when I and E are sinusoidal, which is presented here for the purpose of an exercise. Applying the graphical representation of alternating voltages and currents by means of rotating lines, let AB and AC (Fig. 196) represent respectively the maximum value of the impressed voltage in a circuit and the maximum value of the resulting current. The angle BAC is the angle of lag. If the lines rotate counter-clockwise about the point A, the instantaneous projections of the lines AB and AC upon the axis of Y represent the instantaneous values of the voltage and current, when α is measured from the X axis. It is therefore desired to determine the average value of the products of these projections for the purpose of obtaining the average power for a period. Draw AB' and AC' respectively perpendicular and equal to AB and AC. These lines represent the positions of AB and AC after revolving through 90°. In the figure, the angle BAX represents α, and CAX represents $\alpha - \theta$. Also, the angle $B'AY = BAX$, and $C'AY = CAX$. It is then seen from the figure that

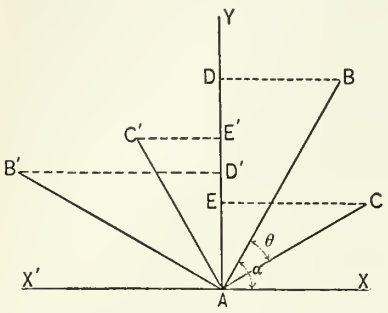


Fig. 196. — Voltage and Current Vectors.

$$AE \times AD = AC \sin CAX \times AB \sin BAX,$$

or

$$ie = i_m \sin(\alpha - \theta) \times e_m \sin \alpha;$$

and in the same way

$$AE' \times AD' = AC' \cos C'AY \times AB' \cos B'AY,$$

or

$$i'e' = i_m \cos(\alpha - \theta) \times e_m \cos \alpha.$$

The mean of these expressions is

$$\frac{ie + i'e'}{2} = \frac{i_m e_m}{2}[\sin \alpha \sin(\alpha - \theta) + \cos \alpha \cos(\alpha - \theta)]$$

$$= \frac{i_m e_m}{2} \cos[\alpha - (\alpha - \theta)] = \frac{i_m e_m}{2} \cos \theta = IE \cos \theta.$$

This is the expression for the mean of the products of e and i for two values of α which are 90° apart. This mean value is independent of the positions of the lines in the figure, and is therefore the mean for all positions.

90. Power expended in a Circuit when the Voltage and Current are Any Single-valued Periodic Functions. — If the current in a circuit is not sinusoidal and is derived from a periodic voltage of irregular form, the instantaneous values of current and voltage are *

$$i = i_{m_1}\sin(\alpha+\beta_1) + i_{m_2}\sin(2\alpha+\beta_2) + i_{m_3}\sin(3\alpha+\beta_3)\cdots + i_{m_n}\sin(n\alpha+\beta_n),$$

and $$e = e_{m_1}\sin(\alpha+\gamma_1) + e_{m_2}\sin(2\alpha+\gamma_2) + e_{m_3}\sin(3\alpha+\gamma_3)\ldots + e_{m_n}\sin(n\alpha+\gamma_n).$$

Also, ie is the instantaneous value of power, and $\frac{1}{T}\int_0^T eidt$ is the average power; therefore,

$$P = \frac{1}{T}\int_0^T eidt = \frac{1}{2\pi}\Big[\int_0^{2\pi} i_{m_1}e_{m_1}\sin(\alpha+\beta_1)\sin(\alpha+\gamma_1)\,d\alpha + \int_0^{2\pi} i_{m_2}e_{m_2}\sin(2\alpha+\beta_2)\sin(2\alpha+\gamma_2)\,d\alpha + \int_0^{2\pi} i_{m_3}e_{m_3}\sin(3\alpha+\beta_3)\sin(3\alpha+\gamma_3)\,d\alpha\cdots + \int_0^{2\pi} i_{m_n}e_{m_n}\sin(n\alpha+\beta_n)\sin(n\alpha+\gamma_n)\,d\alpha\Big].$$

All the other terms obtained by multiplying the values of i and e together are omitted as they become zero when integrated between 0 and 2π.†

Expanding and integrating the above expression gives,

$$P = \tfrac{1}{2}\,i_{m_1}e_{m_1}\cos(\beta_1-\gamma_1) + \tfrac{1}{2}\,i_{m_2}e_{m_2}\cos(\beta_2-\gamma_2) + \tfrac{1}{2}\,i_{m_3}e_{m_3}\cos(\beta_3-\gamma_2)\cdots + \tfrac{1}{2}\,i_{m_n}e_{m_n}\cos(\beta_n-\gamma_n).$$

* Art. 67. † Art. 13.

Since each β represents the angular displacement of a current harmonic from the zero of time and each γ represents the angular displacement of a voltage harmonic from the zero of time, it is apparent that each $(\beta - \gamma)$ represents the angular difference of phase between the voltage and current for the particular harmonic; that is, it corresponds with the angle of lag θ associated with the frequency of the particular harmonic. Therefore, since $\frac{1}{2} i_m e_m = IE$,

$$P = I_1 E_1 \cos\theta_1 + I_2 E_2 \cos\theta_2 + I_3 E_3 \cos\theta_3 + \cdots I_n E_n \cos\theta_n.$$

Each term to the right in this formula is the power expended in the circuit by the particular harmonic and is independent of the other harmonics. The total power expended in the circuit is therefore the sum of the power independently expended by the several harmonics. This theorem also follows from the fact that the product of rotating vectors differing in frequency is equal to zero, and hence $I_m E_n = 0$, where m and n are unequal integers. It will be observed that the power expended in a circuit when the current and voltage are sinusoidal may be derived from the foregoing formula, since, in that case, all of the higher harmonics are zero.

When $\theta = 90°$ for any harmonic, that harmonic is inactive. Also if a particular harmonic of current is present but the corresponding harmonic of voltage is zero, the current harmonic is obviously inactive; and likewise, if a voltage harmonic is not accompanied by a current harmonic of the same frequency, the voltage harmonic does not contribute to average power, though it may contribute to the energy that swashes back and forth in the circuit and balances out for each half period.

The effective current squared in a circuit is

$$I^2 = \frac{i_{m_1}^2}{2} + \frac{i_{m_2}^2}{2} + \frac{i_{m_3}^2}{2} \cdots \frac{i_{m_n}^2}{2} = I_1^2 + I_2^2 + I_3^2 + \cdots I_n^2, *$$

and the power expended in heating a conductor of resistance R when current I flows is represented by

$$I^2 R = I_1^2 R + I_2^2 R + I_3^2 R + \cdots I_n^2 R.$$

* Art. 67.

The active voltage in the circuit may be written for convenience $E \cos \theta$, which, substituted for its equivalent IR in the last formula, gives as before,

$$P = IE \cos \theta = I_1 E_1 \cos \theta_1 + I_2 E_2 \cos \theta_2 + I_3 E_3 \cos \theta_3 + \text{etc.}$$

With irregular waves the phase difference θ cannot be measured directly from the plotted curves, but $\theta = \cos^{-1} \frac{P}{IE}$. It is sometimes convenient to use the formula $P = IE \cos \theta$ in place of the more complicated formula $P = I_1 E_1 \cos \theta_1 + \text{etc.}$, and in that case I and E may be thought of as like sinusoidal functions differing in phase by the angle θ. When so considered I and E are called **Equivalent sinusoids.** Under some circumstances equivalent sinusoids can be substituted for compound curves without error, but as harmonics of different frequencies differ in their effects on circuits containing self-inductance and capacity, the substitution must be made with discretion.

91. Alternating and Pulsating Functions. — It has heretofore been pointed out that commercial alternating currents and voltages ordinarily comprise harmonics of only odd numbers of times the fundamental frequency, namely, 1, 3, 5, 7, etc., in which case the successive half cycles are like in form; but there are nevertheless many instances in which harmonics appear of even numbers of times the fundamental frequency. The foregoing demonstration has therefore been made general in respect to the frequencies of the harmonics, but the following discussion will be confined mostly to the formulas with the even-numbered harmonics omitted. These may easily be reintroduced, however, for use in connection with any particular problem requiring their presence. It is also to be noted that the introduction of a constant term into the general formulas causes a larger average current to flow in one direction than the other or converts the alternating functions into pulsating functions. Then,

$$I^2 = I_0^2 + I_1^2 + I_2^2 + I_3^2 + \cdots I_n^2,$$
$$E^2 = E_0^2 + E_1^2 + E_2^2 + E_3^2 + \cdots I_n^2;$$

the power expended in heating a conductor of resistance R, by

$$I^2R = I_0^2R + I_1^2R + I_2^2R + I_3^2R + \cdots I_n^3R;$$

and, $$P = IE\cos\theta = I_0E_0 + I_1E_1\cos\theta_1 + I_2E_2\cos\theta_2 + I_3E_3\cos\theta_3 + \cdots I_nE_n\cos\theta_n;$$

in which I_0 and E_0 respectively represent the average values of the current and voltage for a whole period.

92. Power Loops. — Power loops or curves may be plotted as in Figs. 197 to 201, in which the ordinates of the dotted curves represent the products of the corresponding ordinates of the current and voltage curves (assumed sinusoidal) plotted in full lines. Figure 197 is for a non-inductive circuit in which the voltage and current reverse their directions at the same time, and the power ordinates are therefore always positive, but their numerical values vary in each half period from 0 to i_me_m and back to 0, so that the power absorbed by the circuit varies continually during each half period. In this case $\theta = 0$, $\cos\theta = 1$, and the average power is $P = IE$. Figure 198 shows the power loops for a reactive circuit in which the angle of lag is 45°. This may be taken to represent equally the condition when the current leads or lags. It will be seen in this case that during a portion of each half period the current and voltage are in

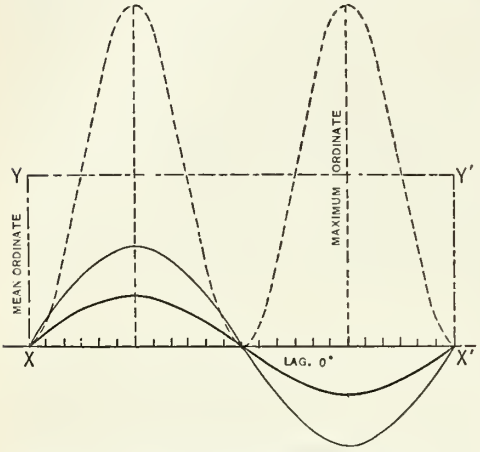


FIG. 197. — Power Loops in Non-reactive Circuit.

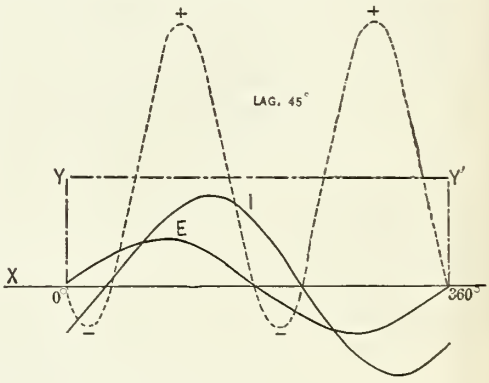


FIG. 198. — Power Loops in a Circuit having a 45° Lag.

opposite directions, and some of the ordinates of the power loops are therefore negative. This must always be the case when the current and voltage do not coincide in phase. During the portion of the half period in which the ordinates of the power loops are positive the circuit absorbs energy, but during the portion in which the ordinates are negative the circuit gives out energy which was stored as magnetic field or condenser charge and returns it to the source.

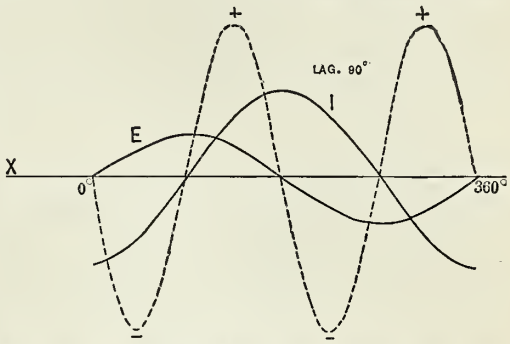


FIG. 199. — Power Loops in a Circuit having a 90° Lag.

The total work given to the circuit during the half period is equal to the difference of that represented by the area of the positive loop and that represented by the area of the negative loop, and the average power absorbed by the circuit is equal to this difference divided by the length of the half period. When

$\theta = 45°$,

$P = IE \cos 45°$

$= .707\, IE.$

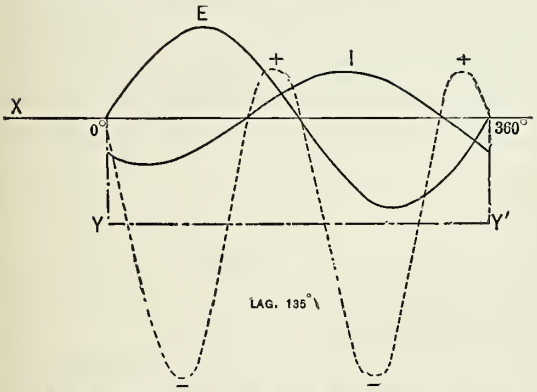


FIG. 200. — Power Loops showing Negative Work. Current lags 135°.

When the ordinates of the loops represent products of instantaneous volts and amperes, the areas represent joules, and the areas divided by their respective lengths measured in seconds give average power in watts.

Figure 199 shows the power loops for a circuit in which the current and voltage differ in phase by 90°. In this case the negative loops are equal to the positive ones, or the circuit and source alternately give and take equal amounts of energy, so

that, taking each half period as a whole, no energy is absorbed by the circuit. In this case $\cos\theta = 0$, and therefore $P = 0$. Figure 200 shows the power loops when the current lags 135°.

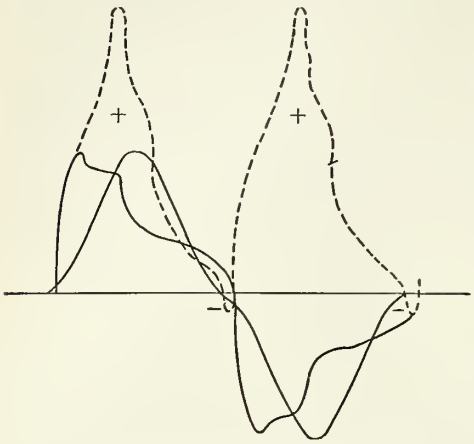


FIG. 201. — Power Loops derived from Irregular Voltage and Current Curves.

Under these circumstances the power is negative, *i.e.* the current is working against the induced voltage, as in a motor.

In case the curves of voltage and current are not sinusoidal, the power loops are not symmetrical. Such loops are shown by the broken lines in Fig. 201.

Power loops are periodic curves of double the frequency of the voltage and current curves, since they are obtained by multiplying current and voltage together, and the argument of the product is equal to $\alpha+(\alpha-\theta)=2\alpha-\theta$. That is, a cycle of the power loops is completed for each half period of current and voltage. The power loops may be expressed by means of Fourier's series in the same manner as any other single-valued periodic curve; thus,

$$p = P + p'_m \sin(2\alpha + \delta_1) + p''_m \sin(4\alpha + \delta_2) + \text{etc.}^*$$

Where p is the instantaneous power corresponding to the angle α, P is the net average power taken over a period, and p'_m, p''_m, etc., are the amplitudes of the harmonics.

Prob. 1. A circuit has a resistance of 8 ohms and a self-inductance of .01 of a henry in series. The voltage impressed is 100 volts, at a frequency of 120 cycles per second. Construct the power loops, assuming the voltage to be sinusoidal.

Prob. 2. A circuit has a resistance of 20 ohms, a capacity of 400 microfarads, and a self-inductance of .01 henry in series. Draw the power loops produced when a sinusoidal voltage of 200 volts at a frequency of 40 cycles per second is impressed thereon.

* Art. 12.

Prob. 3. A circuit of 100 microfarads capacity and 5 ohms resistance, has a sinusoidal voltage of 50 volts at a frequency of 60 periods per second induced in it. A sinusoidal current of 50 amperes is caused by an external impressed voltage to flow through the circuit in exact opposition to the induced voltage. What is the magnitude and relative phase of the impressed voltage? Is the power positive or negative? Construct the power loops.

93. Trigonometrical Proof that Power Curves produced by Sinusoidal Voltages and Currents are Double Frequency Sinusoids with their Axes Displaced. — Sinusoidal current and voltage of equal frequency in a circuit having any fixed phase relation produce instantaneous power that may in general be expressed

$$p = e_m i_m \sin \alpha \sin(\alpha - \theta),$$

where θ is the angle of lag. This expanded gives

$$p = e_m i_m [\sin^2 \alpha \cos \theta - \sin \alpha \cos \alpha \sin \theta].$$

The trigonometrical part of the expression on the right may be written

$$\begin{aligned}&(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\alpha)\cos\theta - (\tfrac{1}{2}\sin 2\alpha)\sin\theta \\ &= \tfrac{1}{2}\cos\theta - \tfrac{1}{2}(\cos 2\alpha \cos\theta + \sin 2\alpha \sin\theta) \\ &= \tfrac{1}{2}\cos\theta - \tfrac{1}{2}\cos(2\alpha - \theta).\end{aligned}$$

Hence, the instantaneous power is

$$\begin{aligned}p &= e_m i_m (\tfrac{1}{2}\cos\theta - \tfrac{1}{2}\cos(2\alpha - \theta)), \\ p &= EI\cos\theta - EI\cos(2\alpha - \theta).\end{aligned}$$

The second term in the right-hand member of this equation is a sinusoid having its maximum ordinate equal to EI, the apparent power. It is displaced with reference to the voltage curve. Since the instantaneous values depend on 2α, it is obvious that the curve goes through two cycles in the period of time required for the current and voltage in the circuit to go through one cycle, and the power therefore has double the frequency of the voltage and current curves.

The first term in the right-hand member of the last equation depends upon the cosine of the lag angle, which is assumed to be constant. This constant term is added algebraically to the

ordinates of the $\cos(2\alpha - \theta)$ curve of the second term, and therefore displaces the horizontal axis of symmetry of the power loops above or below the axis of abscissas of the current and voltage curves. Figure 200 illustrates the arrangement where the voltage curve E has an effective value of 100 volts, and the current curve I has an effective value of 30 amperes and lags 135°. The axis of the power curve P is YY', and is below the axis X in the scale of watts a distance

$$EI\cos\theta = 100 \times 30 \cos 135° = -2121,$$

which is the average power expended in the circuit.

When the angle $\mp\theta < 90°$, the value of $EI\cos\theta$ is positive, and the average power is positive, *i.e.* is absorbed by the circuit, and when $\mp\theta > 90°$, as in the example above, the power is negative, *i.e.* is delivered by the circuit. When $\theta = 0$, the equation becomes $p = EI(1 - \cos 2\alpha)$, in which case the displacement of the axis is maximum and the power loops just touch the axis of abscissas as illustrated in Fig. 197. When $\theta = 90°$, $p = -EI\sin 2\alpha$ and the axis of symmetry of the power loops corresponds with the axis of abscissas, as shown in Fig. 199.

An examination of the power curves shows that an alternator which is connected into a highly reactive circuit, or one in which the phases of current and voltage are far apart, may be under a serious strain even though it may be furnishing very little power to the circuit. This follows from the fact that for one quarter cycle the machine may be delivering large power to the circuit and during the next quarter cycle, when the power is negative, the circuit may be returning nearly as much power to the machine, tending to drive it as a motor. This alternate generator and motor action due to the positive and negative loops is of special importance when there are alternators in the circuit running in parallel or as synchronous motors.*

Prob. 1. Using the formula given on page 323, draw the power loops for a sine voltage of 200 volts and a sine current of 50 amperes when the current lags 90°. Obtain the power loops for the same current and voltage when the lag is 45°, 90°, 135°, and 180°, 0°, − 45°, − 90°, and − 135°, and compare the sets of loops with each other.

* Chap. XII.

Prob. 2. Draw the power loops in which the current and voltage are expressed by the formulas $i = 100 \sin(\alpha + 30°) + 10 \sin(3\alpha + 60°)$, and $e = 200 \sin \alpha + 15 \sin 3\alpha + 5 \sin 5\alpha$.

94. Apparent Power; True Power; Power Factor. — *The product of the effective values of the current and voltage, IE, in a reactive circuit is called the* **Apparent Power** *or* **Volt-amperes** *in the circuit.* The term **Kilo-volt-amperes** is also frequently used (1 kilo-volt-ampere = 1000 volt-amperes).

The reading of a wattmeter applied to the circuit, which gives the value of IE cos θ, gives the **True Power** *or* **Watts** *expended in the circuit.*

The ratio of the watts to the volt-amperes in a circuit is generally called the **Power factor**, *as originally suggested by Fleming.*

The power loops for a circuit are exactly sinusoidal, provided the original current and voltage curves are sinusoids. (Figs. 197 to 200.) When the voltage and current coincide in phase, *i.e.* their phases are **Coincident**, the average ordinate of the power loops is equal to one half of the maximum ordinate, since the maximum ordinate is equal to $i_m e_m$, and the average ordinate is equal to $IE = \frac{i_m e_m}{2}$. When the original curves are sinusoids but their phases are not coincident, the average power ordinate is equal to one half of the difference between the maximum value of the positive ordinate and the maximum value of the negative ordinate. When the curves are not sinusoids, the power loops are not sinusoidal and the average power ordinate does not necessarily depend at all upon the maximum ordinate.

When either or both the current and voltage waves are not sinusoidal, the power factor may be determined by dividing the volt-amperes, obtained from amperemeter and voltmeter readings made in the circuit, into watts obtained from a wattmeter reading; from which $\frac{P}{EI} = \cos \theta$, where $\cos \theta$ is a positive fraction less than unity. Under these circumstances there is no fixed angle of phase difference which may be measured from the relations of the plotted current and voltage curves and called the angle of lag. But the angle $\theta = \cos^{-1} \frac{P}{IE}$ may be called the equivalent angle of lag. It is equal to the angular displacement

between equivalent sinusoids* of voltage and current which produce the same expenditure of power in the circuit.

It must not be assumed that curves of voltage and current which are not sinusoidal and which have no apparent angular displacement with respect to each other, or that dissimilar curves in which the zero ordinates occur simultaneously will necessarily give power factors of unity. In fact, *power factor of unity can be obtained only when the voltage and current curves are exactly similar and have no displacement with reference to each other.* Curves of current and voltage that are not angularly displaced with reference to each other, but are not similar, produce a power factor less than unity. Curves of current and voltage of relative symmetry, such as a semicircle associated with a parabola, yield their maximum power factors when the zeros of current and voltage occur simultaneously. Their maxima then, likewise, occur simultaneously. When the current and voltage curves are not of such relative symmetry, the maxima will not be coincident in time when the zeros are, and the maximum power factor is yielded when neither zeros nor maxima are coincident.

These theorems are easily proved and illustrated.

$$\text{Power factor} = \frac{P}{EI} = \frac{\frac{1}{\pi}\int_0^{\pi} eid\alpha}{\left(\frac{1}{\pi}\int_0^{\pi} e^2 d\alpha\right)^{\frac{1}{2}}\left(\frac{1}{\pi}\int_0^{\pi} i^2 d\alpha\right)^{\frac{1}{2}}}.$$

But $e = e'_m \sin\alpha + e''_m \sin 2\alpha + e'''_m \sin 3\alpha + \text{etc.};$

$$\frac{1}{\pi}\int_0^{\pi} e^2 d\alpha = E_1^2 + E_2^2 + E_3^2 + \text{etc.};$$

$i = i'_m \sin(\alpha - \theta_1) + i''_m \sin(2\alpha - \theta_2) + i_m''' \sin(3\alpha - \theta_3) + \text{etc.};$

$$\frac{1}{\pi}\int_0^{\pi} i^2 d\alpha = I_1^2 + I_2^2 + I_3^2 + \text{etc.};$$

and

$$\frac{1}{\pi}\int_0^{\pi} eid\alpha = E_1 I_1 \cos\theta_1 + E_2 I_2 \cos\theta_2 + E_3 I_3 \cos\theta_3 + \text{etc.}$$

Hence

$$\text{p.f.} = \frac{E_1 I_1 \cos\theta_1 + E_2 I_2 \cos\theta_2 + E_3 I_3 \cos\theta_3 + \text{etc.}}{(E_1^2 + E_2^2 + E_3^2 + \text{etc.})^{\frac{1}{2}}(I_1^2 + I_2^2 + I_3^2 + \text{etc.})^{\frac{1}{2}}}.$$

*Art. 90.

Two conditions must be fulfilled to make this unity:

(1) $\frac{E_1}{E_2} = \frac{I_1}{I_2}$, $\frac{E_1}{E_3} = \frac{I_1}{I_3}$, etc., which is equivalent to saying that the voltage curve and current curve must be alike in form; and

(2) $\theta_1 = 0$, $\theta_2 = 0$, $\theta_3 = 0$, etc., which is equivalent to saying that phases of corresponding harmonics must be in coincidence with each other.

As an example of the association of two curves not of relative

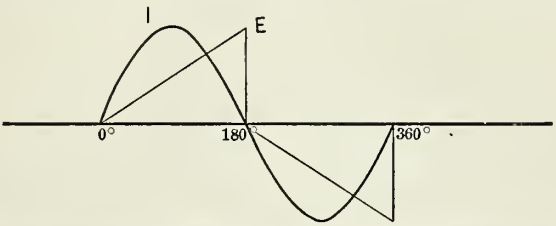


FIG. 202.—Conventional Curves of Voltage and Current with coincident Zeros.

symmetry, consider those illustrated in Fig. 202, which shows a right-angled triangle and a sinusoid with their zeros coincident.

The formulas of the curves for one half period are

$$e = \frac{e_m \alpha}{\pi} \text{ and } i = i_m \sin \alpha.$$

The effective values are,

$$E = \sqrt{av \cdot e^2} = \sqrt{\frac{1}{\pi}\int_0^\pi \frac{e_m^2}{\pi^2}\alpha^2 d\alpha} = \frac{e_m}{\sqrt{3}},$$

and

$$I = \sqrt{av \cdot i^2} = \sqrt{\frac{1}{\pi}\int_0^\pi i_m^2 \sin^2 \alpha d\alpha} = \frac{i_m}{\sqrt{2}}.$$

The volt-amperes therefore become

$$EI = \frac{e_m i_m}{\sqrt{6}}.$$

The power is

$$P = \frac{1}{\pi}\int_0^\pi eid\alpha = \frac{e_m i_m}{\pi^2}\int_0^\pi \alpha \sin \alpha d\alpha = \frac{e_m i_m}{\pi^2}\Big[\sin \alpha - \alpha \cos \alpha\Big]_0^\pi = \frac{e_m i_m}{\pi}.$$

The power factor is $\text{p.f.} = \dfrac{P}{EI} = \dfrac{\sqrt{6}}{\pi} = .78$. The two curves shown in Fig. 202 therefore give a power factor of only 78 per cent when their zero ordinates coincide, and instrument readings

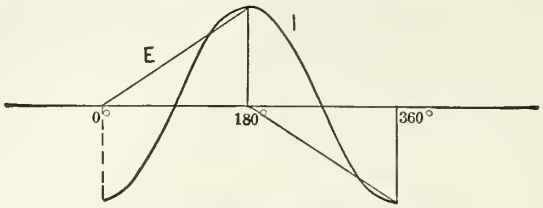


FIG. 203. — Conventional Curves displaced $\frac{\pi}{2}°$.

in the circuit would show an apparent angle of lag of $\theta = \cos^{-1} .78$. Now suppose the current curve is retarded a quarter cycle as shown in Fig. 203. The equations become, for a half period,

$$e = \frac{e_m \alpha}{\pi} \text{ and } i = i_m \sin\left(\alpha - \frac{\pi}{2}\right).$$

The power now is

$$P = \frac{1}{\pi}\int_0^{\pi} \frac{e_m i_m}{\pi}\, \alpha \sin\left(\alpha - \frac{\pi}{2}\right) d\alpha$$

$$= -\frac{e_m i_m}{\pi^2}\Big[\alpha \sin \alpha + \cos \alpha\Big]_0^{\pi} = +\frac{2\, e_m i_m}{\pi^2} = .203\ e_m i_m.$$

Hence the power factor is

$$\text{p.f.} = \frac{P}{EI} = .203\ e_m i_m \times \frac{\sqrt{6}}{e_m i_m} = .497.$$

The power for any displacement κ of the zero points of these curves with respect to each other may be expressed thus:

$$P = \frac{e_m i_m}{\pi^2}\int_0^{\pi} \alpha \sin(\alpha - \kappa)\, d\alpha$$

$$= \frac{e_m i_m}{\pi^2}\int_0^{\pi} \alpha(\sin \alpha \cos \kappa - \cos \alpha \sin \kappa)\, d\alpha$$

$$= \frac{e_m i_m}{\pi^2}\Big[\cos \kappa\left(\sin \alpha - \alpha \cos \alpha\right) - \sin \kappa\left(\cos \alpha + \alpha \sin \alpha\right)\Big]_0^{\pi}$$

$$= \frac{e_m i_m}{\pi^2}\left[\pi \cos \kappa + 2 \sin \kappa\right].$$

The maximum power factor must occur when the power itself is a maximum. Solving the above equation for a maximum gives

$$\frac{dp}{d\kappa} = \frac{d}{d\kappa}\left[\frac{e_m i_m}{\pi^2}(\pi \cos \kappa + 2 \sin \kappa)\right]$$

$$= \frac{e_m i_m}{\pi^2}(-\pi \sin \kappa + 2 \cos \kappa).$$

Equating this to zero and simplifying gives

$$\frac{2}{\pi} \cos \kappa - \sin \kappa = 0.$$

Hence, $\kappa = 32° 30'$ when the power to be derived from these two curves is a maximum, and then $\frac{P}{EI} = .377\, e_m i_m \times \frac{\sqrt{6}}{e_m i_m} = .924$ and the angle of lag indicated by instrument readings would be $\theta = \cos^{-1} .924 = 22° 29'$. The maximum power factor thus becomes 92.4 per cent for curves of these particular forms.

It is obvious that the power factor will be zero when the resultant power is zero or when

$$P = \frac{e_m i_m}{\pi^2}[\pi \cos \kappa + 2 \sin \kappa] = 0,$$

under which conditions the angle κ is $122° 30'$ or $-57° 30'$ and the angle of lag indicated by instrument readings would be $\theta = \cos^{-1} 0 = \pm 90°$.

The latter values of κ are $90°$ different from its value when power and power factor are at their maximum values. This obviously must be so in this instance because one of the curves is a pure sinusoid, and therefore maximum power occurs when its phase and the phase of the fundamental harmonic of the other curve are coincident, and zero power factor must occur when these two are in quadrature.

The above example is especially convenient for purposes of illustration on account of the simplicity of the curve formulas; and the results are comparable to those found in actual electrical circuits. However, for the sake of closer approximation to practical conditions, consider now a peaked curve of voltage and a flat-topped curve of current each composed of two harmonics as shown in Fig. 204, and represented by the following formulas:

$$e = 9 \sin \alpha - 3 \sin 3\alpha,$$

$$i = 9 \sin \alpha + 3 \sin 3\alpha.$$

The effective values of these are

$$E = \sqrt{\frac{9^2}{2} + \frac{3^2}{2}} = 6.7 \text{ and } I = \sqrt{\frac{9^2}{2} + \frac{3^2}{2}} = 6.7.$$

The volt-amperes are, therefore, $EI = 45$.

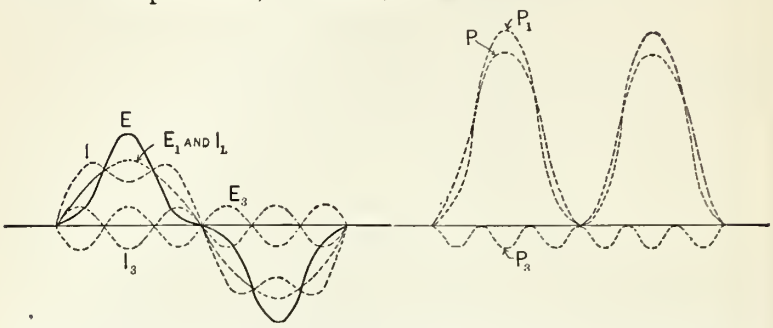


FIG. 204. — Peaked Voltage and Flat-topped Current Curves. Two Harmonics. Angle $\kappa = 0$.

The power is

$$P = \frac{1}{\pi}\int_0^{\pi} eid\alpha = \frac{1}{\pi}\int_0 (9\sin\alpha - 3\sin 3\alpha)(9\sin\alpha + 3\sin 3\alpha)\,d\alpha,$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 81\sin^2\alpha d\alpha - \int_0^{\pi} 9\sin^2 3\alpha d\alpha\right] = 36.$$

Or, using the formulas of Art. 90,

$$P = \tfrac{1}{2} e_{m_1} i_{m_1} \cos\theta_1 + \tfrac{1}{2} e_{m_3} i_{m_3} \cos\theta_3,$$

$$= \tfrac{81}{2}\cos 0° + \tfrac{9}{2}\cos 180° = 40.5 - 4.5 = 36.$$

An inspection of Fig. 204 affords an interpretation of this formula. The first term of the first numerical member is evidently the power given by the primary harmonics which are of coincident phases, and the power loops for which are marked P_1. The second term is the power given by the third harmonics, which is shown as P_3 in the power loops, and is negative as the curves are in opposition or 180° apart. The power factor is p. f. $= \dfrac{P}{EI} = .80$, and the apparent angle of lag is $\theta = \cos^{-1} .80 = 36° 52'$.

In this instance not only are the zeros of current and voltage coincident, but the pairs of harmonics each have unity power factor (positive or negative) as they are either in exact coincidence or exact opposition.

If the current curve is shifted 90° to the right, the formula is

$$P = \frac{81}{2}\cos 90° + \frac{9}{2}\cos 270° = 0, \text{ and p. f.} = \frac{P}{EI} = \cos\theta = 0.$$

Shifting the fundamental of current and voltage from coincidence to quadrature generally does not cause all the harmonics to fall into quadrature as in this example. For instance, if the third harmonic of the current is $i_{m_3} \sin(3\alpha + 90°)$ instead of $i_{m_3} \sin 3\alpha$, the last formulas above become

$$P = \tfrac{81}{2}\cos 90° + \tfrac{9}{2}\cos 0° = 4.5 \text{ and p.f.} = \frac{P}{EI} = \cos\theta = .10.$$

In general the power factor of a circuit, as already demonstrated, may be expressed thus:

$$\cos\theta = \frac{\frac{1}{\pi}\int_0^{\pi} eid\alpha}{EI},$$

where E and I are the effective voltage and current; or in the form

$$\cos\theta = \frac{E_1 I_1 \cos\theta_1 + E_3 I_3 \cos\theta_3 + \cdots + E_n I_n \cos\theta_n}{EI}.$$

The following relations are expressed in the formulas:

a. The power factor in the case of sinusoidal curves is unity when the phases of voltage and current are in coincidence or opposition, and is zero when they are 90° apart.

b. When the voltage and current curves are not sinusoids, the power factor is unity when the curves are in coincidence or opposition, provided the curves are of exactly like forms. In this case, coincidence occurs when the voltage and current curves are in such relative phases that each current harmonic crosses the zero value at the same instant and in the same direction as the corresponding voltage harmonic; that is, each current harmonic is in coincidence with the corresponding voltage harmonic. The current and voltage are in opposition when each current harmonic is in opposition to the corresponding voltage harmonic.

c. If the current and voltage curves are not of exactly like forms, the power factor never becomes as large as unity for any phase relation of the curves; and it may pass through zero when the power factors of some or all of the harmonics are finite,

but the algebraic summation of power produced by the several harmonics is zero.

Power factor is expressed as a matter of convenience and habit in terms of the cosine of an angle between 0° and 180°. The foregoing discussion shows that in the case of sinusoidal current and voltage, this angle is the same as the true angle of lag or difference of phase between the curves. It is therefore usual to call the $\cos^{-1} p.\ f.$ the angle of lag, whatever form may be assumed by current and voltage curves. When the current and voltage are sinusoidal, the angle of lag may be scaled off from a plot of the curves, but this cannot readily be done when the curves are of other forms on account of the effects of the various harmonics.

When the power factor is positive, the circuit is absorbing power from an outside source; when it is negative, the circuit is delivering power to the source of the voltage under consideration.

Prob. 1. A circuit has an effective sinusoidal voltage of 500 volts impressed upon it and a sinusoidal current of 75 amperes flows at a lag of 30°. What are the volt-amperes of the circuit, the power expended, and the power factor?

Prob. 2. Two circuits are in series, one having a resistance of 5 ohms and a self-inductance of .01 of a henry, the other a resistance of 10 ohms and a capacity of 200 microfarads, and a sinusoidal current of 50 amperes flows through these two circuits at a frequency of 40 periods per second. What is the power factor of the combined circuits and of each circuit separately?

Prob. 3. A circuit is composed of four circuits in parallel, the first having a resistance of 20 ohms, the second a resistance of 5 ohms and a self-inductance of .01 of a henry, the third a resistance of 8 ohms and a capacity of 100 microfarads, the fourth a resistance of 2 ohms, a self-inductance of .02 of a henry, and a capacity of 200 microfarads. When 1000 volts (sinusoidal) are impressed upon this circuit at a frequency of 60 periods per second, what is the power factor in the entire circuit, and of each of the four parts? Give the angle of lag or lead in each part of the circuit and in the total circuit.

Prob. 4. A voltage having two harmonics, the first and the third, of which the effective values are respectively 200 volts and 10 volts, is impressed upon a circuit and a current flows having first and third harmonics with effective values of 50

and 5 each in coincidence with the corresponding voltage harmonic. What is the power factor?

Prob. 5. A voltage having two harmonics, the first and the third, with effective values of 200 and 50, is impressed at a frequency of 60 periods per second on a circuit having a capacity of 200 microfarads and a resistance of 5 ohms in series. What current flows in the circuit and what is the power factor?

Prob. 6. The voltage of problem 5 is impressed upon a circuit having a resistance of 10 ohms and a self-inductance of .01 of a henry in series. What is the power factor?

Prob. 7. The voltage of problem 5 is impressed on a circuit having 5 ohms resistance, 200 microfarads capacity, and .2 of a henry self-inductance in series. What is the power factor?

Prob 8. The voltage of problem 5 is impressed upon a circuit having a capacity of 200 microfarads. What is the power factor? What would be the power factor if a sinusoidal voltage of the same effective value were impressed on this circuit?

Prob. 9. The voltage of problem 5 is impressed upon a circuit having a self-inductance of .2 of a henry. What is the power factor? What would be the power factor if a sinusoidal voltage of the same effective value were impressed on the circuit?

Prob. 10. The voltage of problem 5 is impressed upon a circuit having 5 ohms resistance. What is the power factor? What would be the power factor if a sinusoidal voltage of the same effective value were impressed on the circuit?

Prob. 11. What would be the power factor in problem 4 if the third harmonic of current was in opposition to the third harmonic of the voltage?

Prob. 12. A voltage curve having loops in the form of a semicircle and with a maximum value of 200 volts is in coincidence with a sinusoidal current having a maximum value of 100 amperes. What is the power factor?

95. Expression of Power Relations by Means of Vectors. — The product of the vectors of current and voltage may be called **Vector power.** Apparent power, which is given by the product of the readings of the amperemeter and voltmeter in the circuit,

or the " volt-amperes," is numerically equal to the product of the tensors of current and voltage, and is a coefficient in vector power. Apparent power may be resolved into two components in quadrature about the argument θ. One of these components is equal to $IE \cos \theta$, or the reading of a wattmeter in circuit, and is called real power or true power. The component perpendicular thereto is equal to $jIE \sin \theta$, and is the power which is exerted alternately positively and negatively in the circuit, but which balances itself in and out during any half period so that its average is equal to zero. The existence of the second component is dependent on a condition that during one portion of the half period the circuit absorbs power and during the remaining portion the circuit delivers power to an exactly equal average amount, so that a wattmeter reading will be zero with respect to this latter component. Hence comes the absurdly inapplicable term wattless energy or power, which can better be called **Quadrature power** or **Reactive volt-amperes**. It may also be called the quadrature volt-amperes. A mechanical analogy of the reactions of the quadrature power during a half period is found in a cycle consisting of the frictionless raising of a weight and its frictionless return to its initial position. In this case mechanical power is exerted by some source to raise the weight, but the weight delivers back the full power to the source as it returns to its first position. The weight has here had work exerted on it by the source and has then returned an equal amount of work to the source, and a mechanical power meter would afford a zero reading as the result of this operation.

The power vector has the form

$$\bar{P} = \bar{P}\text{cjs}\psi,$$

and it is equal to the product of the current and voltage vectors, or

$$\begin{aligned}\bar{P} = \bar{E}\bar{I} &= E(\cos\alpha + j\sin\alpha)\cdot I[\cos(\alpha-\theta)+j\sin(\alpha-\theta)] \\ &= E\epsilon^{j\alpha}\times I\epsilon^{j(\alpha-\theta)} = IE\epsilon^{j(2\alpha-\theta)} = IE\text{cjs}\,(2\alpha-\theta) \\ &= (IE\cos\theta - jIE\sin\theta)\text{cjs}\,2\alpha.\end{aligned}$$

Cjsψ is therefore equal to cjs 2α, and it is clearly seen that $\bar{P}$ has twice the frequency of $\bar{I}$ and $\bar{E}$, as heretofore explained. Moreover, the apparent power $\bar{P}$ is shown to be composed of two rectangular components. One of these ($IE \cos \theta$) represents the true average power in the period, and the other ($jIE \sin \theta$) is the quadrature power which disappears when

taken over any complete half period and stands for the work that swashes back and forth in the circuit but always gives as much as it takes in any complete half period.

By drawing a curve through the points found by giving various values to α from 0° to 360° in the expression $\bar{P}\text{cjs } 2\,\alpha$, we get the well-known sinusoidal "power loops" which are of twice the frequency of the corresponding vectors of voltage and current.

The true power is zero when

$$\cos\theta = 0,$$

that is, when $\theta = \pm 90°$, in which case $\bar{E}$ and $\bar{I}$ are in quadrature; and the quadrature power is zero when

$$\sin\theta = 0,$$

that is, when $\theta = 0°$ or 180°, in which case $\bar{E}$ and $\bar{I}$ are either in coincidence or in opposition.

The algebraic sign of the angle θ indicates whether the quadrature power is the result of equivalent inductance or equivalent capacity; and the numerical value of θ indicates whether the true power is delivered or absorbed by the circuit, power being absorbed by the circuit when $\pm\theta < 90°$, and delivered by the circuit when $\pm\theta > 90°$. The application of the commutative, associative, and distributive laws of algebra to the vector formulas is not affected by doubling the frequency in the equation.

These same deductions can be made, using the ordinary complex form of the vectors. In this case the product of current and voltage gives:

$$\begin{aligned}\bar{E}\bar{I} &= (a+jb)(c+jd) = ac - bd + j(ad+bc)\\ &= EI\{[\cos\alpha\cos(\alpha-\theta) - \sin\alpha\sin(\alpha-\theta)]\\ &\qquad + j[\sin\alpha\cos(\alpha-\theta) + \cos\alpha\sin(\alpha-\theta)]\}\\ &= EI[\cos(2\alpha-\theta) + j\sin(2\alpha-\theta)].\end{aligned}$$

This expression is the complex expression in which are shown the horizontal and vertical components of the complete power vector; that is,

$$ac - bd = EI\cos(2\alpha-\theta)$$

and

$$ad + bc = EI\sin(2\alpha-\theta).$$

This is not what we desire, as the valuable information to be derived from the equation is the relative magnitude of the rec-

tangular components of the apparent power corresponding to the argument θ, since the average of the variable angle α over a period is zero and θ is constant. The forms given earlier have shown that apparent power is a coefficient in the vector power and is multiplied by $(\cos\theta - j\sin\theta)\text{cjs}\ 2\alpha$ to give the vector power. We now desire to transform $EI(\cos\theta - j\sin\theta)$ into terms of the rectangular components of voltage and current a, b, c, and d. But,

$$\begin{aligned}\cos\theta &= \cos[\alpha-(\alpha-\theta)] \quad \text{(see Fig. 205)}\\ &= \cos\alpha\cos(\alpha-\theta)+\sin\alpha\sin(\alpha-\theta)\\ &= \frac{a}{E}\cdot\frac{c}{I}+\frac{b}{E}\cdot\frac{d}{I}=\frac{ac+bd}{EI},\end{aligned}$$

and, therefore,

$$EI\cos\theta = ac+bd.$$

Also,
$$\begin{aligned}\sin\theta &= \sin[\alpha-(\alpha-\theta)]\\ &= \sin\alpha\cos(\alpha-\theta)-\cos\alpha\sin(\alpha-\theta)\\ &= \frac{b}{E}\cdot\frac{c}{I}-\frac{a}{E}\cdot\frac{d}{I}=\frac{bc-ad}{EI},\end{aligned}$$

and, therefore,

$$jEI\sin\theta = j(bc-ad).$$

The true power is therefore equal to $ac+bd$ and not to $ac-bd$; and the quadrature power to $bc-ad$ and not to $bc+ad$.

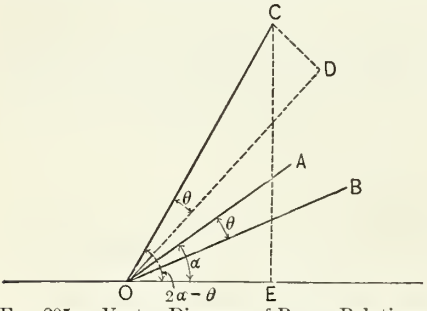


Fig. 205.—Vector Diagram of Power Relations.

It is to be remembered that the true power, $ac+bd$, is the average value for any full period of a periodic quantity of double frequency, $P\text{cjs}\ 2\alpha$, the origin of which is displaced from the X-axis by a distance equal to $EI\cos\theta$; and the quadrature power $bc-ad$ is the quadrature component of $\overline{EI}$ obtained by subtracting $(ac+bd)^2$ from $(EI)^2$.

The conditions are illustrated in Fig. 205, in which OA and OB, respectively, represent vector voltage and current. OC represents vector power, and OD and DC, respectively, represent $IE\cos\theta = ac+bd$ and $IE\sin\theta = bc-ad$; OE and EC repre-

sent the horizontal and vertical components of vector power obtained by the multiplication of the complex quantities, *i.e.* $ac - bd$ and $ad + bc$.

The expression $EI(\cos\theta - j\sin\theta)$ is of the nature of a vector operator, similar in character to $\overline{Y} = \frac{I}{E}(\cos\theta - j\sin\theta)$; and $EI(\cos\theta - j\sin\theta) = E^2\overline{Y} = \frac{E^2}{\overline{Z}}$.

Similar deductions are reached when the expressions are still further generalized by assigning an initial fixed angle to the voltage vector, in which case $\overline{E} = E\text{cjs}(\alpha - \phi)$ and $\overline{I} = I\text{cjs}(\alpha - \phi - \theta)$.

Since the power factor is equal to true power divided by volt-amperes, it must be equal to $\cos\theta$. In the same way the inductance factor which is equal to quadrature power divided by volt-amperes must be equal to $\sin\theta$.

We see from what goes before that the volt-amperes of a compound circuit may not be equal to the algebraic sum of the volt-amperes in the parts of the circuit, but they must always be equal to the vector sum of the volt-amperes in the parts of the circuit.

Inasmuch as true power and quadrature power in the several parts of a circuit are the rectangular components of the respective volt-amperes taken with respect to the corresponding arguments θ_1, θ_2, etc., it is clear that the true power in any circuit is equal to the algebraic sum of the powers in the parts of the circuit, and the quadrature power in the total circuit equals the algebraic sum of the quadrature power in the parts.

The conditions that are here set forth hold regardless of whether the parts of the circuit are in series or in parallel.

Power will be assumed to be positive in every case where it is absorbed by a circuit and negative when it is delivered from the same.

Considering the semicircle diagrams of Figs. 142 and 143, the impressed voltage is supposed to be constant and drawn vertically upwards, and the vertical component of current in each instance is the active component. Then the power expended in the circuit is equal to the voltage multiplied by the vertical component of current. Consequently, the vertical component of the current vector for each position in the semicircle diagram is proportional to true power, since voltage is assumed to be constant; and the power corresponding to each current

vector may be laid off on that vector of a length from O which is proportional to the vertical component. Following this process for the Figs. 142 and 143, gives Figs. 206 and 207, in which the power vector locus is represented by an oval-like loop within each semicircle. The power pertaining to any current vector is equal to the intercept on that current between O and the oval cutting it, as, for instance, the power corresponding to the current OI with the given voltage is represented by OP, which is numerically equal to the vertical component of current OI when E is unity. It will be observed that the ovals do not give any new information about the numerical value of the power for each current vector, other than that to be obtained directly from the active current components in the semicircle diagrams, but the ovals show in a graphic manner the way in which the power varies with the current over the entire range of conditions considered.

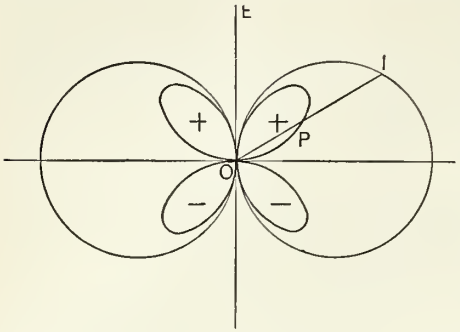


Fig. 206. — Locus of the Power Vector in a Series Circuit where Resistance varies and Reactance is maintained Constant.

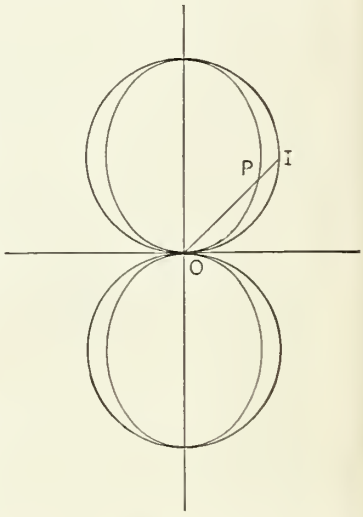


Fig. 207. — Locus of the Power Vector in a Series Circuit where Reactance varies and Resistance is Constant.

It will be observed that each semicircle in Fig. 206 contains its own oval-like loop. Of the four ovals occurring in this figure, two are positive and two are negative as indicated. The positive ovals represent conditions in which power is absorbed by and expended in the circuit by conversion into heat, light, or mechanical power; while the negative ovals represent the

conditions when the circuit generates and gives up energy as in the case of an electrical generator. In each of these figures the semicircles to the right of the vertical axis represent conditions when the current lags behind the voltage, and the other semicircles represent conditions when the current leads the voltage. In passing from one oval to another in either figure, it will be observed that either the current or the power factor passes through zero.

It will be observed that the ovals of Fig. 206 show the maximum value of power when the angle of lag in the circuit is either $\pm 45°$, or $\pm 135°$. It is easy to prove that the condition of maximum possible power transferred through an alternating-current circuit with a fixed voltage and with fixed reactance and variable resistance, as illustrated by Fig. 206, comes when the angle of lag is $\pm 45°$ or $\pm 135°$. The demonstration is as follows:

Since $P = IE\cos\theta$ and since $\cos\theta = \frac{R}{Z}$ and $I = \frac{E}{Z}$, there results the equation:

$$P = \frac{E^2 R}{R^2 + X^2}.$$

This is obviously a maximum when $R = X$, at which time $\theta = 45°$ and the power factor is 70.7 per cent. It is equally easy to prove that the condition of maximum possible power transferred through an alternating current circuit with a fixed voltage and with fixed resistance and variable reactance, as illustrated in Fig. 207, comes when the angle of lag is 0° or 180°. In this case, as before,

$$P = \frac{E^2 R}{R^2 + X^2},$$

which is obviously a maximum when $X = 0$. Under these circumstances the current is in coincidence with the voltage ($\theta = 0°$) when the circuit absorbs power, and in opposition to the voltage ($\theta = 180°$) when the circuit delivers power.

It may be noted in this connection that the condition of maximum power is a condition of little practical importance, as the highest practicable operating efficiency or *plant efficiency* is usually desired in the operation of electrical circuits and machinery. A high plant efficiency is ordinarily incompatible with a low power factor.

96. Quadrature Components of Power Loops. — If θ, the angle of lag in a circuit, were $\pm 90°$, the total current would be in quadrature with the voltage, and therefore its vector product with the voltage would be zero, as already indicated. A quadrature current may do a considerable amount of work in one quarter period, since $\int_0^{\frac{1}{2}\pi} \sin\alpha \sin(\alpha - 90°) d\alpha = \frac{1}{2}$, but during the next quarter period the circuit returns an equal amount, and the total work for the period is zero. (Compare power loops, Fig. 199.) It is possible to make the ratio of inductive reactance, $2\pi fL$, so great in comparison with the true resistance R of a circuit that the lag is nearly 90°. It is also possible to make the capacity of a circuit so great in comparison with its resistance that there is a lead of nearly 90°. The latter condi tion is one not ordinarily met in practice, but the former may quite easily be brought about in circuits including underloaded transformers of poor design.

The effect of the quadrature element of a current lagging less than 90° may be shown by components as indicated in Fig. 208, where the full line curves E, I, I_A, and I_q, respectively, represent the voltage, current, active component of current, and the quadrature component of current, while the dash-dot line represents the power loops and the dotted lines the components of the power loops provided respectively by the active and quadrature components of current.

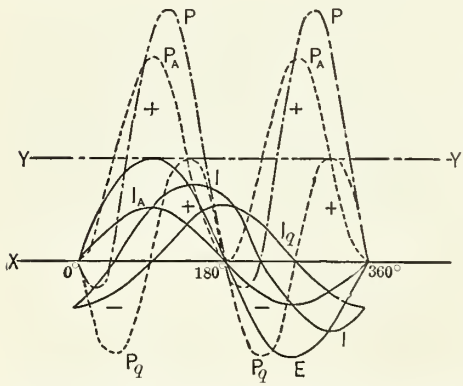


FIG. 208. — Power Loops caused by Active and Quadrature Components of the Current I. Lag 45°.

It is here seen that during each half period equal amounts of power are received and given out on account of the quadrature component, while the active component causes energy to be absorbed by the circuit throughout the period. The power loops due to the quadrature component really represent the work stored in the magnetic field as the current rises, or stored

in the electrostatic field as the voltage rises, and given out as it falls.

Figure 209 represents the component power loops formed by the product of the current respectively with the active and quadrature components of the voltage. The total current, total voltage, and total power curves are identical with those of Fig. 208.

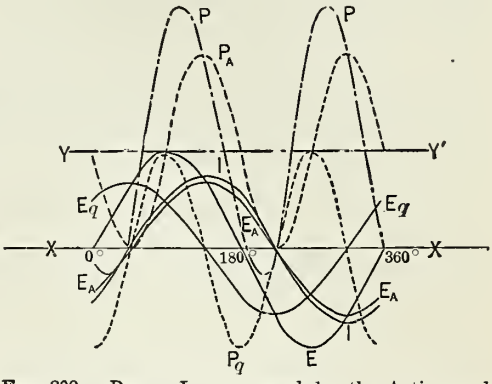


FIG. 209. — Power Loops caused by the Active and Quadrature Components of the Voltage E, Lag 45°.

The power loops representing total instantaneous power are sinusoidal when E and I are sinusoids, and they vary in position with respect to the X-axis from just touching that axis when $\theta = 0°$ to a symmetrical location thereon when $\theta = 90°$. The power loops due to the active component of current or voltage represent the irreversible power (the average of which over each period we call true power) and are always located so as to just touch the X-axis; while the loops due to the quadrature component of current or voltage represent the self-compensating quadrature or reactive power and are always located symmetrically on the X-axis with equal areas above and below the axis.

Prob. 1. A circuit has a reactance of 10 ohms and a resistance of 10 ohms. When a voltage of 2000 volts, at a particular frequency required to give the reactance named, is impressed upon it, what are the values of the active and quadrature components of the current? Work this and the following problems by means of complex quantities.

Prob. 2. A total current of 50 amperes at the given frequency flows through the circuit in problem 1. What is the impressed voltage, and what are the values of the active and quadrature components of the current?

Prob. 3. A circuit composed of three parts in series, the first having a resistance of 10 ohms, the second a self-inductance of

.02 of a henry, and the third a capacity of 100 microfarads, has a sinusoidal voltage of 500 volts with a frequency of 120 periods per second impressed upon it. What are the values of the active and quadrature currents in the total circuit and in each part taken respectively with reference, (1) to the voltage on the total circuit, and (2) to that on the particular part considered?

97. Reactive Factor; Table of Reactive and Power Factors. — The value of the power factor of a circuit is equal numerically to cos θ, and the total current in a circuit multiplied by

TABLE OF POWER FACTORS AND REACTIVE FACTORS

Lag Angle θ $\pm$	Power Factor cos θ	Reactive Factor sin θ $\pm$		Lag Angle θ $\pm$	Power Factor cos θ	Reactive Factor sin θ $\pm$	
Degrees			Degrees	Degrees			Degrees
0	1.0000	.0000	90	23	.9205	.3907	67
1	.9998	.0175	89	24	.9135	.4067	66
2	.9994	.0349	88	25	.9063	.4226	65
3	.9986	.0523	87	26	.8988	.4384	64
4	.9976	.0698	86	27	.8910	.4540	63
5	.9962	.0872	85	28	.8829	.4695	62
6	.9945	.1045	84	29	.8746	.4848	61
7	.9925	.1219	83	30	.8660	.5000	60
8	.9903	.1392	82	31	.8572	.5150	59
9	.9877	.1564	81	32	.8480	.5299	58
10	.9848	.1736	80	33	.8387	.5446	57
11	.9816	.1908	79	34	.8290	.5592	56
12	.9781	.2079	78	35	.8191	.5736	55
13	.9744	.2249	77	36	.8090	.5878	54
14	.9703	.2419	76	37	.7986	.6018	53
15	.9659	.2588	75	38	.7880	.6156	52
16	.9613	.2756	74	39	.7771	.6293	51
17	.9563	.2924	73	40	.7660	.6428	50
18	.9511	.3090	72	41	.7547	.6561	49
19	.9455	.3256	71	42	.7431	.6691	48
20	.9397	.3420	70	43	.7313	.6820	47
21	.9336	.3584	69	44	.7193	.6946	46
22	.9272	.3746	68	45	.7071	.7071	45
	$\pm$ sin θ Reactive Factor	cos θ Power Factor	$\pm$ θ Lag Angle		$\pm$ sin θ Reactive Factor	cos θ Power Factor	$\pm$ θ Lag Angle

the power factor is, therefore, equal to the active component of the current. A factor which in the same way is proportional to the quadrature current, may be called the **Reactive factor** of a circuit. It is evidently equal in numerical value to sin θ.

Any table which, like the foregoing, gives natural sines and cosines of the circular angles also gives the numerical values of the reactive factors and power factors of the angles.

This table is given for angles of lag; that is, the values of θ are positive. When the current leads, that is, θ is negative, the numerical value of the power factor ($\cos \theta$) corresponding with the particular value of θ is unchanged, but reactive factor ($\sin \theta$) changes from positive to negative value. The reactive factor has sometimes been called the **Induction Factor**, especially in inductive circuits.

Prob. 1. What is the reactive factor in a circuit having a resistance of 10 ohms, a capacity reactance of 10 ohms, and a self-inductive reactance of 10 ohms in series, when a voltage of 100 volts at a frequency of 120 periods per second is impressed?

Prob. 2. The volt-amperes in a circuit equal 10,000 and the reactive factor is .9. What is the power?

98. Method of Measuring Angle of Lag and Power Factor. — The power factor, or $\cos \theta$, whether the curves of voltage and current are sinusoidal or not, may be determined by using voltage, current, and power readings taken simultaneously, as already explained.* The method for determining the angle of lag is as follows: Measure the current flowing in a circuit by a correct amperemeter; measure the voltage at its terminals by an electrostatic voltmeter or some type of non-inductive voltmeter of relatively very high resistance. Finally, measure the power absorbed in the circuit by means of a wattmeter, the voltage coil of which is non-reactive and of relatively high resistance. The power in watts determined by the wattmeter, when divided by the product of volts and amperes, gives the cosine of the angle of lag. If the curves of current and voltage are of irregular form, this measurement will give the power factor which is equal to the angle of lag between the equivalent sine curves for the particular condition of the circuit.† We will later take

* Art. 94. † Art. 90.

up the effect of reactance in the voltage coil of the wattmeter.*

Prob. 1. The no-load terminal voltage of a generator having a frequency of 40 periods per second is 2200 volts, as shown by a voltmeter. When the machine is fully loaded on a non-inductive load, the voltmeter measures 2000 volts, the amperemeter measures 100 amperes, and the wattmeter measures 200 kilowatts. The resistance of the armature is 1 ohm. What is the angle of lag between the induced voltage (2200 volts) and the current when the machine is fully loaded? Is the wattmeter reading needed in this instance? Why?

99. Methods for Measuring the Power in an Alternating-current Circuit. — It can be readily understood that the power in an alternating-current circuit may be measured most accurately and expeditiously by means of a wattmeter. The other methods here given may serve a purpose in special cases or when a wattmeter of the proper range is not at hand. They are all well worth studying as problems.

1. *Wattmeter Methods.* Any instrument which directly measures the true power in a circuit is called a wattmeter. The common form of a wattmeter is an electrodynamometer† with one of its coils connected across the terminals of the circuit under test and the other in series therewith. Such instruments of modern form are usually encased in a suitable box to make them portable, and are proportioned so as to give direct readings. The electrodynamometer in its ordinary amperemeter arrangement with the two coils in series measures the value of $\frac{1}{T}\int_0^T i^2 dt$, since the torque exerted on the movable coil at each instant is proportional to the product of the currents in the movable and fixed coils. The average torque is therefore proportional to I^2, since $\frac{1}{T}\int_0^T i^2 dt$, as has been earlier shown, is equal to I^2; and the deflection of the movable coil, if its motion is opposed by a spring, is $D = KI^2$, when K has a value fixed by the conformation and character of the parts of the instrument. To make such an instrument direct reading, it must be provided

*Art. 99, 1 *a*. † Art. 24.

with a square-root scale, and the scale consequently is quite open at the higher portion, but contracts badly toward the lower parts, because the units of graduation are proportional to the square root of the torque. As such a scale is inconvenient for use, these instruments, when arranged for use as voltmeters or amperemeters, are sometimes constructed so that the conductors of the movable coil are located in the densest portions of the magnetic field of the fixed coil when the deflections are small. This arrangement manifestly counteracts in some degree the contraction in the lower part that is inherent in the pure square-root scale.

The assumption that the torque on the movable coil of any such instrument is directly proportional to the product of the currents in the two coils may be vitiated by various disturbing influences, — particularly if the movable coil in its deflection moves so as to appreciably alter the relations to each other of the magnetic fields of the two coils, and if eddy currents can be set up in any metal parts of the instrument in such a manner as to introduce an additional and extraneous magnetic field. If no such disturbing influences enter the situation, the calibration of such an instrument by means of currents of one frequency, or even by means of direct currents, is sufficient to standardize it for use with currents of any ordinary frequency ; provided, however, that if one or both of the instrument coils are to be used across the line as voltage coils they must be non-reactive, or the calibration will be affected by frequency.

An additional explanation of these conditions is contained in Art. 24.

When an electrodynamometer is arranged for use as a wattmeter, one coil (ordinarily the movable coil) is planned for connection between the terminals of the circuit within which the power is to be measured and may be called the **Voltage coil**, and the other coil is planned to be connected in series with the circuit and may be called the **Current coil**. The connections are illustrated in Fig. 51. In this case the torque on the moving coil, at each instant, is still proportional to the product of the corresponding instantaneous currents in the two coils, and the average torque is therefore proportional to $\frac{1}{T}\int_0^T ii_1 dt$, where i represents the instantaneous current in the current coil and i_1

the corresponding instantaneous current in the voltage coil. If the coil is non-reactive or the frequency is constant, the current in the voltage coil is directly proportional to the voltage impressed upon the coil, which is the same as the voltage impressed on the main circuit, and therefore i_1 is proportional to e, where e is the instantaneous impressed voltage. Hence the average torque acting on the movable coil of the wattmeter is proportional to $\frac{1}{T}\int_0^T iedt = P$; or, $D = KP$, if the motion of the coil is opposed by a spring and its deflection is represented by D. In this case, P is the power expended in the circuit to which the wattmeter is connected and K is a constant of the instrument with a value fixed by the conformation and character of the parts of the device. It will be observed that the deflection is here proportional to the first power of P, and a uniformly graduated scale is therefore used on such an instrument except as modifications are required to correct for errors of construction of the parts. The torque of a wattmeter of this type, where the coils are non-reactive, is therefore directly proportional to the power, while the torque of instruments of the same type used with the coils in series is proportional to the square of the effective current. In the usual arrangement, shown in Fig. 51, wattmeters of this class have a current coil of a few turns of thick wire, which is placed in series with the circuit to be measured. The voltage coil is composed of a few turns of fine wire in series with non-reactive resistance, and is connected across the terminals of the circuit.

a. Errors in Wattmeter Readings on Account of the Reactance of the Coils. It is essential that the voltage coil of the wattmeter be of entirely negligible inductance and capacity, or, where the voltage and current of the circuit are harmonic, that these constants be so mutually adjusted that the time constant is practically zero. If this is not the case, the current in the voltage coil is equal to $\frac{E\cos\theta_1}{R_1}$ instead of $\frac{E}{R_1}$, where E is the voltage in the circuit, θ_1 the angle of lag in the voltage coil which is dependent on the relation $\tan\theta_1 = \frac{X_1}{R_1}$, and R_1 and X_1 are respectively the resistance and reactance of the voltage coil. The currents in voltage and current coils now have

a difference of phase which is equal to $\theta - \theta_1$ instead of θ, where θ is the angle of lag in the main circuit. This is illustrated by Fig. 210, where the vectors E and I are the impressed voltage and the current of the circuit to which the wattmeter is attached, and I_1 is the current flowing in the voltage coil and lagging θ_1 degrees behind the voltage. In case the voltage coil is affected by capacity reactance, the angle θ_1 is negative, and should be so used in the computations; that is, current I_1 of Fig. 210 would lead voltage E and the angle between I_1 and I would become $\theta - (-\theta_1) = \theta + \theta_1$. The reading of a wattmeter in which the voltage coil is reactive is therefore proportional to $IE \cos \theta_1 \cos (\theta - \theta_1)$, since $Z_1 = \frac{R_1}{\cos \theta_1}$ and therefore $I_1 = \frac{E \cos \theta_1}{R_1}$ instead of $\frac{E}{R_1}$. A correct reading is proportional to $IE \cos \theta$. The readings of such a wattmeter must therefore be multiplied by a factor equal to

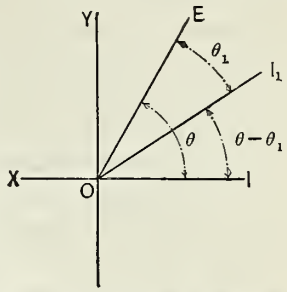


FIG. 210. — Phase Diagram showing Relation of Currents and Voltage in Wattmeter with Reactive Voltage Coil. Lagging Current in Main Circuit.

$$F = \frac{\cos \theta}{\cos \theta_1 \cos (\theta - \theta_1)} = \frac{\cos \theta}{\cos \theta_1 (\cos \theta \cos \theta_1 + \sin \theta \sin \theta_1)},$$

in order that they may give the true power. This multiplier may be called the "correcting factor" of a reactive wattmeter, and is given here with the following discussion for the purpose of illustrating the relation of the currents in the coils rather than for use in measurement.

Since

$$\cos \theta = \frac{R}{\sqrt{R^2 + X^2}}, \qquad \sin \theta = \frac{X}{\sqrt{R^2 + X^2}},$$

$$\cos \theta_1 = \frac{R_1}{\sqrt{R_1^2 + X_1^2}}, \qquad \sin \theta_1 = \frac{X_1}{\sqrt{R_1^2 + X_1^2}},$$

the correcting factor reduces to

$$\frac{RR_1^2 + X_1^2 R}{RR_1^2 + XX_1R_1} = \frac{1 + \tan^2 \theta_1}{1 + \tan \theta \tan \theta_1}.$$

The formulas show that *when* $\tan^2 \theta_1$ *is negligibly small* (in which case θ_1 is practically equal to zero), *or* $\tan \theta_1$ *is equal to* $\tan \theta$ (in which case $\theta_1 = \theta$), the correcting factor reduces to unity, and *the readings of the wattmeter are directly proportional to power. When* $\tan \theta_1$ *is algebraically smaller than* $\tan \theta$, the correcting factor is less than unity, and *the wattmeter reads too high*, and when $\tan \theta_1$ *is algebraically greater than* $\tan \theta$, the correcting factor is greater than unity, and *the wattmeter reads too low.* The indications of a wattmeter with reactive voltage coil may, therefore, be either correct, too high, or too low, depending upon the algebraic value of the time constant of the circuit upon which measurements are being made. When $\theta = 90°$, the correcting factor is equal to zero and the true power is zero even though the wattmeter gives a finite reading, as it will do under these circumstances if $\tan \theta_1$ has any appreciable value, though it may be very small. As a general rule, the $\tan \theta$ of the circuit is likely to be positive and greater than that of the wattmeter coil, so that the readings of such a wattmeter are generally found in practice to be too high; but in ordinary measurements it is impracticable to make an advance determination of the value of $\tan \theta$, so that the correcting factor of the wattmeter is unknown. If the main circuit has a leading current, — is a capacity circuit, — and the voltage coil of the wattmeter is reactive, the vector relations are as in Fig. 211. If the voltage coil were also affected by capacity, θ_1 would be negative and I_1 would lie, in Fig. 211, between E and I.

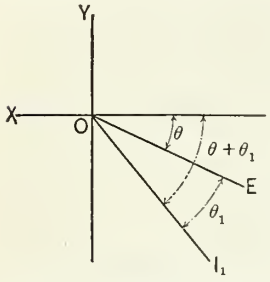


FIG. 211. — Phase Diagram showing Relations of Currents and Voltage in Wattmeter with Reactive Voltage Coil. Leading Current in Main Circuit.

It is evident that the correcting factors for either leading or lagging currents depart far from unity when θ approaches 90° and that the indications of the instrument are likely to be far from accurate under those circumstances for even microscopic values of θ_1. Indeed, as a rule ordinary electrodynamometer wattmeters are not to be depended upon for accurate measurement of power in circuits in which the lead or lag of the main current approaches 90°.

The only safety in wattmeter measurements of power in alternating-current circuits, therefore, lies in the use of a wattmeter with such a very small time constant in the voltage coil that it may be considered absolutely negligible. This construction can be accomplished by placing a relatively very large non-reactive resistance in series with the voltage coil, due care being taken to avoid appreciable capacity between conductors of the resistance coil, and also by making proper use of the mutual induction between the coils.* It is very seldom that there is appreciable self-inductance in the current coil of a wattmeter; but if there is, the effect merely increases the current lag in the main circuit or decreases its lead by a small amount and is therefore taken into account in the above discussion. In ordinary commercial instruments the capacity of the current coil is insignificant.

If the voltage is not sinusoidal, the inductance of the wattmeter will cause a reactance having a value for each harmonic proportional to the frequency of the harmonic. Thus the harmonics of current through the voltage coil will each have different phase angles when compared with the harmonics of the impressed voltage. As the power is equal to the sum of the products of the effective values of the current and voltage of each harmonic with the cosine of their phase difference, or $\Sigma E_n I_n \cos \theta_n$, it is evident that a separate correcting factor is necessary for each harmonic. The large effect of any reactance on the higher harmonics may cause extremely untrue indications of a wattmeter with reactive voltage coil where it is used to measure power in a circuit of low power factor. This may occur with values of the voltage coil reactance that might be considered negligible when the voltage and current are sinusoidal. The relation $\cos \theta = F \cos \theta_1 \cos (\theta - \theta_1)$ does not hold if equivalent sinusoids are substituted for the irregular currents in the current and voltage coils and for the line voltage, for it will be found that $(\theta - \theta_1) + \theta_1 > \theta$. Under these circumstances the current vector I_1 of Fig. 210 cannot be in the plane of the vectors I and E, but the three vectors must form a triangular pyramid in which the faces give the angles $IOE = \theta$, $IOI_1 = \theta - \theta_1$, and $I_1OE = \theta_1$. A like relation holds for Fig. 211.

b. Correction of Wattmeter Readings on Account of the Power

* Art. 23.

Absorbed by the Instrument. Another correction due to the power used by the wattmeter itself is also necessary. Thus, if the voltage coil is connected to the circuit between the current coil and the test circuit (Fig. 212), it is evident that the power measured includes that absorbed by the voltage coil. If the current coil is included between the point of connection of the voltage coil and the test circuit (Fig. 213), the power measured includes that absorbed by the current coil. In either case this power should be small and usually may be neglected; but when this is not the case, it is easily determined from the resistance of the coil included, if the voltage or current is known. In some wattmeters a special correcting coil wound over the series coil is introduced in series with the voltage coil, which corrects for the current in the voltage coil, — the instrument being connected as in Fig. 212. (Example: Weston wattmeter.)

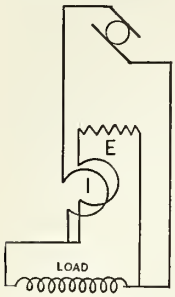


FIG. 212. — Wattmeter connected so that the Power used in Voltage Coil is added to that of the Circuit being Tested.

c. Effect of Eddy Currents in Wattmeter Frame. As in the case of any electrodynamometer or other instrument operated by electrodynamic action, it is necessary that a wattmeter of the type here discussed shall have no metal in its frame in which eddy currents may be developed.* If this precaution is not carefully looked after, the constant of the instrument will vary with the frequency, and a calibration is necessary for every frequency.

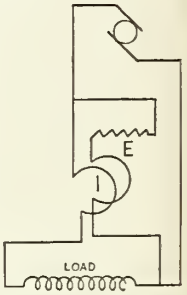


FIG. 213. — Wattmeter connected so that the Power used in the Current Coil is added to that of Circuit being Tested.

The magnetic effects of any eddy currents which are set up in the metal supports of an instrument may have quite a marked effect upon the magnetic fields around the coils, and as the intensities of the eddy currents are dependent upon the frequency of the inducing current, the above statement about calibration is evidently correct. For a properly built wattmeter, as said above, which is used at a point near which there are no masses

* Art. 111.

of metal, a single calibration with direct currents is sufficient.

2. *Electrometer Method.* If the two pairs of quadrants of a quadrant electrometer are connected with points of potential, respectively, V_1 and V_2, and the needle is connected with a point of potential V_3, then the deflection of the needle is theoretically

$$d = k\,(V_1 - V_2)\left(V_3 - \frac{V_1 + V_2}{2}\right),$$

where k is the constant of the electrometer. An explanation of the truth of this formula may be made as follows. In Fig. 214 the electrometer needle is well under the quadrants, and, if the needle moves slightly, the capacity afforded by the quadrants toward which the needle moves and the portion of the needle under them is increased slightly, due to the greater surface exposed to the inductive effect, while the quadrants from which the needle moves and the portion of the needle influenced by them decrease in capacity. If c is the change in capacity of either pair of quadrants and the accompanying portion of the needle for each very small angular movement of the needle, then the amount of charge gained by the quadrants toward which the needle moves is $c\alpha\,(V_3 - V_1)$ and by the needle is $c\alpha\,(V_1 - V_3)$, where α is the angular distance moved. Likewise, the charge lost by the quadrants from which the needle moves is $c\alpha\,(V_3 - V_2)$, and that lost by the needle is $c\alpha(V_2 - V_3)$. These expressions come from the fundamental equation, $Q = CE$. Equal charges of opposite signs must flow to quadrants and needle.

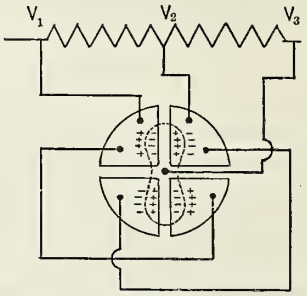


FIG. 214. — Diagram of Electrometer Connections.

It has been proved earlier that the work stored in a charge is $\frac{1}{2}QV$, where V is the absolute potential of the charge. Therefore, the energy gained by the first pair of quadrants is $\frac{1}{2}c\alpha\,(V_3 - V_1)\,V_1$, and by the needle $\frac{1}{2}c\alpha\,(V_1 - V_3)\,V_3$. Likewise, the energy lost by the second pair of quadrants is $\frac{1}{2}c\alpha\,(V_3 - V_2)\,V_2$, and by the needle $\frac{1}{2}c\alpha\,(V_2 - V_3)\,V_3$. Subtracting the energy lost from the energy gained, there results

$$w = c\alpha\,(V_1 - V_2)\left(V_3 - \frac{V_1 + V_2}{2}\right),$$

but the couple tending to turn the needle is equal to the work done divided by the angular displacement. Therefore, the torque F, tending to turn the needle, is

$$F = c\,(V_1 - V_2)\left(V_3 - \frac{V_1 + V_2}{2}\right).$$

If the angle α is small, so that the surface distribution of the charges is not materially altered, this torque should be constant throughout the range of the deflection ; and when the restraining force of the needle suspension is proportional to the angle of deflection, the deflection is proportional to the applied torque. As a result,

$$d = k\,(V_1 - V_2)\left(V_3 - \frac{V_1 + V_2}{2}\right),$$

as stated above. These conditions, however, cannot be rigorously obtained in ordinary electrometers because the distribution of the charges will not remain unchanged during the operations.

If v_1, v_2, and v_3 represent the instantaneous values of the potentials at the points when varying alternatingly, similar reasoning shows that the deflection becomes

$$d = \frac{k}{T}\int^{T} (v_1 - v_2)\left(v_3 - \frac{v_1 + v_2}{2}\right)dt.$$

If it is desired to measure the power absorbed by a reactive circuit, the electrometer may be used in the following manner:

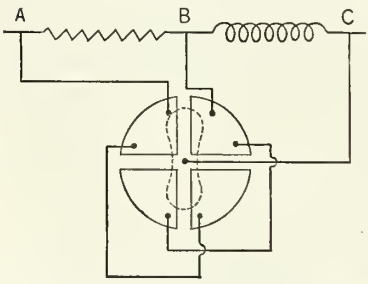


FIG. 215.—Electrometer in First Position for obtaining a Power Reading.

The reactive resistance BC is connected in series with the non-reactive resistance AB (Figs. 215 and 216). Let the potential of the points A, B, and C at any instant be represented respectively by v_1, v_2, and v_3, when B is the junction between the reactive and non-reactive resistance. Then, if a quadrant electrometer is connected with its quadrants to A and B, and its needle and case to C (Fig. 215), the deflection is

$$d = \frac{k}{T}\int_0^T (v_1 - v_2)\left(v_3 - \frac{v_1 + v_2}{2}\right)dt.$$

If the connection of the needle is interchanged so that it is connected to B while the connections of the quadrants remain unchanged (Fig. 216), this becomes

$$d' = \frac{k}{T}\int_0^T (v_1 - v_2)\left(v_2 - \frac{v_1 + v_2}{2}\right)dt.$$

By subtraction, this results in

$$d' - d = \frac{k}{T}\int_0^T (v_1 - v_2)(v_2 - v_3)dt.$$

Dividing this by kR, where R is the resistance of AB, gives

$$\frac{d' - d}{kR} = \frac{1}{T}\int_0^T \frac{v_1 - v_2}{R}(v_2 - v_3)dt.$$

Now $\frac{v_1 - v_2}{R}$ is equal to the instantaneous value of the current passing through the circuit, and $v_2 - v_3$ is the corresponding instantaneous value of the voltage between the terminals of the reactive resistance BC.

Consequently, $\frac{d' - d}{kR} = \frac{1}{T}\int_0^T iedt = P$, where P is the power absorbed by the reactive part of the circuit.

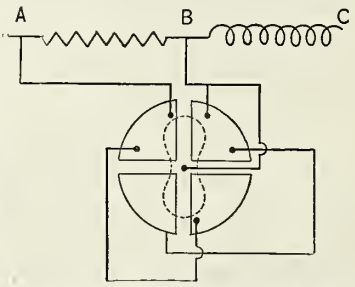


FIG. 216. — Electrometer in Second Position for obtaining a Power Reading.

On account of structural defects, the deflections of electrometer needles do not always follow the theoretical law, as aforesaid. Consequently, it is necessary to determine how great the deviation is before the instrument may be relied upon. Or, the instrument may be calibrated by the use of direct currents passing through known resistances, which are so adjusted that v_1, v_2, and v_3 are nearly the effective values of the tests.

3. *Electrostatic Wattmeter.* A modification of the quadrant electrometer may be made which reads directly as a watt-

meter.* In this case the needle box is divided diametrically into two parts instead of into quadrants. The needle consists of a disc divided diametrically into two parts (Fig. 217). The points A and B of the circuit are connected to the two halves of the needle, and B and C to the two halves of the needle box.

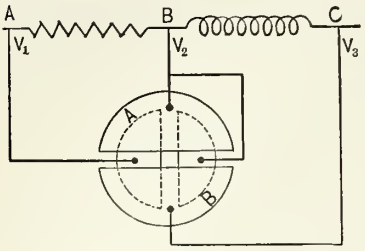


FIG. 217.—Diagram of an Electrostatic Wattmeter.

Then the force which causes the deflection of the needle is theoretically proportional to the difference which is found by subtracting the total loss of energies of the quadrants and needles from the total gain due to a small movement, exactly as was done in the previous case for the electrometer.

Hence,

$$P = \frac{d}{kR} = \frac{1}{T}\int_0^T \frac{(v_1 - v_2)}{R}(v_2 - v_3)dt,$$

since $\frac{(v_1 - v_2)}{R}$ is the instantaneous current and $(v_2 - v_3)$ the instantaneous voltage in the circuit.

This instrument may also be calibrated, as explained above, by passing a known direct current through a known resistance. Wattmeters of this type have been designed and constructed, but they are not very satisfactory.

4. *Three-voltmeter Method.*† As in the previous method, a non-reactive resistance must be connected in series with the reactive circuit to be tested (Fig. 218). Non-reactive voltmeters are then respectively connected between the points A and B, B and C, and A and C. Letting

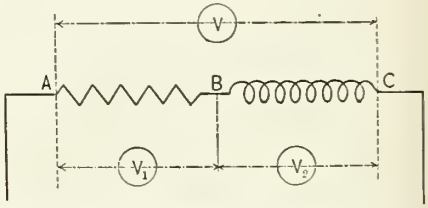


FIG. 218.—Connections for Three-voltmeter Measurement of Power.

* Gerard's *Leçons sur l'Électricité*, 3d ed., vol. I, p. 611; Hospitalier's *Traité de l'Énergie Électrique*, vol. I, pp. 205 and 507.

† Suggested by Ayrton and Sumpner, *London Electrician*, vol. 26, p. 736; *Electrical World*, vol. 17, p. 329.

e_1, e_2, and e represent instantaneous voltages at the three voltmeters, then $e = e_1 + e_2$, whence

$$e^2 - {e_1}^2 - {e_2}^2 = 2\, e_1 e_2.$$

But the instantaneous value of the power in the inductive circuit is $p = ie_2 = \frac{e_1}{R} e_2$. Substituting the value of $e_1 e_2$ already found gives $p = \frac{1}{2\,R}(e^2 - {e_1}^2 - {e_2}^2)$, and the mean power absorbed during a period is

$$P = \frac{1}{T}\int_0^T p\,dt = \frac{1}{2\,RT}\int_0^T (e^2 - {e_1}^2 - {e_2}^2)\,dt$$

$$= \frac{1}{2\,R}(E^2 - {E_1}^2 - {E_2}^2),$$

where E, E_1, and E_2 are the respeetive readings of the voltmeters. If R is not known and the value of the current I is known, the formula may be written

$$P = \frac{I}{2\,E_1}(E^2 - {E_1}^2 - {E_2}^2).$$

The letters E representing the effective voltages replace the letters e representing the instantaneous voltages in the formula after the integration is performed on the formula, because

$$\frac{1}{T}\int_0^T e^2 dt = av \cdot e^2 = E^2,$$

as has already been proved.*

In order that the results of measurements by this method may be the most accurate possible, E_1 should equal E_2, which makes the method inconvenient for use in ordinary testing. Neither is the method sufficiently accurate to compensate for its disadvantages. The accuracy of any particular measurement made by this method may be checked by inserting a known non-reactive resistance in place of the reactive circuit. This method, and methods 5 to 8, inclusive, are not desirable for ordinary measurements and only prove useful under special conditions.

* Art. 5.

5. *Three-ammeter Method.* Instead of putting a non-reactive resistance in series with the reactive circuit, it may be placed in parallel with it. In this case amperemeters must replace the voltmeters of the preceding method (Fig. 219). One amperemeter measures the whole current I, another measures the current I_1 in the non-reactive resistance R, and another measures the current I_2 in the reactive circuit. It is evidently essential that the amperemeters are of negligible reactance. Supposing i, i_1, i_2 are the instantaneous values of the currents at any moment, we have

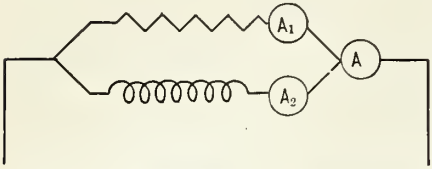


FIG. 219. — Connections for Three-ammeter Measurement of Power.

$$i = i_1 + i_2 \text{ and } i^2 - {i_1}^2 - {i_2}^2 = 2\, i_1 i_2,$$

while
$$p = e i_2 = R i_1 i_2 = \frac{R}{2}(i^2 - {i_1}^2 - {i_2}^2).$$

Whence
$$P = \frac{1}{T}\int_0^T p\,dt = \frac{R}{2}(I^2 - {I_1}^2 - {I_2}^2).$$

This may also be written

$$P = \frac{E}{2 I_1}(I^2 - {I_1}^2 - {I_2}^2).$$

In this case the greatest accuracy is given when I_1 and I_2 are about equal; but, at the best, the method is not very exact. Its accuracy may be checked, as in the previous case, by replacing the reactive circuit by a suitable non-reactive resistance.

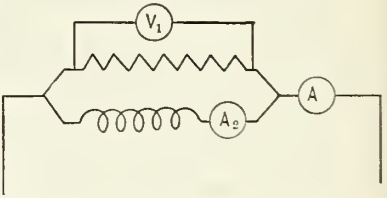


FIG. 220. — Connections for measuring Power using Two Ammeters and One Voltmeter.

6. *Other Three-instrument Methods.* Various modifications of the last two methods have been suggested by Ayrton, Sumpner, Blakesley, and others. One of the obvious arrangements is to omit the amperemeter in series with the non-reactive resistance of the fifth method, and connect a volt-

meter across the circuit as in Fig. 220. In this case the power becomes

$$P = \frac{R}{2}\left(I^2 - I_2^2 - \frac{E_1^2}{R_1^2}\right).$$

7. *Split Dynamometer Methods.* If separate alternating currents of the same frequency are passed through the two coils of an electrodynamometer, its reading will be proportional to $\frac{1}{T}\int_0^T i_1 i_2 dt$. This is equal to $I_1 I_2 \cos\theta$. For

$$i_1 = \sqrt{2}\, I_1 \sin\alpha \text{ and } i_2 = \sqrt{2}\, I_2 \sin(\alpha - \theta),$$

where θ is the angle between the two current waves. Substituting gives

$$\frac{1}{T}\int_0^T i_1 i_2 dt = \frac{2}{T}\int_0^T I_1 I_2 \sin\alpha \sin(\alpha - \theta)\, dt,$$

which is equal to

$$\frac{2\, I_1 I_2}{\pi}\int_0^\pi \sin\alpha \sin(\alpha - \theta)\, d\alpha = I_1 I_2 \cos\theta.$$

An electrodynamometer used in this manner is called a **Split dynamometer.** Now suppose we determine the values of I_1 and I_2 by means of a dynamometer used as an amperemeter or by other instruments, then the value of $\cos\theta$ is at once found. If the measurements are all made by the same electrodynamometer, its constant does not need to be known. Suppose the readings in the two circuits are $I_1^2 = k\delta_1$, and $I_2^2 = k\delta_2$, and the reading as a split dynamometer is $I_1 I_2 \cos\theta = k\delta_3$, then $\cos\theta = \frac{\delta_3}{\sqrt{\delta_1 \delta_2}}$. This plan was first suggested by Blakesley.*

8. *Blakesley's Split Dynamometer Method.* Blakesley planned various methods for using a split dynamometer in measuring the power absorbed by a reactive circuit.† In one of the methods a non-reactive resistance is connected in parallel with the reactive circuit to be tested (Fig. 221), and a split dynamometer is connected so that one coil

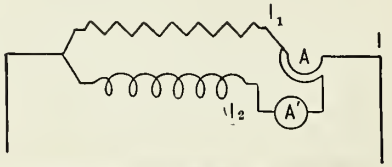


FIG. 221. — Measurement of Power by a Split Dynamometer.

* *Alternating Currents of Electricity*, 2d ed., p. 97.
† Phil. Mag., vol. 31, p. 346.

carries the total current, and the other carries the current of the reactive branch. An amperemeter is also placed in the reactive branch. Calling i the instantaneous value of the total current, and i_1, i_2, respectively, the instantaneous currents in the non-reactive and reactive circuits, the following relations hold: the reading of the split dynamometer is proportional to $I I_2 \cos \theta$, and that of the amperemeter gives I_2; but $i_1 i_2 = (i - i_2) i_2 = i i_2 - i_2^2$, and $R i_1 i_2 = R(i i_2 - i_2^2)$. $R i_1$ is equal to the instantaneous voltage between the terminals of the non-reactive resistance, and therefore $R i_1 i_2$ is equal to the instantaneous value of the power absorbed by the reactive circuit. Integrating gives

$$P = \frac{R}{T}\int_0^T i_1 i_2 dt = \frac{R}{T}\int_0^T i i_2 dt - \frac{R}{T}\int_0^T i_2^2 dt$$
$$= R(I I_2 \cos \theta) - R I_2^2 = kRD - R I_2^2,$$

where D is the scale reading of the split dynamometer, and k is its constant. Hence, the power absorbed by the reactive circuit is equal to R times the difference between the reduced split dynamometer reading and the square of the current in the reactive circuit.

A similar result may be gained by putting the amperemeter in the non-reactive branch, provided the instrument is itself non-reactive.

CHAPTER VIII

POLYPHASE CIRCUITS AND THE MEASUREMENT OF POWER THEREIN

100. Polyphase Systems. — A **Polyphase system** is a system comprising a number of simple alternating-current circuits carrying currents of different phases used conjointly to obtain the advantage of combined vector relations. When the simple circuits number more than two, the several line voltages are capable of forming a closed vector polygon, and the several line currents of likewise forming a closed vector polygon. The circuits of such a system may be operated as separate single-phase circuits, or jointly as a polyphase circuit.

A polyphase system is said to be **Balanced** when the voltages of the principal single-phase circuits are numerically equal to each other and alike in form, and the currents of the same circuits are also numerically equal to each other and alike in form; with the additional proviso that when the simple circuits number only two, the difference in phase between the two voltages is 90° and the difference in phase between the currents is also 90°. When the number of single circuits composing the balanced polyphase system is greater than two, the phase difference between the successive voltages or successive currents is $\frac{2\pi}{n}$, where n is the number of phases (*i.e.* the number of simple circuits composing the polyphase system). Polyphase systems are usually operated with either two currents with approximately 90° difference of phase, called quarter-phase or two-phase currents, or three currents with approximately 120° phase difference, called three-phase currents.

The transmission circuits for quarter-phase currents may be arranged to be entirely independent of each other, four wires being then required (Fig. 37); or, three wires may be used, in which case one of them is common to the two currents (Fig. 38) and the current in the third or common wire, at any instant, is equal to the algebraic sum of the currents in the other two.

The algebraic sum of the instantaneous currents in the three wires is always equal to zero, the total return current being of course equal to the total outgoing current at every instant. The effective current in the common return wire is equal to the vector sum of the two circuit currents; and it is, therefore, $\sqrt{2}\,I$, where I is the effective current in one circuit, provided the currents are equal in the two circuits and have a phase difference of 90°, which is the condition when the system is properly designed and symmetrically loaded or **Balanced**. The voltage between the two outside wires of the quarter-phase system with common return is the vector sum of the two circuit voltages, and is, therefore, $\sqrt{2}\,E$ in a balanced system, where E is the voltage between one side and the common return. The common current in a balanced system is 45° from the phase of the current in either of the independent wires and the joint voltage is 45° from the voltage of either phase. Figure 222 shows the graphical composition of the voltages. A and B are the two line voltages, and R is the resultant voltage measured across the outside wires.

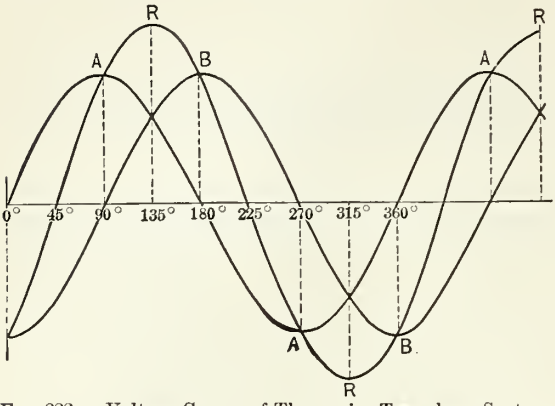


FIG. 222. — Voltage Curves of Three-wire Two-phase System.

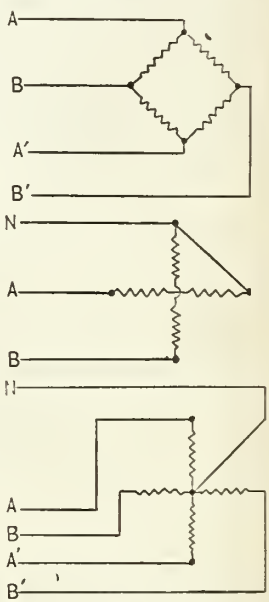


FIG. 223. — Methods of Connection—Two-phase Armatures.

The coils of quarter-phase machines may be entirely independent of each other, in which case an armature requires four circuit terminals, or the circuits may be joined so as to require only three terminals. In a

quarter-phase machine with rotating armature the armature may be wound with the equivalent of a series-path direct-current winding, and four collector rings and independent circuits are then required to avoid short-circuiting portions of the armature. Figure 223 shows diagrammatically various ways of connecting the coils of quarter-phase machines. (See also Art. 22.)

It is possible, in three-phase systems, to use three entirely

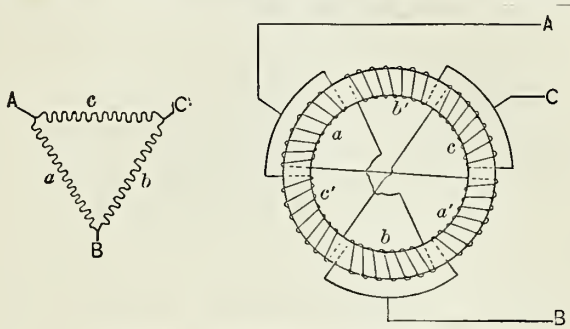


FIG. 224. — Three-phase Delta Connection.

independent circuits, each consisting of two wires, and carrying currents of 120° difference of phase; but in practice the circuits are almost invariably combined so as to use three line wires.

The three armature circuits of three-phase machines may be connected together so that they form the sides of a delta with the transmission wires connected to the three corners of the triangle (Fig. 224), or one end of each of the three circuits

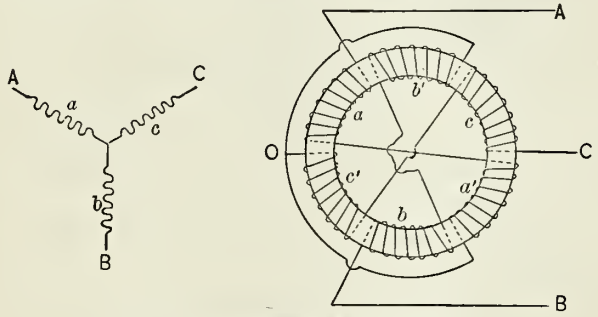


FIG. 225. — Three-phase Wye Connection.

may be individually connected to the transmission wires, the free ends of the three circuits being connected together (Fig. 225). In either case the number of transmission wires is three,

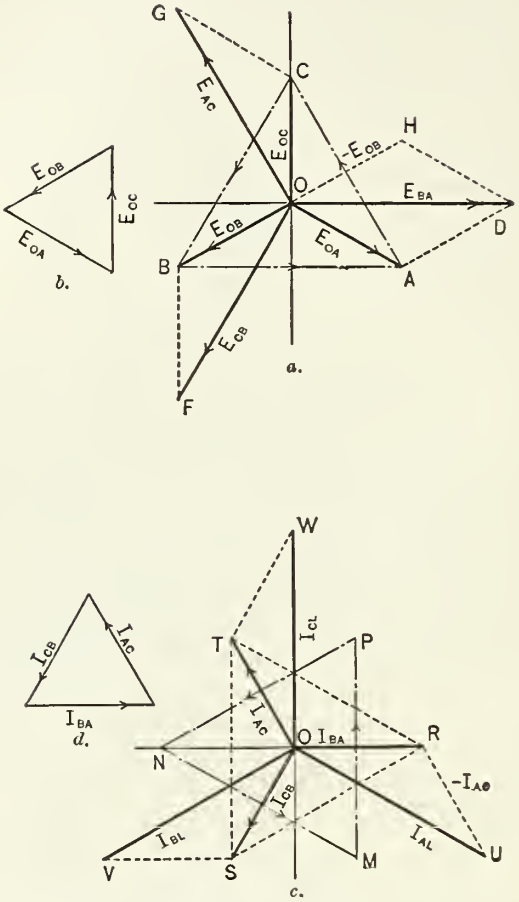


FIG. 226. — Vector Relations in Star-connected System.

and *the algebraic sum of their instantaneous currents is always equal to zero.*

In the latter, called the **Wye** or **Star** arrangement, which is represented by the symbol Y, the voltage between any two line wires in a balanced system is $\sqrt{3}\,E$, where E is the voltage in one circuit of the machine. Thus, in Fig. 226 *a*, the triangle ABC represents the vector polygon of line voltages, and the arrowheads the directions of the vectors which are of successively 120° difference in phase.* The phase diagram of these vector voltages (denominated $\bar{E}_{AC}$, $\bar{E}_{CB}$, and $\bar{E}_{BA}$) is laid out from the center of rotation O, with the vectors of the same dimensions and directions as the sides of the triangle ABC. The wye armature, or load branch, voltages in a balanced system are OA, OB, and OC (denominated $\bar{E}_{OA}$, $\bar{E}_{OB}$, and $\bar{E}_{OC}$). This must be the case since the three branches are equal and meet at a central or neutral point, and their free terminals have the same instantaneous potentials as the points A, B, and C. It is evident, then, from Fig. 226 *a* that the branch voltages $-\bar{E}_{OA}$ and $\bar{E}_{OB}$ form a closed vector triangle with line voltage $\bar{E}_{BA}$, and that the vector difference of $\bar{E}_{OA}$ and $\bar{E}_{OB}$ (*i.e.* the vector sum of $\bar{E}_{OA}$ and $-\bar{E}_{OB}$) is therefore equal to $\bar{E}_{BA}$†; likewise, branch voltages $-\bar{E}_{OB}$ and $\bar{E}_{OC}$ form a closed vector triangle with line voltage $\bar{E}_{CB}$, and $\bar{E}_{CB}$ is therefore equal to the vector sum of $\bar{E}_{OB}$ and $-\bar{E}_{OC}$; and, in the same manner, it will be observed that $\bar{E}_{AC}$ is equal to the vector sum of $-\bar{E}_{OA}$ and $\bar{E}_{OC}$. It is sometimes more convenient to illustrate these relations by laying down $-\bar{E}_{OB}$ as at OH, and completing the parallelogram on the sides $\bar{E}_{OA}$ and $-\bar{E}_{OB}$, whence it is at once seen that the diagonal, OD, is equal by construction to $\bar{E}_{BA}$. Consequently $\bar{E}_{BA}$ must be the vector sum of $\bar{E}_{OA}$ and $-\bar{E}_{OB}$. The vector $-\bar{E}_{OB}$ is of course the same as vector $\bar{E}_{BO}$. The line OD shows the position of $\bar{E}_{BA}$ in the phase diagram of branch and line voltages for a balanced three-phase Y-connected circuit. The lines OG and OF likewise show the positions of $\bar{E}_{AC}$ and $\bar{E}_{CB}$ in the phase diagram.

Figure 226 *b* shows the vector polygon of the voltages $\bar{E}_{OA}$, $\bar{E}_{OB}$, and $\bar{E}_{OC}$, which, being numerically equal and 120° apart in phase for a balanced system, make a closed equilateral triangle.

Returning to Fig. 226 *a*, since the angle BOA in the phase diagram equals angle OAD and OB and OA are of equal length, it is

* Art. 103. † Art. 79.

obvious that $OD = \sqrt{3}\ OA$ and, therefore, that $E_{BA} = \sqrt{3}\, E_{OA} = \sqrt{3}\, E_{BO}$ when these are given in effective values. Similar relations holding for the other two line voltages, it is seen that $E_L = \sqrt{3}\, E$, when E_L is the effective value of a line voltage and E is the effective value of a branch voltage. The figure shows that line voltage $\bar{E}_{BA}$ leads branch voltage $\bar{E}_{OA}$ by 30° and branch voltage $\bar{E}_{OB}$ by 150° (*i.e.* lags behind $\bar{E}_{BO}$ by 30°); line voltage $\bar{E}_{AC}$ leads branch voltage $\bar{E}_{OC}$ by 30° and branch voltage $\bar{E}_{OA}$ by 150°; and line voltage $\bar{E}_{CB}$ leads branch voltage $\bar{E}_{OB}$ by 30° and $\bar{E}_{OC}$ by 150°.

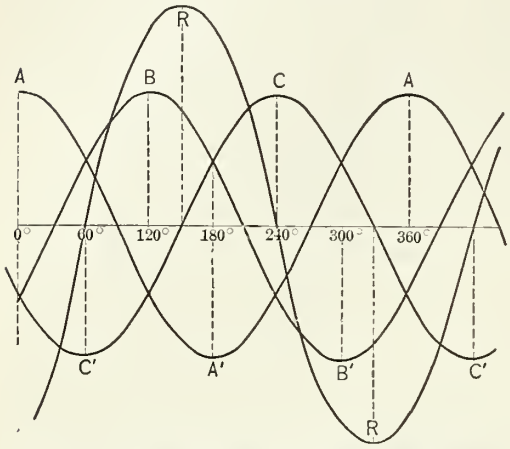


FIG. 227. — Voltage Curves of a Three-phase System.

In Fig. 227, the curve R shows the potential difference between A and B.

The line current in any star-connected circuit must always be the same as that passing through the branch in which the line terminates.

In the **Delta** or **Mesh** winding, which is often represented by the symbol Δ, the voltage between wires is evidently that generated by one coil, and the current in the line wire is the combination of those in two adjacent coils, or $\sqrt{3}\ I$ in a balanced system, where I is the current in a coil. Thus, in Fig. 226 *c*, the triangle MNP is the vector polygon of the line currents from the delta corners A, B, and C, of Fig. 224, the arrowheads representing the vector directions.* The phase diagram of these vector currents, called $\bar{I}_{AL}$, $\bar{I}_{BL}$, and $\bar{I}_{CL}$, respectively, is laid out from the center of rotation O with the vectors of the same directions and dimensions as those forming the sides of the triangle MNP. The branch currents between BA, AC, and CB, Fig. 224, called $\bar{I}_{BA}$, $\bar{I}_{AC}$, and $\bar{I}_{CB}$, respectively, are numerically equal and of successively 120° difference in phase. But $\bar{I}_{BA}$ and $\bar{I}_{AC}$ join with $\bar{I}_{AL}$ at the delta corner A and the

* Art. 103.

three must therefore form a closed vector triangle, since the algebraic sum of their instantaneous values must always be equal to zero in accordance with Kirchoff's law of currents at junction points. Similar relations exist between the branch and line currents at the other corners of the delta. To fulfill these conditions OR, OS, and OT may represent the vector currents $\bar{I}_{BA}$, $\bar{I}_{CB}$, and $\bar{I}_{AC}$.* Then, from the parallelogram $OTRU$, it is seen that $\bar{I}_{AL}$ is the vector difference of $\bar{I}_{BA}$ and $\bar{I}_{AC}$ (*i.e.* the vector sum of $\bar{I}_{BA}$ and $-\bar{I}_{AC}$); parallelogram $ORSV$ shows $\bar{I}_{BL}$ is equal to the vector sum of $-\bar{I}_{BA}$ and $\bar{I}_{CB}$; and parallelogram $OSTW$ shows that $\bar{I}_{CL}$ is equal to the vector sum of $-\bar{I}_{CB}$ and $\bar{I}_{AC}$. These relations are in accord with the principle enunciated by Kirchoff that the algebraic sum of the currents meeting in a point is zero, when those flowing toward the point are considered of one sign and those flowing away from the point of the opposite sign. At any point of meeting such as A, Fig. 224, the sum of the instantaneous currents must always be zero, and the geometric sum of their vectors must therefore be zero. But we have taken $\bar{I}_{BA}$ to be from B to A and $\bar{I}_{AC}$ and $\bar{I}_{CB}$ must therefore be taken in the directions, respectively, from A to C and C to B. Hence $\bar{I}_{BA}$ has a direction toward the junction A, and $\bar{I}_{AC}$ has a direction away from A and must be reversed in order that it may add to $\bar{I}_{BA}$ to give $\bar{I}_{AL}$. This is expressed in the vector triangle OUR, where the sum of $\bar{I}_{BA}$ and $-\bar{I}_{AC}$ is vectorially equal to $\bar{I}_{AL}$, or $\bar{I}_{AL} = \bar{I}_{BA} + (-\bar{I}_{AC})$. The same relations exist for the other corners. From Fig. 226 *c* it is seen that $I_L = \sqrt{3}\, I$, where I_L is the effective value of a line current in amperes and I the effective value of a branch current. It is also seen that $\bar{I}_{AL}$ lags behind $\bar{I}_{BA}$ by 30° and $\bar{I}_{AC}$ by 150°, $\bar{I}_{BL}$ lags behind $\bar{I}_{CB}$ by 30° and $\bar{I}_{BA}$ by 150°, and $\bar{I}_{CL}$ lags behind $\bar{I}_{AC}$ by 30° and $\bar{I}_{CB}$ by 150°. Figure 226 *d* shows a vector polygon of the branch currents.

Figure 228 shows ways in which the coils of three-phase machines may be connected to the external circuit. The arrangements are either of the wye or delta connection. The point of common connection O in the wye arrangement is called the **Neutral point** of the winding. A line wire may or may not be led from this point, depending on the conditions of use of the current.

* Art. 103.

In the case of a two-phase system, it is obvious that the algebraic sum of the instantaneous currents must always be zero at any section taken across the line wires when the two phases are kept separate by the use of four main line wires, because the instantaneous incoming current is equal to the instantaneous outgoing current in each single-phase circuit. When three wires are used for such a circuit, the joint wire is a common return for the other two, and the algebraic sum of the instantaneous currents in the three wires is, therefore, clearly equal to zero.

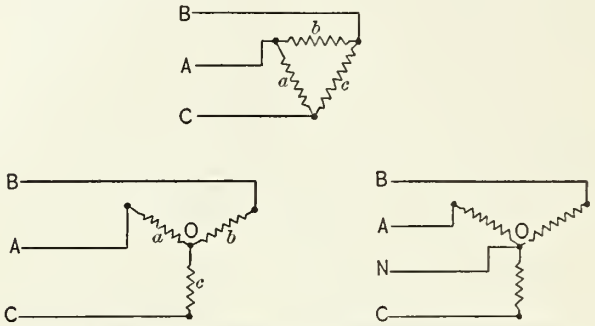


FIG. 228. — Methods of Connection of Three-phase Machines or Loads.

In other polyphase systems, the vectors of the several line currents are capable of making a closed vector polygon, and it therefore follows that at any section taken across the line wires,

$$I_1 \sin \alpha + I_2 \sin (\alpha - \beta_2) + \cdots + I_n \sin (\alpha - \beta_n) = 0,$$

if $I_1, I_2, \cdots I_n$ are the effective line currents and $\beta_2 \cdots \beta_n$ are the angular differences between the several current vectors. This must be equally true for each harmonic of current. But each of the terms on the left-hand side of this equation multiplied by $\sqrt{2}$ gives the instantaneous value of the current, and the algebraic sum of the instantaneous currents is therefore always equal to zero.

This is plainly to be seen in the case of star-connected windings, when it is remembered that the electric current acts like the flow of an incompressible fluid and the aggregate current flowing from the neutral point through one or more of the windings must at each instant be equal to the aggregate current entering the neutral point at that instant through the other windings. This is in accordance with what is called Kirchoff's law of current flow at a junction point. In case of the use of

a neutral wire, or other extra wire in a polyphase system, the instantaneous current flowing therein, across the section taken, must be included in the algebraic sum.

When the system is balanced and has n phases,

$$i_1 = i_m \sin \alpha;$$

$$i_2 = i_m \sin\left(\alpha - \frac{2\pi}{n}\right);$$

$$i_3 = i_m \sin\left(\alpha - \frac{4\pi}{n}\right);$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$i_n = i_m \sin\left(\alpha - 2\frac{(n-1)\pi}{n}\right).$$

Hence, $i_1 + i_2 + i_3 + \cdots + i_n = i_m \left\{\sin \alpha + \sin\left(\alpha - \frac{2\pi}{n}\right) + \sin\left(\alpha - \frac{4\pi}{n}\right) + \cdots + \sin\left(\alpha - 2\frac{(n-1)\pi}{n}\right)\right\};$

but evidently, $\sin \alpha + \sin\left(\alpha - \frac{2\pi}{n}\right) + \sin\left(\alpha - \frac{4\pi}{n}\right) + \cdots + \sin\left(\alpha - 2\frac{(n-1)\pi}{n}\right) = 0,$

and therefore $i_1 + i_2 + i_3 + \cdots + i_n = 0$.

101. Uniform Power in Polyphase Systems. — In general, the power transferred in a balanced polyphase circuit is uniform throughout each period, and the torque exerted by balanced polyphase machinery is uniform. This is different from the conditions in single-phase circuits, where the power has been shown to vary between a maximum and a minimum during every quarter period.* In the case of a single-phase circuit the power at any instant is $i_m e_m \sin(\alpha - \theta) \sin \alpha = i_m e_m \sin^2 \alpha \cos \theta - i_m e_m \sin \alpha \cos \alpha \sin \theta$, which varies with α. In a balanced two-phase circuit the instantaneous power is $i_m e_m \sin^2 \alpha \cos \theta + i_m e_m \sin^2(\alpha - 90°) \cos \theta = i_m e_m \cos \theta (\sin^2 \alpha + \cos^2 \alpha) = i_m e_m \cos \theta$, which is constant. In the same way the power in a balanced three-phase circuit is, when i_m and e_m represent maximum values of branch currents and voltages, $i_m e_m \cos \theta \{\sin^2 \alpha + \sin^2(\alpha - 120°) + \sin^2(\alpha - 240°)\} = \frac{3}{2} i_m e_m \cos \theta$, which is constant; and, in general, the power in any balanced polyphase circuit in which the

* Art. 92.

phase differences are equal to $\frac{2\pi}{n}$, where n is the number of phases, is, when i_m and e_m are maximum values of branch currents and voltages,

$$i_m e_m \cos\theta \left\{ \sin^2\alpha + \sin^2\left(\alpha - \frac{2\pi}{n}\right) + \sin^2\left(\alpha - \frac{4\pi}{n}\right) + \cdots + \sin^2\left(\alpha - \frac{2(n-1)\pi}{n}\right) \right\},$$

which is equal to $\left(\frac{n}{2}\right) i_m e_m \cos\theta$, and is constant, since

$$\sin^2\alpha + \sin^2\left(\alpha - \frac{2\pi}{n}\right) + \sin^2\left(\alpha - \frac{4\pi}{n}\right) + \cdots + \sin^2\left(\alpha - \frac{2(n-1)\pi}{n}\right) = \frac{n}{2}.$$ *

This being true for one harmonic it is true for all harmonics where the currents and voltages in a balanced circuit are not sinusoidal.

The uniformity of power in a balanced polyphase circuit may also be directly deduced from the proposition that the resultant of n equal harmonic motions acting in lines having successive angular differences of $\frac{2\pi}{n}$ is a uniform circular motion with an amplitude equal to $\frac{n}{2}$ times the amplitude of the components.

It is to be observed that uniform transfer of energy is inherent in direct current and balanced polyphase alternating-current circuits, but that this attribute is not possessed by single-phase circuits nor in general by polyphase circuits which are unbalanced.

102. Delta and Wye Connections for more than Three Phases.— With more than two phases the number of line wires may be equal to the number of phases, as illustrated in Figs. 228 and 229 *a* to *h*, which show corresponding delta and star arrangements of operating circuits and the connection of line wires thereto, for various numbers of phases. Turning to the illustrations in Fig. 229, *a* to *h* inclusive, which show simple mesh and star connections for five, six, seven, and eight phases, it will be observed that there are always as many corners in the mesh connection as there are operative circuits or phases, and it is obvious that an equal number of line wires individually con-

* Todhunter's *Plane Trigonometry*, p. 243.

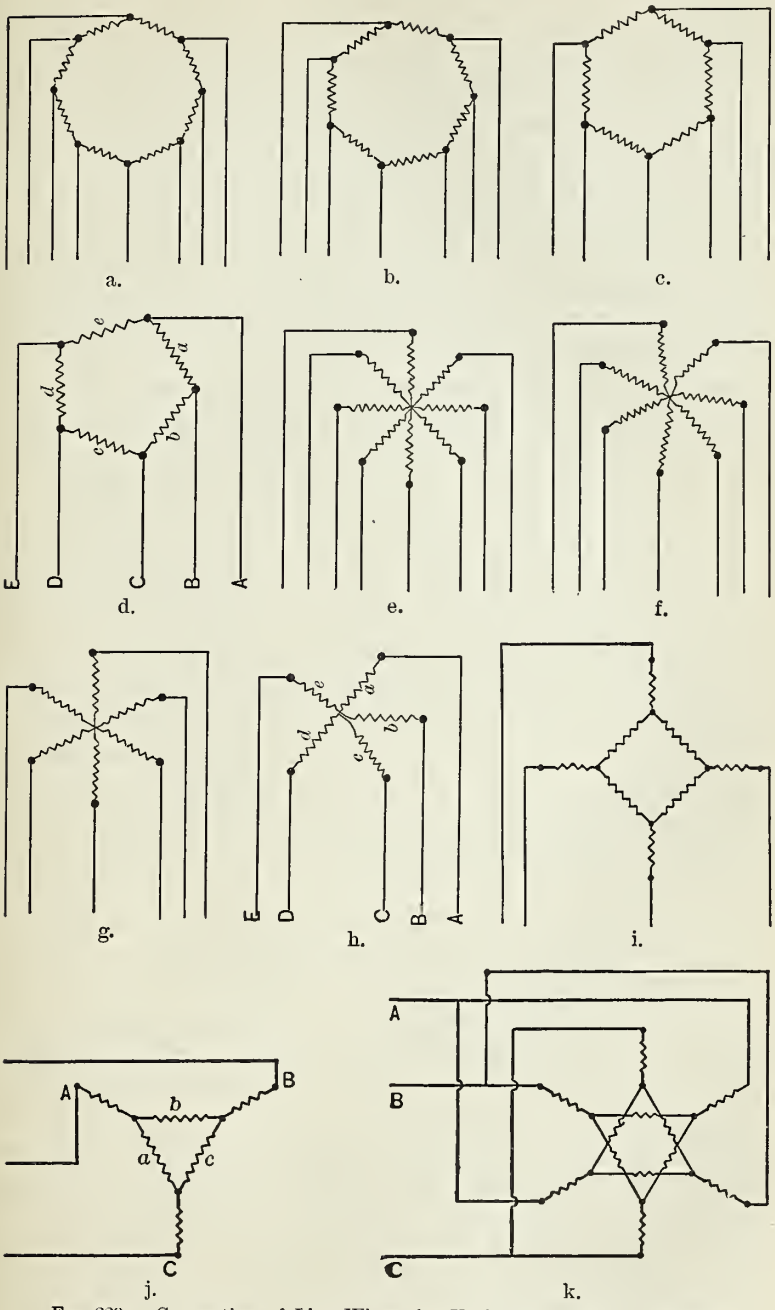


FIG. 229. — Connection of Line Wires, for Various Numbers of Phases.

nected to these corners as shown will convey all of the currents of the operative circuits. Each line wire will convey the vector difference of the currents in two of the mesh circuits. That is, the current in line wire B is the vector difference of currents in a and b, and so on. Four wires may thus be used for a four-phase system, five wires for a five-phase system, etc.

When the star arrangement is used, one end of each operative circuit has a line wire connected to it and the minimum practicable number of line wires is again equal to the number of operative circuits. The free ends of the operative circuits are connected with each other at the neutral point, and each operative circuit utilizes all the other circuits and lines as its return lines. In this case, the current in a line wire is equal to the current in the operative circuit to which it is connected, but the voltage between any two line wires is the vector difference of the voltages of the corresponding two operative circuits.

An additional line conductor may be led from the neutral point, if desired, and this may then act as a partial return wire for all the circuits in case the system is unbalanced.

Figure 229 i, j, and k show three combinations of mesh and star loads, so that six, eight, and twelve phase loads may be fed from three, two (using four wires), and three phase lines respectively; the value of such combinations is explained later.

103. Relations between Currents and Voltages. — In both the graphical and analytical representation of polyphase currents or voltages, it is particularly desirable to adopt a notation which logically and conveniently indicates the vector relations in the circuit. This may be accomplished by following the methods described in Chapter V and exemplified in Art. 100. A diagram showing the connection of circuits (whether mesh or star) may be drawn, and phase diagrams of currents and voltages then laid down. The connection diagram being carefully lettered, each vector in a phase diagram may be distinguished by the two corresponding terminal letters taken from the diagram of connections, and the vectors in analytical expressions may be identified by the same letters used as subscripts. Vector voltages may be considered as falling in the direction from the point corresponding to the first subscript toward the point corresponding to the second. For instance, in a circuit from A to B, where A is taken as the point of positive or higher

potential, the voltage is $\bar{E}_{AB}$, the positive direction being from A to B. Likewise vector currents may be considered as flowing from the point corresponding to the first subscript to the point corresponding to the second subscript when caused to flow by the difference of potential of those two points. For instance, in a circuit having the voltage $\bar{E}_{AB}$ the current would be under these circumstances $\bar{I}_{AB}$. If, however, the current is to be represented as flowing from the point of low to the point of high potential, under the influence of an electromotive force in the circuit itself or other cause, it may be considered as flowing from the point corresponding to the second subscript to the point corresponding to the first subscript; and in such a case, the circuit having a voltage $\bar{E}_{AB}$, the current would be $\bar{I}_{BA}$; if the subscripts are not changed, this may be written $-\bar{I}_{AB}$, *i.e.* $\bar{I}_{BA} = -\bar{I}_{AB}$.

To illustrate further, Fig. 230 shows a wye-connected three-phase generator armature and the corresponding phase diagram of voltages. The voltages of the coils a, b, c, are respectively $\bar{E}_{AO}$, $\bar{E}_{BO}$, $\bar{E}_{CO}$, and successively differ 120° in phase. The terminal voltages become, vectorially:

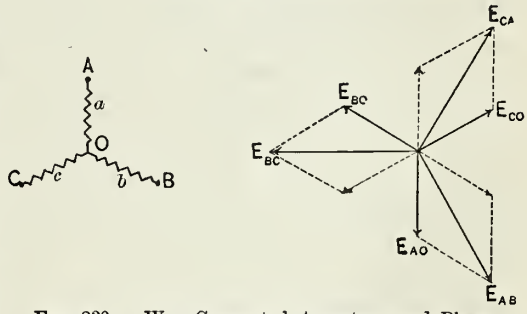


FIG. 230. — Wye Connected Armature and Phase Diagram of Voltages.

$$\bar{E}_{AB} = \bar{E}_{AO} + \bar{E}_{OB} = \bar{E}_{AO} + (-\bar{E}_{BO});$$
$$\bar{E}_{BC} = \bar{E}_{BO} + \bar{E}_{OC} = \bar{E}_{BO} + (-\bar{E}_{CO});$$
$$\bar{E}_{CA} = \bar{E}_{CO} + \bar{E}_{OA} = \bar{E}_{CO} + (-\bar{E}_{AO}).$$

The currents in the coils are $\bar{I}_{AO}$, $\bar{I}_{BO}$, and $\bar{I}_{CO}$, and lag behind their respective voltages by angles of lag fixed by their respective circuits.

The following are the relations between the currents and the voltages in the lines and coils of a balanced three-phase system developed from the earlier discussion and illustrated in Fig. 228:

1. *Star Connection.* $I_{\text{line}} = I_{\text{coil}}$; $E_{AB} = E_{BC} = E_{CA} = \sqrt{3}\, E_{\text{coil}}$. Line voltage $\bar{E}_{AB}$ is the vector sum of coil voltages $\bar{E}_{AO}$ and $\bar{E}_{OB}$ (note that $\bar{E}_{AO} = -\bar{E}_{OA}$) and is 30° behind the phase of coil

voltage $\bar{E}_{OB}$. Line voltage $\bar{E}_{AC}$ is the vector sum of coil voltages $\bar{E}_{AO}$ and $\bar{E}_{OC}$ and is 30° behind the phase of coil voltage $\bar{E}_{AO}$. Similar relations hold for the other two corners. For instance, line voltage $\bar{E}_{CA}$ (which is equal to $-\bar{E}_{AC}$) is the vector sum of coil voltages $\bar{E}_{CO}$ and $\bar{E}_{OA}$ and is 30° behind the phase of $\bar{E}_{OA}$ and 30° ahead of $\bar{E}_{CO}$; and $\bar{E}_{CB}$ (equal to $-\bar{E}_{BC}$) is the vector sum of $\bar{E}_{CO}$ and $\bar{E}_{OB}$, and is 30° behind the phase of $\bar{E}_{CO}$ and 30° ahead of the phase of $\bar{E}_{OB}$.

2. *Delta Connections.* $E_{AB} = E_{BC} = E_{CA} = E_{\text{coil}}$, $I_{\text{line}} = \sqrt{3}\, I_{\text{coil}}$. $\bar{I}_{LA}$ is the vector sum of branch currents $\bar{I}_{AC}$ and $\bar{I}_{AB}$ $(= -\bar{I}_{BA})$ and is 30° ahead of $\bar{I}_{AC}$. $\bar{I}_{LB}$ is the vector sum of branch currents $\bar{I}_{BA}$ and $\bar{I}_{BC}$ $(= -\bar{I}_{CB})$ and is 30° ahead of $\bar{I}_{BA}$. $\bar{I}_{LC}$ is the vector sum of branch currents $\bar{I}_{CB}$ and $\bar{I}_{CA}$ $(= -\bar{I}_{AC})$ and is 30° ahead of $\bar{I}_{CB}$.

Figures 226 *a* and 226 *c* show the relations that exist if *non-reactive* wye and delta circuits are placed on the same line wires. Thus, the line voltage between each pair of wires is in phase with the delta current between the same wires, so that $\bar{I}_{BA}$, $\bar{I}_{CB}$, and $\bar{I}_{AC}$ of *c* are parallel, respectively, to $\bar{E}_{BA}$, $\bar{E}_{CB}$, and $\bar{E}_{AC}$ of *a*. Also, the voltage and current in each branch of the wye are in the same phase, so that $\bar{E}_{OA}$, $\bar{E}_{OB}$, and $\bar{E}_{OC}$ of *a* are parallel, respectively, to $\bar{I}_{AL}$, $\bar{I}_{BL}$, and $\bar{I}_{CL}$ of *c*. Also, the wye branch voltages in a balanced three-phase circuit successively differ in phase from each other by angles of 120°, and they occupy certain specific relations to the delta or line voltages, which are illustrated in Figs. 226 and 230. It will be observed from Fig. 226 *a* that $\bar{E}_{BA}$, which is in phase with $\bar{I}_{BA}$, leads $\bar{E}_{OA}$ by 30°. But it will also be observed from Fig. 226 *c* that $\bar{I}_{AL}$ lags 30° behind $\bar{I}_{BA}$, and $\bar{I}_{AL}$ is therefore in the same phase as $\bar{E}_{OA}$. That is, the line current flowing from a corner of the delta has the same phase as the voltage measured from the neutral point of the circuit to the same corner, when the delta circuit is non-reactive. Now, the branch voltage $\bar{E}_{OA}$ of the wye circuit is identical with the voltage $\bar{E}_{OA}$ measured from the neutral point to the corner *A* of the delta circuit, and they are therefore in the same phase. Hence, $\bar{I}_{OA}$ of the wye circuit, which is in phase with $\bar{E}_{OA}$, is also in phase with $\bar{I}_{AL}$ of the delta circuit. By the same reasoning, $\bar{I}_{OB}$ of the wye circuit is in phase with $\bar{I}_{BL}$ of the delta circuit, and $\bar{I}_{OC}$ of the wye circuit is in phase with $\bar{I}_{CL}$ of the delta circuit. These relations are always true

for balanced wye and delta circuits when the phase of each branch current coincides with the phase of the corresponding branch voltage.

If the delta and wye branches of the balanced circuits, on the other hand, are reactive, but all of the same power factor, the voltage and current in each branch differ in phase, but the displacement of the line current from the phases of the branch currents combining at a delta corner is not altered, and the delta line currents $\bar{I}_{\text{line}}$ therefore are still in phase with the corresponding wye-branch currents $\bar{I}_{\text{coil}}$. If the power factor of the delta branches differs from the power factor of the wye branches, the delta-line currents $\bar{I}_{\text{line}}$ are then out of phase with the wye-branch currents $\bar{I}_{\text{coil}}$, and the main-line currents feeding the wye and delta jointly are vector resultants of the delta and wye line currents.

If the circuits of utilization in a polyphase system are not machines (for instance, are incandescent lamps), the devices must be connected exactly as would be the coils of a machine; unless transformers intervene, in which case the secondary circuits may be independent; but the load should be uniformly distributed to keep the system balanced.

The following examples deserve careful study, as they exhibit the phase and quantity relations of currents and voltages in polyphase circuits which are *unbalanced*. The voltages and currents are assumed to be sinusoidal to avoid undue complexity in exhibiting the underlying principles. If irregular curves are dealt with, a rigorous solution requires that they be analyzed and their sinusoidal components be used in the solution; but in many instances the substitution of equivalent sinusoids for the irregular waves is sufficiently accurate. The first two examples deal with three-phase unbalanced wye loads, in which the position of the neutral point of the wye with reference to the three-line voltages must be determined.*

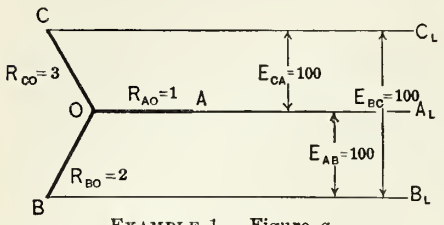
Example 1. — A three-phase line ($A_L B_L C_L$, Fig. *a*) is connected to a non-reactive wye circuit containing resistance of one ohm in the branch AO, two ohms in BO, and three ohms in CO. The voltage is balanced, being 100 volts on each phase.

(*a*) What are the values of the current, voltage, and angle of lag in each branch?

* H. P. Wood. *Electrical World*, vol. 55, p. 1597.

(b) What are the current values in the line wires AA_L, BB_L, and CC_L and their phase relations respectively with the voltages $\overline{E}_{AB}$, $\overline{E}_{BC}$, and $\overline{E}_{CA}$?

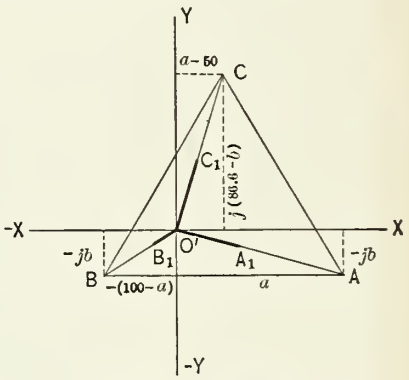
(c) What is the total power absorbed?



EXAMPLE 1.—Figure a.

Figure a diagrammatically represents the load.

(a) The triangle ABC (Fig. b) is drawn to scale representing, by the length and positions of its sides, the line voltages $\overline{E}_{AB}$, $\overline{E}_{BC}$, and $\overline{E}_{CA}$. For convenience, the side AB is made horizontal. The point O' is taken at any convenient position in the plane; and the lines $O'A$, $O'B$, and $O'C$ are drawn, representing the corresponding voltages $\overline{E}_{O'A}$, $\overline{E}_{O'B}$, and $\overline{E}_{O'C}$ from O' to A, B, and C. The position of the point O' is assumed for convenience in the process of determining the location of the neutral point O of the circuit, with reference to the points A, B, and C. Rectangular axes are erected with the origin at O' and the axis of abscissas parallel to AB.



EXAMPLE 1.—Figure b.

From Fig. b we obtain the following vector equations, in which the true values of a and b are to be derived:

$$\overline{E}_{O'A} = a - jb.$$

$$\overline{E}_{O'B} = -(100 - a) - jb.$$

$$\overline{E}_{O'C} = a - 50 + j(86.6 - b).$$

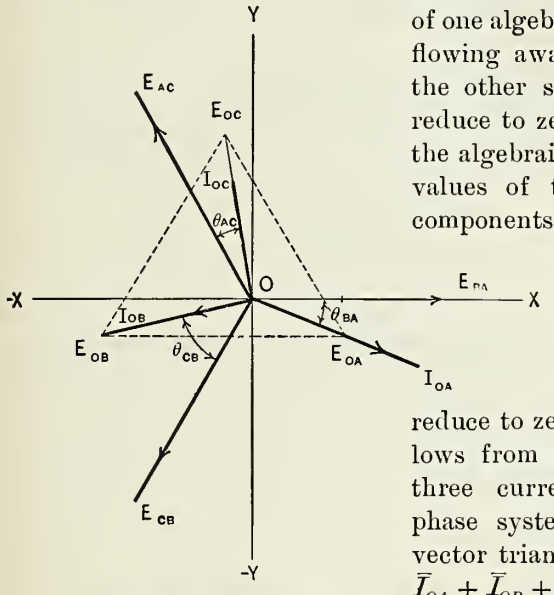
As the circuit is non-reactive, the directions of the currents, $\overline{I}_{O'A}$, $\overline{I}_{O'B}$, $\overline{I}_{O'C}$, may be represented by the lines $O'A_1$, $O'B_1$, and $O'C_1$, coincident with the lines $O'A$, $O'B$, and $O'C$. Dividing each vector equation for voltage given above by the resistance of its respective branch, gives:

$$\bar{I}_{OA} = a - jb.$$

$$\bar{I}_{OB} = -\left(\frac{100}{2} - \frac{a}{2}\right) - j\frac{b}{2}.$$

$$\bar{I}_{OC} = \frac{a}{3} - \frac{50}{3} + j\left(\frac{86.6}{3} - \frac{b}{3}\right).$$

Since the algebraic sum of the instantaneous currents meeting at the neutral point of the system (currents flowing towards the neutral point being taken of one algebraic sign and those flowing away being taken of the other sign) must always reduce to zero, it is plain that the algebraic sum of the scalar values of the three vertical components, and likewise of the three horizontal components, in these three vector equations must each reduce to zero. This also follows from the fact that the three currents of a three-phase system make a closed vector triangle, and therefore $\bar{I}_{OA} + \bar{I}_{OB} + \bar{I}_{OC} = 0$. To correctly locate the neutral point it is therefore only required to solve for a and b in the following equations:



EXAMPLE 1.—Figure *c*.

$$a - \left(\frac{100}{2} - \frac{a}{2}\right) + \frac{a}{3} - \frac{50}{3} = 0,$$

and

$$-b - \frac{b}{2} + \frac{86.6}{3} - \frac{b}{3} = 0;$$

whence $a = 36.4$ and $b = 15.8$.

The values of the wye currents and the location of the neutral point O are determined by substituting these values in the foregoing vector equations of current, and are plotted in Fig. *c*.

In this figure the three line-voltages are also drawn from O, making a phase diagram. The numerical values of the voltages $\bar{E}_{OA}$, $\bar{E}_{OB}$, and $\bar{E}_{OC}$ are determined by substituting the values of a and b, just found, in their vector equations and obtaining the tensors in the usual manner, thus:

$$E_{OA} = \sqrt{(36.4)^2 + (15.8)^2} = 39.7 \text{ volts.}$$

$$E_{OB} = \sqrt{(63.6)^2 + (15.8)^2} = 65.5 \text{ volts.}$$

$$E_{OC} = \sqrt{(13.6)^2 + (70.8)^2} = 72.1 \text{ volts.}$$

The numerical values of the currents are found in the same way from their equations, or by dividing the voltages just determined by the accompanying branch impedances with the result that

$$I_{OA} = 39.7 \text{ amperes.}$$

$$I_{OB} = 32.8 \text{ amperes.}$$

$$I_{OC} = 24.0 \text{ amperes.}$$

The angles of lag of the currents with respect to the voltages of the respective branches are zero, since the branches are non-reactive.

(*b*) The currents in the line wires, $\bar{I}_{AL}$, $\bar{I}_{BL}$, and $\bar{I}_{CL}$ are, of course, equal to respectively and in phase with the branch currents $\bar{I}_{OA}$, $\bar{I}_{OB}$, and $\bar{I}_{OC}$.

The phase angle between $\bar{E}_{BA}$ and $\bar{I}_{AL}$ $(= \bar{I}_{OA})$ is, from the vector equation of the current, $\theta_{BA} = -\tan^{-1}\left(\frac{-15.8}{36.4}\right) = 23° 28'$. The minus sign is applied to $\tan^{-1}\left(-\frac{15.8}{36.4}\right)$, because by convention the phase angle between current and voltage is positive when measured *from the current to the voltage in counter-clockwise direction, i.e.* for a lagging current. In this case we have taken the voltage as the initial line, and measuring from it gives a reversed angle.

Moreover, since $\bar{E}_{CB}$ lags 120° behind $\bar{E}_{BA}$ or what is the same thing leads by 240°, the angle between $\bar{I}_{OB}$ and $\bar{E}_{CB}$ is

$$\theta_{CB} = \left[240° - \tan^{-1}\left(\frac{-15.8}{-(100 - 36.4)}\right)\right] = 46° 3'.$$

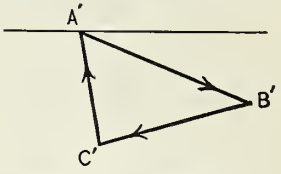
In like manner the phase angle is

$$\theta_{AC} = \left[120° - \tan^{-1}\left(\frac{86.6 - 15.8}{36.4 - 50}\right)\right] = -19° 9'.$$

Here $\overline{E}_{AC}$ lags 240° behind $\overline{E}_{BA}$, or, for simplicity, in the formula may be counted as leading by 120°.

It must be understood that these angles are the angles between the respective line currents and line voltages, and do not determine the power factor, which is fixed by the phase relations of the branch currents with respect to the branch voltages and is unity in this case.

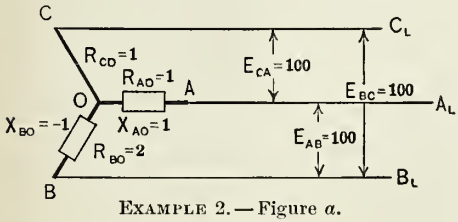
(*c*) The power in the circuit is $E_{OA}I_{OA} + E_{OB}I_{OB} + E_{OC}I_{OC} = 5460$ watts, since the branches are non-reactive.



EXAMPLE 1. — Figure *d*.

A check can be applied to the computations made in finding the currents by constructing a vector diagram as in Fig. *d*. Here, $A'B'$ is equal and parallel to I_{OA}, $B'C'$ is equal and parallel to I_{OB}, and $C'A'$ is equal and parallel to I_{OC}. Had the triangle not closed, it would have been an indication of error, as in such a case the instantaneous sum of the branch currents could not be zero at the neutral point *O*, Fig. *a*.

Example 2. — A three-phase line ($A_LB_LC_L$, Fig. *a*) is connected to a wye circuit having the following characteristics: In branch *OA*, $R_{OA} = 1$ ohm in parallel with $X_{OA} = +1$ ohm; in branch *OB*, $R_{OB} = 2$ ohms in parallel with $X_{OB} = -1$ ohm; and in branch *OC*, $R_{OC} = 1$ ohm (Fig. *a*). Equal voltages of 100 volts are impressed on the three phases. Answer the questions *a*, *b*, and *c* of Ex. 1.



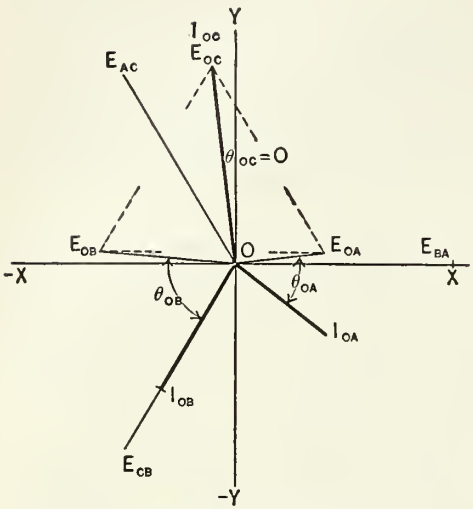
EXAMPLE 2. — Figure *a*.

The same preliminary or trial diagram as was previously used (Ex. 1, Fig. *b*) may be employed here for constructing the vector equation of branch voltages. They are as before:

$$\overline{E}_{OA} = a - jb.$$

$$\overline{E}_{OB} = -(100 - a) - jb.$$

$$\overline{E}_{OC} = a - 50 + j(86.6 - b).$$



EXAMPLE 2. — Figure *b*.

(*a*) To obtain the vector expression for branch currents, the voltage equations must be divided by the respective branch impedances or multiplied by the respective admittances. As a matter of convenience, multiply by the admittances, which are

$$\overline{Y}_{OA} = 1 - j1;$$

$$\overline{Y}_{OB} = \tfrac{1}{2} + j1;$$

and $\overline{Y}_{OC} = 1.$*

Then

$$\overline{I}_{OA} = (a - jb)(1 - j1) = a - b - j(a + b).$$

$$\overline{I}_{OB} = (a - 100 - jb)(\tfrac{1}{2} + j1) = \tfrac{1}{2}a - 50 + b + j(-\tfrac{1}{2}b + a - 100).$$

$$\overline{I}_{OC} = (a - 50 + j(86.6 - b))(1) = a - 50 + j(86.6 - b).$$

By placing the vertical and horizontal components respectively equal to zero, there results

$$a = 40 \text{ and } b = -5.3,$$

and Fig. *b* can now be plotted to scale as shown.

The scalar values of the branch voltages are:

$$E_{OA} = \sqrt{(40)^2 + (5.3)^2} = 40.3 \text{ volts.}$$

$$E_{OB} = \sqrt{(60)^2 + (5.3)^2} = 60.2 \text{ volts.}$$

$$E_{OC} = \sqrt{(10)^2 + (92)^2} = 92.5 \text{ volts.}$$

* Arts. 74 and 78.

The currents may be obtained in the same manner from the vector equations of currents, and are:

$$I_{OA} = \sqrt{(40 + 5.3)^2 + (40 - 5.3)^2} = 57.2 \text{ amperes.}$$

$$I_{OB} = \sqrt{(20 - 50 - 5.3)^2 + \left(\frac{5.3}{2} + 40 - 100\right)^2} = 67.4 \text{ amperes.}$$

$$I_{OC} = \sqrt{(40 - 50)^2 + (86.6 + 5.3)^2} = 92.5 \text{ amperes.}$$

The angles of lag between the currents and voltages of the branches are found directly from the admittances and are:

$$\theta_{OA} = -\tan^{-1}\frac{-1}{1} = 45°; \quad \theta_{OB} = -\tan^{-1}\frac{1}{\frac{1}{2}} = -63° 26';$$

$$\theta_{OC} = -\tan^{-1}\frac{0}{1} = 0°.$$

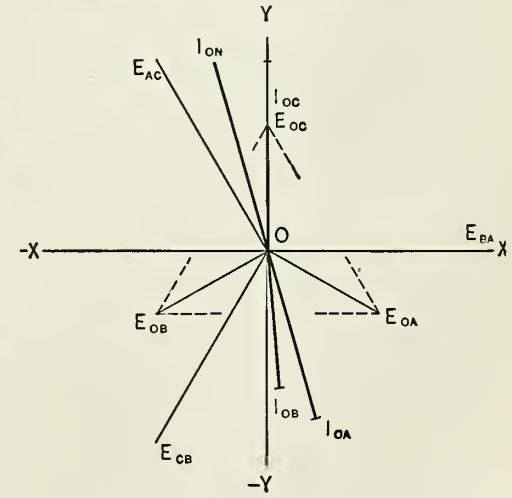
b. The currents in the branches are the same as in the lines to which they are connected.

c. The power in the circuit is

$$I_{OA}\, E_{OA} \cos\theta_{OA} + I_{OB}\, E_{OB} \cos\theta_{OB} + I_{OC}\, E_{OC} \cos\theta_{OC} = 12{,}000 \text{ watts.}$$

The way in which the branch voltages are influenced by the power factors of the branches, as well as by the magnitudes of the branch currents, is clearly seen in this example.

Example 3. — A neutral wire is connected to the point *O* in the wye of Ex. 2, Fig. *a*, and is maintained at fixed equal voltages, 120° apart, relatively to the respective potentials of the points *A*, *B*, and *C*. Answer questions *a* and *c* of Ex. 1, and calculate the current in the neutral wire.



Example 3.—Figure *a*.

(*a*) In this case the point *O* in the trial

vector diagram of Ex. 2, Fig. *a*, is fixed at the center of the triangle by the effect of the neutral wire; therefore, the effective voltages on the branches are equal numerically to each other and of value $\frac{100}{\sqrt{3}} = 57.7$ volts. Figure *a* shows the relations of the voltages. The vector currents in the branches, each referred to its branch voltage as the initial line, also shown in Fig. *a*, are as follows:

$$\bar{I}_{OA} = E_{OA}\,\overline{Y}_{OA} = 57.7\,(1 - j\,1) = 57.7 - j\,57.7.$$

$$\bar{I}_{OB} = E_{OB}\,\overline{Y}_{OB} = 57.7\,(\tfrac{1}{2} + j\,1) = 28.9 + j\,57.7.$$

$$\bar{I}_{OC} = E_{OC}\,\overline{Y}_{OC} = 57.7\,(1 + j\,0) = 57.7 + j\,0.$$

The scalar values and the angles of lag are:

$$I_{OA} = \sqrt{(57.7)^2 + (57.7)^2} = 81.7 \text{ amperes.} \qquad \theta_{OA} = 45°.$$

$$I_{OB} = \sqrt{(28.9)^2 + (57.7)^2} = 64.6 \text{ amperes.} \qquad \theta_{OB} = -\,63°\,26'.$$

$$I_{OC} = 57.7 \text{ amperes.} \qquad \theta_{OC} = 0.$$

The neutral current $\bar{I}_{ON}$ must have such a value that the algebraic sum of instantaneous currents meeting at O will be equal to zero, but each of the vector equations of currents just given has its components referred to its own branch voltage for the horizontal axis. These are 120° apart; the coördinates of the current vectors must, therefore, be changed so as to refer to a single pair of axes before they can be added. We will take as the axes for this OX and OY, where OX coincides with the line voltage $\bar{E}_{BA}$. It conduces to simplicity to use one of the line voltages as the initial line in this manner.

Then, by using the proper direction coefficients,

$$\bar{I}'_{OA} = (57.7 - j\,57.7)\ \text{cjs}^*\ (-30) = (57.7 - j\,57.7)(.866 - j\,.5)$$
$$= 21.1 - j\,78.8.$$

$$\bar{I}'_{OB} = (28.9 + j\,57.7)\ \text{cjs}\ (-150°) = 3.87 - j\,64.4.$$

$$\bar{I}'_{OC} = (57.7 + j\,0)\ \text{cjs}\ (-270°) = 0 + j\,57.7.$$

But $$\bar{I}'_{OA} + \bar{I}'_{OB} + \bar{I}'_{OC} + \bar{I}'_{ON} = 0;$$

* Art. 76.

hence substituting the right-hand sides of the above equations in this and transposing gives,

$$\bar{I}'_{ON} = -25 + j\,85.5,$$

with a scalar value of $I_{ON} = 89.1$ amperes, and an angle with reference to the direction axis OX of $\theta'_{ON} = 106° 18'$.

(*c*) The power may be found by taking the product of the current, voltage, and cosine of angle of lag of each branch and adding the three products together.

This and the previous problems show clearly the advantage of using a neutral lead wire on a star load for the purpose of keeping the voltages in the branches as nearly as possible equal, and to prevent severe insulation strains and poor service.

The check may be obtained as in Ex. 1 by drawing the vector polygons of voltages or currents. The vector polygon of line voltages is partly dotted-in, in Fig. *a*. In making the check polygon of current vectors, the currents should be drawn parallel to and of equal magnitude to their vectors in the phase diagram; and should, to avoid confusion, be preferably taken in the order in which they appear on that diagram when following it around in a clockwise direction. Thus, to $\bar{I}_{OA}$ add $\bar{I}_{OB}$, then add $\bar{I}_{ON}$, and then $\bar{I}_{OC}$. In order that the solution may be correct, it is evidently necessary for the neutral current to close an open polygon of branch currents.

Example 4. — A star circuit having the same impedances in its branches and the same arrangement as in Ex. 2 has impressed voltages as follows: $E_{AB} = 141.4$; $E_{BC} = 100$; $E_{CA} = 100$. Answer the questions of Ex. 2.

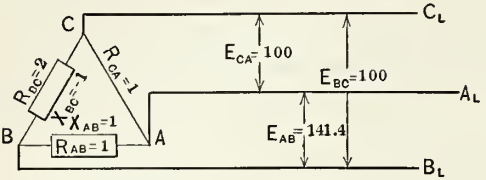
Suggestion: Lay out the vector polygon of voltages as in Ex. 2, making $\bar{E}_{BA}$ horizontal. Assume a point O' and solve for the position of the neutral point as before. Lay out the phase diagram and proceed as in Ex. 2, using care to plot the current phase angles correctly. The problem is in essence the same and as simple as that of Ex. 2, and should be solved without difficulty. These same unbalanced voltages are used in Ex. 5, which has to do with a delta system.

Example 5. — The branch-load impedances of Ex. 2 are connected in the form of a delta as shown in Fig. *a*; 141.4 volts

are impressed on the phase AB and 100 volts are impressed upon each of the other phases.

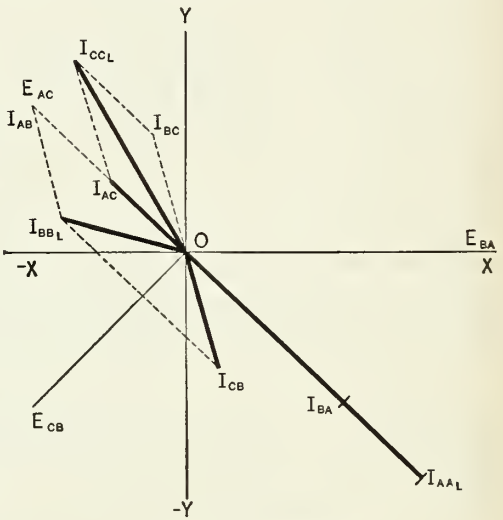
(a) What is the current and angle of lag in each branch?

(b) What is the current in each line and its lag angle with reference to one of the line voltages?



EXAMPLE 5. — Figure a.

(a) First the vector polygon of voltages is made and then the phase diagram by drawing the voltage vectors parallel to those of the vector polygon; or as the angles of the triangle and their supplements can readily be obtained by the elementary formulas of trigonometry, the vector polygon may be omitted. Using the latter method, we find that $\bar{E}_{CB}$ lags 135° behind $\bar{E}_{BA}$, and that $\bar{E}_{AC}$ lags 225° behind $\bar{E}_{BA}$. These voltages are laid out in Fig. b with $\bar{E}_{BA}$ on the initial line OX, as heretofore. The vector equation of each branch current, with components taken with reference to its particular branch voltage, is obtained by multiplying the branch voltage by the corresponding branch admittance; so that,



EXAMPLE 5. — Figure b.

$$\bar{I}_{BA} = 141.4 - j\,141.4.$$

$$\bar{I}_{CB} = 50 + j\,100.$$

$$\bar{I}_{AC} = 100.$$

From this the scalar values of the currents in amperes and the angles of lag are found to be:

$I_{BA} = 200$ amperes. $\qquad \theta_{BA} = 45°.$

$I_{CB} = 111.8$ amperes. $\qquad \theta_{CB} = -63° 26'.$

$I_{AC} = 100$ amperes. $\qquad \theta_{AC} = 0°.$

(*b*) The currents in the three line conductors may be found by combining the branch currents graphically as shown in Fig. *b*, or they may be determined more accurately by adding their vectors geometrically. Thus, referring each of the branch currents to the axes OX, OY, there results,

$$\bar{I}'_{BA} = \bar{I}_{BA} \text{ cjs } (0°) = 141.4 - j\,141.4.$$

$$\bar{I}'_{CB} = \bar{I}_{CB} \text{ cjs } (-135°) = 35.3 - j\,106.$$

$$\bar{I}'_{AC} = \bar{I}_{AC} \text{ cjs } (-225°) = -70.7 + j\,70.7.$$

Remembering that, by the convention adopted in this book, current at a junction point is considered positive when it flows toward the point, the two currents meeting at a corner of the mesh may be added, giving the following results:

$$\bar{I}'_{AL} = \bar{I}'_{BA} + \bar{I}'_{CA} = \bar{I}'_{BA} + (-\bar{I}'_{AC}) = 212.1 - j\,212.1.$$

Likewise, $\qquad \bar{I}'_{BL} = \bar{I}'_{CB} + \bar{I}'_{AB} = -106.1 + j\,35.4,$

and $\qquad \bar{I}'_{CL} = \bar{I}'_{AC} + \bar{I}'_{BC} = -106 + j\,176.7.$

The scalar values of these currents from these equations are:

$I_{AL} = 300$ amperes; $\quad I_{BL} = 112$ amperes; $\quad I_{CL} = 206$ amperes.

These currents have the following phase relations with the line voltages: $\bar{I}_{AL}$ lags behind $\bar{E}_{BA}$ by 45°; $\bar{I}_{BL}$ lags behind $\bar{E}_{CB}$ by 63° 26′; and $\bar{I}_{CL}$ lags behind $\bar{E}_{AC}$ by 14° 5′. These angles are obtained from the vector equations of current representing $\bar{I}'_{AL}$, $\bar{I}'_{BL}$, and $\bar{I}'_{CL}$.

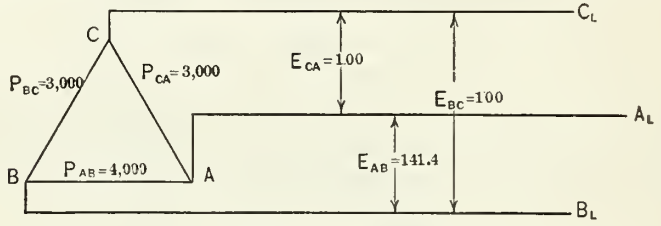
Example 6. — The wye circuits of Ex. 1 and Ex. 2 are placed in parallel on a set of 100-volt, three-phase mains of the frequency assumed in those examples. What total current flows in the mains?

All that is necessary is to geometrically add the currents already determined in those problems. (A line is drawn under the subscripts of the symbols from Ex. 1 to distinguish them from those of Ex. 2.) Thus,

$$\bar{I}_{\underline{OA}} + \bar{I}_{OA} = (36.4 - j\,15.8) + (45.3 - j\,34.7)$$
$$= 81.7 - j\,50.5,$$

from which $I_{AL} = 96$ amperes. The total currents in the other lead wires can be obtained in the same manner.

Example 7.—A load of 10,000 watts is absorbed in a delta three-phase system in the following way: in the branch AB, 4000 watts at 90 per cent power factor, current leading; in the branches BC and CA, 3000 watts each at 80 per cent power



EXAMPLE 7.—Figure *a*.

factor, current lagging. What are the currents in the branches and in the lines, the voltage between A and B being 141.4 volts, between B and C, 100 volts, and between C and A, 100 volts?

From the data given we have,

$$I_{BA} = \frac{4000}{141.4 \times .9} = 31.4 \text{ amperes.} \qquad \theta_{BA} = -25^\circ\ 50'.$$

$$I_{CB} = \frac{3000}{100 \times .8} = 37.5 \text{ amperes.} \qquad \theta_{CB} = 36^\circ\ 52'.$$

$$I_{AC} = \frac{3000}{100 \times .8} = 37.5 \text{ amperes.} \qquad \theta_{AC} = 36^\circ\ 52'.$$

The combined currents can be obtained graphically as shown in Fig. *b*, but more accurately by adding their vectors referred to a common axis, OX, as has been done heretofore, care being

used to give the dividing currents at a branch junction the correct sign. The vector equations are:

$$\bar{I}'_{BA} = I_{BA}\,\epsilon^{j\,(25^\circ\,50')} = 28.3 + j\,13.7.$$

$$\bar{I}'_{CB} = I_{CB}\,\epsilon^{-j\,(36^\circ\,52' + 135^\circ)} = -37.1 - j\,5.30.$$

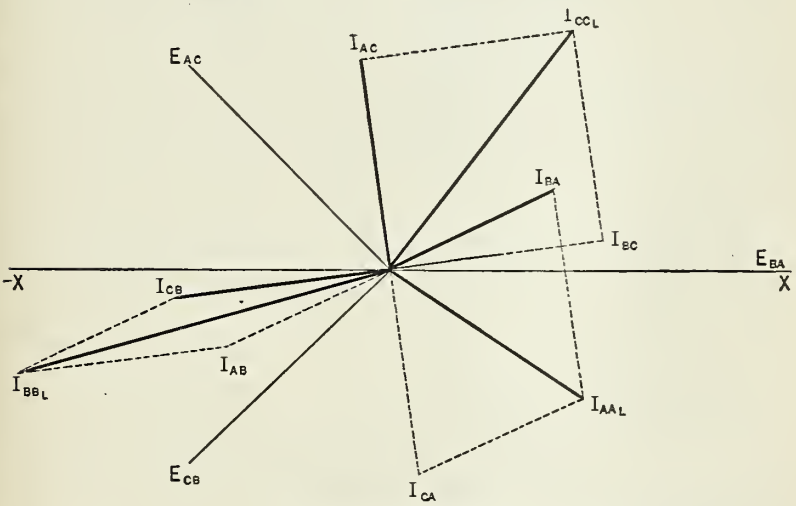
$$\bar{I}'_{AC} = I_{AC}\,\epsilon^{-j\,(36^\circ\,52' + 225^\circ)} = -5.30 + j\,37.1.$$

The exponential form of the complex expression is used here as a matter of convenience in computation, as might have been done also in earlier examples. From these expressions of the vector-branch currents we obtain

$$\bar{I}_{AL} = \bar{I}'_{BA} + \bar{I}'_{CA} = 33.6 - j\,23.4.$$

$$\bar{I}_{BL} = \bar{I}'_{CB} + \bar{I}'_{AB} = -65.4 - j\,19.0.$$

$$\bar{I}_{CL} = \bar{I}'_{AC} + \bar{I}'_{BC} = 31.8 + j\,42.4.$$



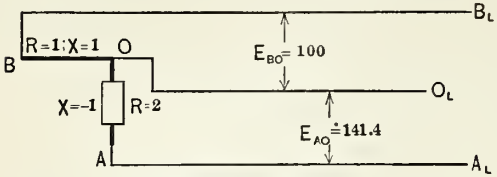
EXAMPLE 7. — Figure *b*.

The scalar values of the line currents then are:

$$I_{AL} = 40.8;\ I_{BL} = 68.1;\ \text{and}\ I_{CL} = 53\ \text{amperes.}$$

It should be noted that where the voltages or impedances are unequal, as in this case and in Ex. 5, the delta branch currents do not necessarily form a closed vector polygon, though the three line currents do.

Example 8. — A two-phase receiver having a common return wire (Fig. *a*) has the following characteristics: In OA, $R = 2$ in parallel with $X = -1$; and in OB, $R = 1$ in series with $X = 1$. Between the line wires A_L and O_L, and B_L and O_L, respectively, 141.4 and 100 volts are impressed. What are the currents in the three line wires and the voltage between the lines A_L and B_L when $\bar{E}_{OA}$ leads $\bar{E}_{OB}$ by 90°.



EXAMPLE 8. — Figure *a*.

$$\bar{Y}_{OA} = \tfrac{1}{2} + j\,1 \quad \text{and} \quad \bar{Y}_{OB} = \tfrac{1}{2} - j\,\tfrac{1}{2},$$

and the vectors of current referred to their respective voltages are

$$\bar{I}_{OA} = E_{OA}\,\bar{Y}_{OA} = 70.7 + j\,141.4,$$

$$\bar{I}_{OB} = E_{OB}\,\bar{Y}_{OB} = 50 - j\,50;$$

from which, $I_{OA} = 158.1$ amperes and $\theta_{OA} = -63°\ 26'$; and

$$I_{OB} = 70.7 \text{ amperes and } \theta_{OB} = 45°.$$

Referring the vectors $\bar{I}_{OA}$ and $\bar{I}_{OB}$ to the axes OX, OY,

$$\bar{I}'_{OA} = I_{OA}\,\epsilon^{j(63°\,26')} = 70.7 + j\,141.4,$$

and $$\bar{I}'_{OB} = I_{OB}\,\epsilon^{-j(45° + 90°)} = -50 - j\,50.$$

Adding these vectors gives,

$$\bar{I}_{LO} = \bar{I}'_{OA} + \bar{I}'_{OB} = 20.7 + j\,91.4.$$

The scalar value of I_{LO} is 93.7 amperes.

The voltage between A_L and B_L is

$$\bar{E}_{BA} = \bar{E}_{BO} + \bar{E}_{OA} = 141.4 + j\,100,$$

or $E_{BA} = 173$ volts at an angle leading E_{OA} by $35°\ 16'$.

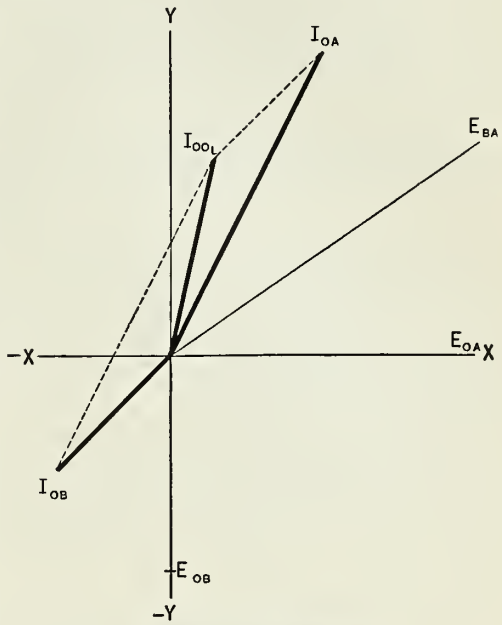
Prob. 1. A three-phase wye circuit has in series in each branch a resistance of two ohms and an inductive reactance of one ohm When a line voltage of 100 volts is impressed on

each phase, what current flows in each branch, what are the branch voltage and angle of lag, and what are the power expended in the circuit and the power factor?

(In these problems lay out the results of computations in phase diagrams.)

Prob. 2. If the power of 10,000 watts is expended in the wye system of Prob. 1, what must be the voltage across the mains, and the currents in the mains and branches?

Prob. 3. A three-phase delta circuit has a resistance of one ohm and capacity reactance of one ohm in series in each branch. When 500 line volts are impressed on each phase, how much current flows in each branch and line? How much power is expended in the circuit, and what is the power factor?



EXAMPLE 8. — Figure *b*.

Prob. 4. When 25 Kw. are expended in the delta of Prob. 3, what currents flow in the branches and line conductors and what are the line voltages?

Prob. 5. If two wye circuits, having the characteristics and being under the conditions named in Prob. 1, are connected in parallel to the same line wires, will current flow through a conductor joining their neutral points? Prove the answer.

Prob. 6. A three-phase transmission line receives from a generating plant 1000 Kw., the line voltage being 5000 volts in each phase. How much current flows in each wire if the circuit is balanced and of 90 per cent power factor?

Prob. 7. The generator of Prob. 6 is wye-connected. What voltage must be produced by the coils of each phase?

Prob. 8. A balanced load of 500 Kw. is transmitted over a two-phase line of three wires, at a voltage of 2000 volts per phase, to induction motors having an average power factor of 85 per cent. What current flows in each of the three wires and what is the voltage between the two independent wires?

Prob. 9. A three-phase wye circuit has resistance of one ohm and capacity reactance of .25 ohms in series in each branch. The line voltages E_{AB}, E_{BC}, and E_{CA}, respectively, are 200, 220, and 240 volts. What are the values of current and voltage in each branch, and what are the angles of lag of the currents with respect to the corresponding branch voltages?

Prob. 10. A three-phase wye circuit has the following vector impedances in its branches: $\bar{Z}_{OA} = 1 + j\,1$; $\bar{Z}_{OB} = \frac{1}{2} + j\,\frac{1}{2}$; and $\bar{Z}_{OC} = \frac{1}{2} - j\,1$. What are the branch voltages, currents, and angles of lag when a line voltage of 500 volts is maintained on each phase? What power is absorbed by the circuit?

Prob. 11. The impedances of Prob. 10 are connected in delta to the same mains. What are the branch currents and angles of lag, and what are the line currents? What power is absorbed by the circuit?

Prob. 12. A neutral wire is connected to the wye junction of Prob. 10 without changing the other conditions, except that, by proper regulators, the voltages between the neutral point and the line conductors are maintained numerically equal. What current flows in the neutral wire, and what power is absorbed by the circuit?

Prob. 13. A transmission line supplies a three-phase wye circuit with 100 Kw. in the proportion of 50 Kw. at 85 per cent capacity power factor in one branch and 25 Kw. at unity power factor in each of the other branches. The line voltage of each phase is 240 volts. What are the values of the branch currents, voltages, angles of lag, kilowatts, and kilovolt-amperes, what are the line currents, and what are the total kilovolt-amperes of the circuit?

Prob. 14. A transmission line supplies the three branches of a delta circuit with 1000 Kw. in the proportion of 250, 350, and 400 Kw., the voltage and power factor for each branch being 2000 volts and 90 per cent power factor. What are the branch currents, the line currents, the circuit power, and the circuit kilovolt-amperes?

Prob. 15. A three-phase transmission line of 2000 line volts on each phase supplies the following circuit: a three-phase delta circuit taking 20 Kva.* in each branch at 80 per cent lagging power factor; a three-phase wye circuit taking 50 Kva. in each branch at 90 per cent leading power factor; and a single-phase load from one phase, of 25 Kw. at unity power factor. How much current flows in each line wire?

Prob. 16. A balanced load of 100 Kw. at 100 per cent power factor is absorbed by a six-phase mesh-connected receiver. The voltage between line wires is 250 volts. What is the current per line wire?

Prob. 17. What would the branch voltages and line currents be if the load of Prob. 16, at the same line voltages, was absorbed by a six-phase star circuit?

104. Measurement of Power in Two- and Three-phase Circuits. — The principles underlying the methods of measuring power in polyphase circuits differ in no respect from those already deduced in relation to single-phase circuits,† but it is desirable to apply them in such a way as to reduce the number of necessary readings to a minimum. For satisfactory measurements, non-reactive wattmeters are of essential importance.

A. Two-phase Systems

A 1. *Independent Circuits.* In a two-phase system with separate circuits, independent wattmeter readings are taken in each circuit and the total power is the sum of the readings. One wattmeter placed in each circuit, as shown in Fig. 231, from which simultaneous readings are taken, is the best arrangement; but if two wattmeters are not to be had, one may be inserted successively in the two circuits, and the sum of the readings is equal to the power in the system, provided the load does not vary while the readings are being taken. If the circuit is perfectly balanced, twice the reading of a wattmeter in one circuit is equal to the power, but this is a condition which cannot be relied upon.

A 2. *Circuits with Common Return.* Two wattmeters may here be used, one for each circuit, connected in the way shown

* Kva. is an abbreviation for kilovolt-amperes. † Art. 99.

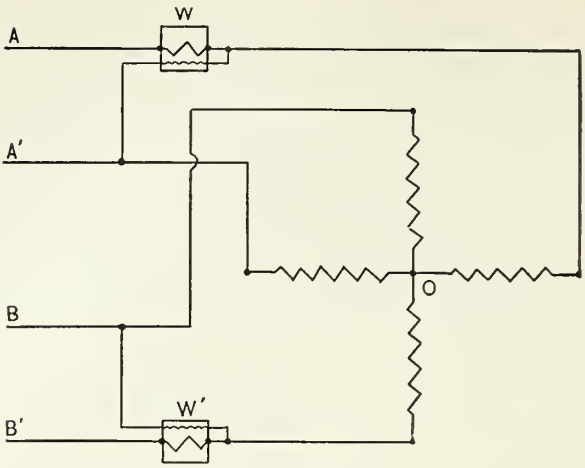


FIG. 231. — Measurement of Power in Two-Phase System with Separate Circuits.

in Fig. 232. The arrangement shown in Fig. 233 is equivalent to a single wattmeter connected as in Fig. 234, and is only correct for a system in exact balance. When the single watt-

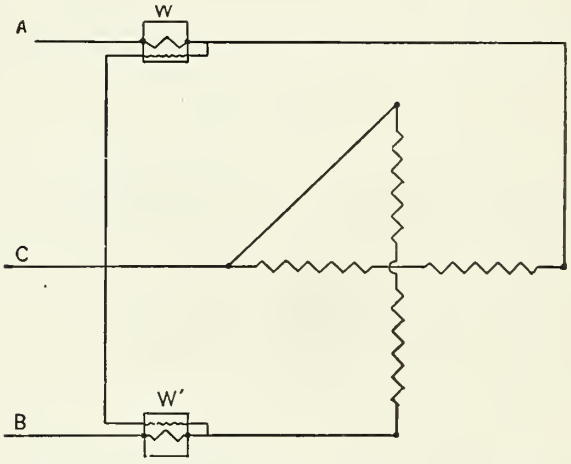


FIG. 232. — Measurement of Power in Two-phase System with Common Return.

meter is used in a balanced system, the current coil is placed in the common wire, and a reading is taken with the free end of the voltage coil connected to one outside wire. The voltage coil terminal is then quickly transferred to the other outside

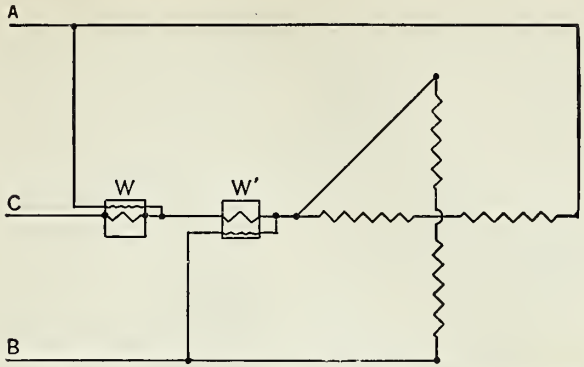


FIG. 233. — Measurement of Power in Two-phase System in Exact Balance.

wire and a new reading taken. The condition of exact balance is not to be relied upon, so that the arrangement of Fig. 232

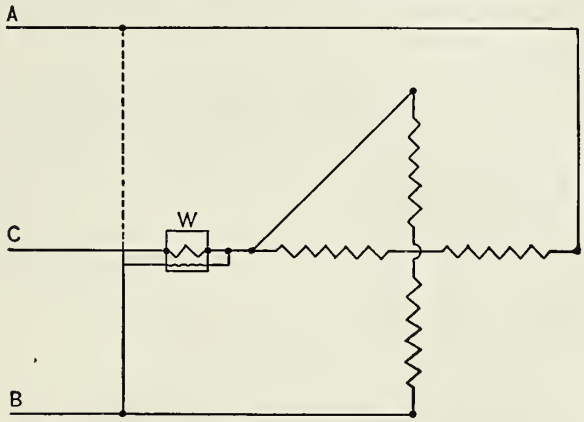


FIG. 234. — Equivalent Connection for Fig. 233, using One Wattmeter.

must ordinarily be used. The sum of the readings of the two wattmeters then gives the power in the system.

B. Three-phase Systems

B 1. *Three Wattmeters.* *a.* If the power delivered by a generator, or absorbed by a motor or other device, which is connected in wye fashion, is to be measured, three wattmeters may be used, connected as shown in Fig. 235, provided the common or neutral point is accessible. It is evident that each

wattmeter measures the power in one branch so that the sum of the readings gives the power in the system. If the system is exactly balanced, three times the reading of one wattmeter gives the power.

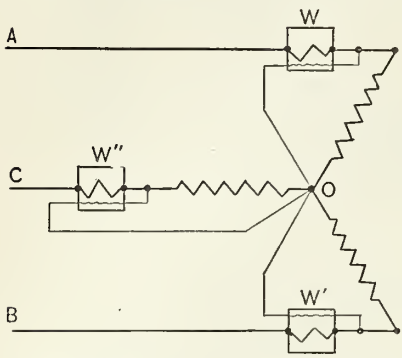


FIG. 235. — Measurement of Power in Three-phase System, Wye Connected. Three Wattmeters used.

b. If the devices are connected delta fashion, three wattmeters may still be used, provided the current coils of the wattmeters can be inserted directly in the branch circuits as shown in Fig. 236. The power in the circuit is equal to the sum of the three wattmeter readings, and if the circuit is exactly balanced, three times the reading of one wattmeter gives the power.

c. When it is impossible to insert the wattmeters in the coil circuits of a device with delta connection, the three-wattmeter method may still be used by the creation of an artificial neutral point as shown in Fig. 237. For this purpose, three equal non-reactive resistances are connected together at one end of each, and the other ends are connected to the respective corners of the mesh circuit. In a balanced circuit, the voltage between the neutral point and either corner is equal to $\frac{E}{\sqrt{3}}$, where E is the voltage of one branch, and the phase of this voltage is θ degrees in advance of the current entering the corner. A wattmeter with its current coil inserted in the circuit wire leading to the corner carries a current equal to $\sqrt{3}\,I$, where I is the branch

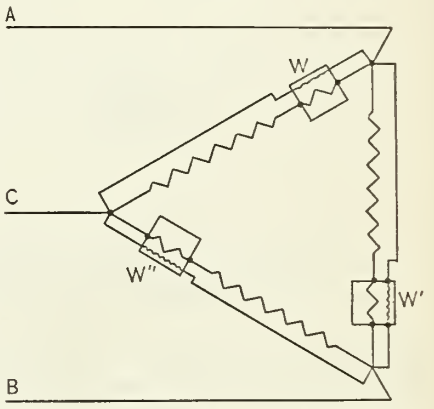


FIG. 236. — Measurement of Power in Three-phase System, Delta Connected. Three Wattmeters used.

current, and if the free end of the voltage coil is connected to the neutral point, the power reading of the wattmeter is

$$\frac{\sqrt{3}\,IE\cos\theta}{\sqrt{3}} = IE\cos\theta,$$

which is the power in the branch. Care must be taken that the resistances of the wattmeter voltage coils are so large compared with the three auxiliary resistances that connecting them in circuit does not disturb the voltage of the neutral point.

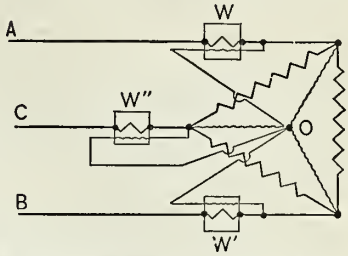


FIG. 237. — Measurement of Power in Three-phase System, Delta Connected, with Artificial Neutral.

d. If the free ends of the three voltage coils are connected directly together, the sum of the instrument readings gives the total power, although the readings may not be individually significant unless the resistances of the voltage coils are exactly equal.

These methods are independent of the condition of balance in the system or the current lag.

B 2. *Two Wattmeters.* The algebraic sum of the readings of two wattmeters inserted in a three-phase circuit as shown in Fig. 238, gives the power in the system with entire independence of the balance of the system or current lag. When the current lag in the circuit is less than 60°, *i.e.* the power factor is greater than .50, the arithmetical sum of the readings is equal to the power in the circuit ; but if the lag is greater than 60° (the power factor is less than .50), the relation of the currents in the current and voltage coils of one of the wattmeters causes it to have a negative reading, and the arithmetical difference of the readings of the two instruments gives the power. There is some difficulty in distinguishing which condition exists in some cases, especially when the power absorbed by partially loaded induction motors, in which the power factor is low, is under measurement. As a general rule,

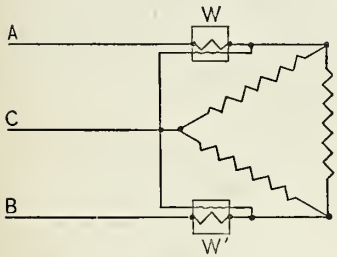


FIG. 238. — Measurement of Power in Three-phase System. Two Wattmeters used.

if the conditions do not make the case evident, the truth may be discovered by interchanging the positions of the instruments without altering the relative connections of their main and voltage coils. If the deflections of both needles are reversed, the difference of the original readings represents the power, but if the deflections are in the same direction as before, the sum of the readings is correct. The proof of this theorem is given in Art. 105.

A double wattmeter, consisting of two movable coils on one spindle and two fixed coils, can be used in measuring power by the two-wattmeter method. Such an instrument of itself sums up the double reading algebraically, and a single reading gives the power. Integrating wattmeters based upon this principle are very useful in the commercial sale of power from two-phase and three-phase circuits.

B 3. *One Wattmeter.* In a balanced circuit one wattmeter may be very conveniently used by connecting the current coil in one wire and connecting the free terminal of the voltage coil alternately to the other two leads (Fig. 239), when the sum of the readings gives the power. For, the power reading of the wattmeter in its first position is $\sqrt{3}\,IE\cos(\theta + 30°)$, and in its second position,

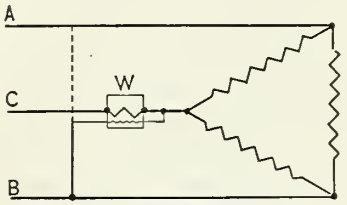


FIG. 239. — Measurement of Power in Three-phase Balanced System. One Wattmeter used.

$$\sqrt{3}\,IE\cos(\theta - 30°),$$

and the sum of the readings,

$$\sqrt{3}\,IE\{\cos(\theta + 30°) + \cos(\theta - 30°)\} = 3\,IE\cos\theta,$$

where I, E, and θ are the current, voltage, and lag in a branch: but $IE\cos\theta$ is the power in one branch and $3\,IE\cos\theta$ is the total power of the three branches, hence the one wattmeter gives correct indications provided θ is the same for all the branches, and the load is uniformly distributed. A wattmeter having two independent voltage coils could be used as a direct reading instrument for this purpose. A similar wattmeter could also be used in one-wattmeter measurements of power in two-phase circuits.

Thus, when one wattmeter is used as explained above in a balanced three-phase circuit, the sum of the two readings gives

$3\,EI\cos\theta$. In like manner it may be shown that the difference of the same two readings gives $\sqrt{3}\,EI\sin\theta$. From which it follows that

$$\tan\theta = \sqrt{3}\,\frac{w_1 - w_2}{w_1 + w_2},$$

where w_1 and w_2 are the two readings of the wattmeter. This relation is sometimes convenient for obtaining the value of θ and thus the power factor of a balanced three-phase circuit. It may also be shown by a similar process of reasoning that, when the circuit is balanced, two wattmeters connected in the manner illustrated in Fig. 238 give readings which may also be used in this formula.

Wye Box. The power in a balanced three-phase system may also be measured by using one wattmeter and what is called a Y-box. This consists of two equal non-reactive resistance coils with one end of each connected to a common binding post and the other end of each connected to an individual binding post. If each coil is of resistance equal to the resistance of the voltage coil of a wattmeter with which it is designed to be used, inserting the wattmeter current coil in one leg of the circuit, connecting the individual binding posts of the wye box respectively to the other legs of the circuit, and connecting the free end of the wattmeter voltage coil to the common binding post of the wye box, affords an arrangement equivalent to one wattmeter of Fig. 240. The power for the balanced system is three times the wattmeter reading. If the arms of the wye box are not each equal in resistance to the voltage coil of the wattmeter, a correction must be made.

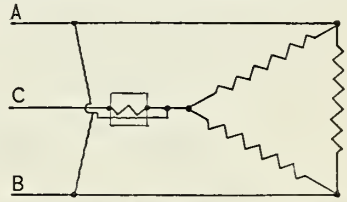


FIG. 240. — Measurement of Power in Three-phase System. One Wattmeter used with Non-reactive Wye-box.

105. Measurement of Power in Any Polyphase Circuit. — In the case of any polyphase system of n phases and n conductors, it was originally proved by Blondel that the power in the circuit may be measured by $n-1$ wattmeters. Supposing A, B, C, D, etc., are points where the n conductors of a polyphase supply circuit connect to the circuits under test, then, as has been already proved,* $\Sigma i = 0$, if i represents the instantaneous cur-

* Art. 100.

rent in any conductor. The power being supplied through the A conductor at any instant is equal to $\frac{dq_a}{dt}\, v_a = i_a v_a$, where q_a is the quantity of electricity brought to A during a time dt, and v_a is the absolute electrical potential of A.

The average power transferred through conductor A during a complete period is $\frac{1}{T}\int_0^T i_a v_a\, dt$, and the total power in the circuit, $P = \Sigma \frac{1}{T}\int_0^T ivdt$.

The absolute potentials of the points are inconvenient to measure, and it is desirable to introduce into the formula the difference between the potentials at the points and some fixed point of potential v'. Since $\Sigma i = 0$, we also have $\Sigma iv' = 0$, and $\Sigma iv'$ may therefore be directly inserted in the formula without destroying the equality, or

$$P = \Sigma \frac{1}{T}\int_0^T (iv - iv')\, dt = \Sigma \frac{1}{T}\int_0^T i(v - v')\, dt.$$

Writing e for $v - v'$ (the instantaneous difference of potential between the fixed point and any given point in the system) gives

$$P = \Sigma \frac{1}{T}\int_0^T iedt.$$

The fixed point may be taken at one of the corners of the circuit, A for instance, since $\Sigma iv' = 0$ holds equally for it, and the power formula becomes

$$P = \frac{1}{T}\int_0^T i_b e_{ab} dt + \frac{1}{T}\int_0^T i_c e_{ac} dt + \text{etc.} + \frac{1}{T}\int_0^T i_n e_{an} dt,$$

or

$$P = I_b E_{ab} \cos\theta' + I_c E_{ac} \cos\theta'' + \text{etc.}$$

where θ', θ'', etc., are the differences of phase between the respective line currents and the corresponding voltages to point A. The terms on the right of the equation are the familiar forms representing the power readings of a wattmeter, so that if $n - 1$ wattmeters are inserted in circuit with their current coils respectively in the $n - 1$ conductors, B, C, D, etc., and the free ends of their voltage coils all connected to A, the algebraic sum of their readings is equal to the power in the system.

When the circuit includes three phases only, the formula is $P = I_b E_{ab} \cos \theta' + I_c E_{ac} \cos \theta''$, and only two wattmeters, connected as in Fig. 238, are required to give the power in the circuit, but due regard must be had to the relative signs of $\cos \theta'$ and $\cos \theta''$.

From the relative phases of the currents I_b and I_c and voltages E_{ab} and E_{ac} it is easy to see that in a balanced three-phase system $\theta' = \theta_1 - 30°$, and also that $\theta'' = \theta_2 + 30°$, and therefore

$$P = I_b E_{ab} \cos (\theta_1 - 30)° + I_c E_{ac} \cos (\theta_2 + 30°),$$

in which θ_1 and θ_2 are the angles of lag of the branch currents, and the subscripts indicate the respective branches. The formula shows that the first term at the right, which represents the reading of one wattmeter, is positive within the limits $\theta_1 = + 90°$ and $\theta_1 = - 60°$, and that its value is negative between the limits $\theta_1 = - 60°$ and $\theta_1 = - 90°$. The second term, which represents the reading of the second wattmeter, is positive between the limits $- 90°$ and $+ 60°$ and negative between $+ 60°$ and $+ 90°$. Consequently, *if the current lags equally in the circuits*, i.e. $\theta_1 = \theta_2$: Both wattmeters have a positive reading and *the power in the circuit is the sum of the readings, for angles of lag between* $+ 60°$ *and* $- 60°$. *If the angle of lag is* $+ 60°$ (the current lags behind the voltage), the second wattmeter reading is zero, and *the power in the circuit is equal to the reading of the first wattmeter. If the lag is greater than* $+ 60°$, the reading of the first wattmeter is positive and the second is negative, and *the power in the circuit is equal to the difference of the two readings.* Again, *if the angle of lag is* $- 60°$ (the current leads the voltage), the first wattmeter reading is zero, and *the power in the circuit is equal to the reading of the second instrument. If the lead is more than* 60°, the reading of the first instrument is negative and of the

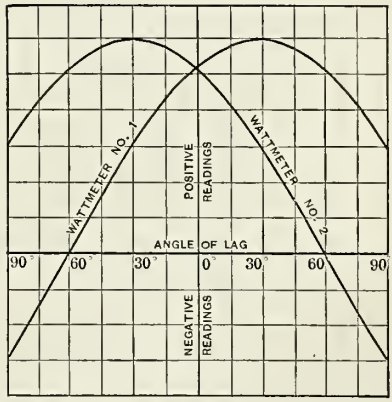


FIG. 241. — Curves showing Relation of Wattmeter Readings to Lag Angle in Balanced Three-phase Circuit.

second positive, and *the power in the circuit is equal to the difference of the two readings.* When the angle of lag is $\pm 90°$, the readings of the two instruments are equal, but one is posi-

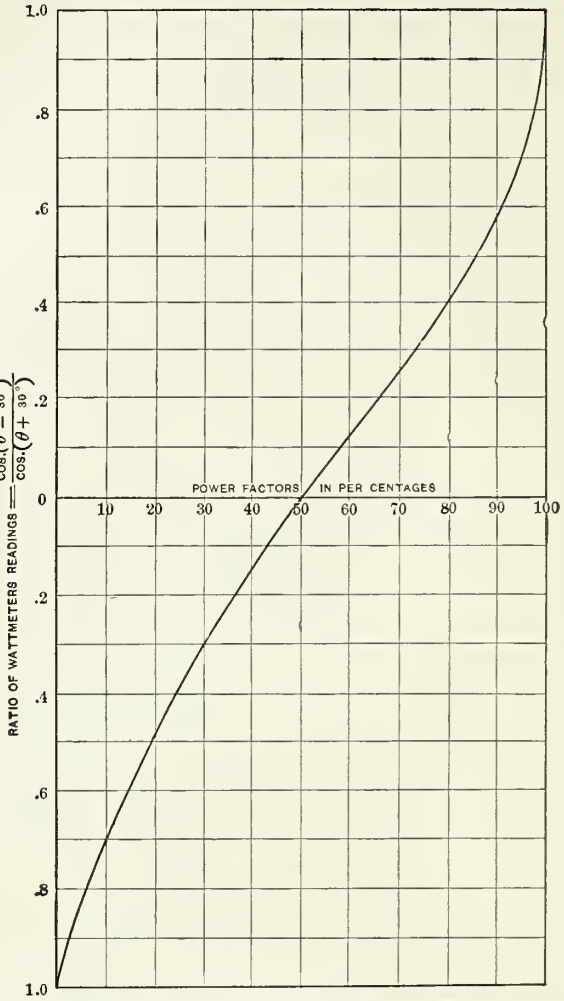


Fig. 242. — Curve showing Relation of the Ratio of the Wattmeter Readings to Power Factor in Balanced Three-phase Circuit.

tive and the other negative. The relations of the wattmeter readings to the angle of lag between $+ 90°$ and $- 90°$ are shown by the curves in Fig. 241. These are two equal sinusoids with a phase difference equal to 60°. The readings of two watt-

meters in a balanced three-phase circuit at any value of the lag are in the proportion of the corresponding ordinates of the two curves. Figure 242 shows similar relations plotted in a different manner. In this case the ratios of the readings of the wattmeters are plotted as ordinates and the power factors as abscissas. This curve is derived directly from the curves of Fig. 241.

The same conditions obtain in an unbalanced circuit, provided equivalent angles of lag are considered.

CHAPTER IX

HYSTERESIS AND EDDY CURRENT LOSSES

106. The Energy Losses Caused by Magnetic Hysteresis. — When the number of lines of force passing through a coil is changed, a voltage is generated in the coil the value of which is $e = \frac{-nd\phi}{10^8 dt}$, in which $\frac{nd\phi}{dt}$ is the change per second of the number of magnetic linkages through the turns of the coil. If a current is flowing in the coil when the change occurs, the work dw, done upon or by the lines of force, is equal to $eidt$, and therefore, $dw = -\frac{nid\phi}{10^8}$. But in case the magnetic circuit is of uniform material and its cross section is of uniform area $\phi = AB$ and $d\phi = AdB$, and $n = \frac{10\,M}{4\pi i}$, where ϕ is the magnetic flux in lines of force, B is the magnetic density (lines of force per square centimeter), A is the cross section of the magnetic circuit in square centimeters, M is magnetomotive force in Gilberts (one Gilbert $= \frac{10}{4\pi}$ ampere-turns), and n is the number of turns in the coil. From this we have $dw = iedt = \frac{-AMdB}{10^7 \times 4\pi}$. But, in any space sufficiently limited so that H, the field strength or magnetizing force, is uniform, $M = HL$, where L is the length of the magnetic circuit, and consequently $dw = \frac{-AL}{10^7 \times 4\pi} HdB$. It will be observed that AL is equal to the volume of the magnetic circuit under consideration, and the coefficient of HdB is therefore proportional to that volume. By integrating this we get the expression

$$W = \int iedt = -\frac{AL}{10^7 \times 4\pi} \int HdB.$$

The power in watts which is expended during the change is of course the average rate of doing the work shown above and is equal to the work divided by t. This work is taken from the electric circuit as the magnetism rises and returned to the circuit as the magnetism falls, when the varying magnetism is caused by varying the exciting current. The value of $\int HdB$ equals an area between curves of rising and falling magnetism, such as $Ogjs$ of Fig. 243, when the operation starts with both H and B of zero value, proceeds to a value of H equal to rj and the corresponding value of B equal to Or, and returns to the condition of H equal to zero and the state of remanent magnetism (as might be the case if the magnetic circuit was composed of iron) when B equals Os. This work is stored in the magnetic linkages or expended in heating the metal. That some of the work remains stored in the magnetic linkages under the conditions indicated by Fig. 243 is manifest by the finite value Os of the magnetism at the end of the procedure; and part of this stored energy is recoverable, but it is impossible to say from the given premises what proportion of the whole it constitutes. However, when the $\int HdB$ is carried through a closed (cyclic) curve, such as is illustrated in Fig. 244 for instance, the available energy of the magnetic linkages at the end of the cycle is the same as at the beginning, and in that case all of the energy represented by $\frac{AL}{10^7 \times 4\pi} \int HdB$ is converted into heat. In Fig. 244 the cyclic curve is $JSBjsbJ$. After the iron part of a magnetic circuit has come into a cycle of this form through continuous reversals of the magnetism between the fixed limits of J and j, the curve repeats itself with each reversal, and $\int HdB$ has a fixed

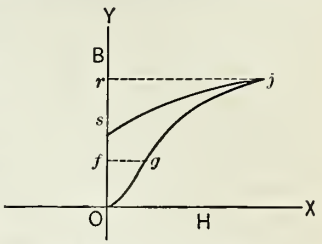


Fig. 243. — Curve of Rising and Falling Magnetism.

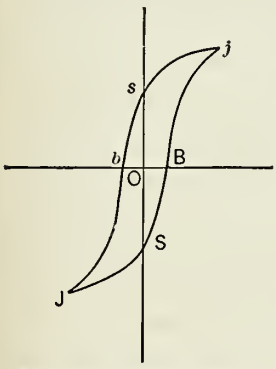


Fig. 244. — Cyclic Curve of Magnetism.

value per cycle. Upon first starting the cyclic magnetization, $B = f(H)$ travels over a line such as Oqj of Fig. 243, but it takes up the fixed cyclic curve when the iron is subjected to complete cycles or reversals of magnetism.

A mental picture of these phenomena may be formed by considering that work is done upon the magnetic molecules in order to swing them into line when a piece of iron is magnetized. Part of this work is expended in overcoming an opposition to the swinging of the molecules which is akin to friction, and this work is converted into heat; but another part is expended in overcoming the live forces of mutual attraction and repulsion between the molecules or groups of molecules, and this part is returned to the electric circuit by any molecules which swing back upon the withdrawal of the magnetizing force. As no energy is required to maintain magnetic flux when once established, there would be no loss of energy from magnetizing and then demagnetizing a piece of iron if the curves of magnetism ascending and descending were the same, for in this case $\int HdB$ for the descending curve would be equal and opposite to $\int HdB$ for the ascending curve, and the two would cancel. But in fact there is a difference between the ascending and descending branches of the curves of magnetization of iron and other magnetic metals, which is caused by **Hysteresis** and there must be a loss of energy due to the operation of magnetizing and demagnetizing, which is proportional to the area between the ascending and descending curves. In Fig. 243 the area Orj is equal to $\int HdB$ going up the curve and srj is equal to $\int HdB$ going down the curve. The work which was expended in going up the curve to magnetizing force H and magnetic density B is proportional to area Orj: and the work which was recovered in going down the curve to zero magnetizing force is proportional to area srj. The net work which has been expended in the operation, part of which may yet be recoverable, though a portion of it has been expended in overcoming what may be called magnetic molecular friction and is not recoverable, is proportional to the area Ojs. The work proportional to the area Ojs is $W = \frac{-AL}{10^7 \times 4\pi} \times \int_{B=0}^{B=Os} HdB$. The recoverable part of this is

retained by the remanent (residual) magnetic flux which has a density $B = Os$. Of this part, all may be recovered except that which is absorbed in molecular friction when the coercive force is overcome and the magnetism is forcibly brought to zero.

If the piece of iron is carried through a cyclic curve of magnetization so that it coincidently returns to the same values of magnetization and magnetizing force as those from which it started, all the work which has been expended must have been done at the expense of the magnetising circuit and is irrecoverable, as already explained. Consequently, in a cyclic curve like Fig. 244, the area of the cyclic curve is directly proportional to the lost energy, and is equal to the lost energy in *ergs* when H is measured in gilberts per centimeter, B is given in lines of force per square centimeter, L is given in centimeters, and A is given in square centimeters. If the cycles are produced by reversals of the magnetization between equal positive and negative values by rotating a cylindrical piece of iron between fixed magnet poles or subjecting the iron to the magnetizing effect of a coil carrying alternating currents, the irrecoverable energy has a fixed value per cycle as shown by the formula, which is proportional to the area of the cyclic curve of magnetization or "hysteresis loop," and this area bears a close relation to the maximum magnetic density between the limits of which the cycle of magnetism is carried.

This energy loss due to hysteresis was shown by Steinmetz, as the result of a very brilliant series of experiments,* to be closely proportional to the 1.6 power of the maximum magnetic density, within the ranges of density used in ordinary engineering practice. Later experiments carried on by Ewing and others, including Steinmetz, have shown that the exponent need not be exactly 1.6, but commonly approximates thereto, and for most practical purposes a figure between 1.55 and 1.6 is sufficiently close. At magnetic densities exceeding 20,000 lines of force per square centimeter, the exponent seems to become considerably smaller; and it varies for different steels.

Warburg called early attention to the fact that the actual loss of energy, when a metal subject to hysteresis is carried through a complete magnetic cycle, is proportional to the area

* *Trans. Amer. Inst. Elect. Eng.*, 1892, Vol. IX.

of the cyclic curve of magnetism. This was also demonstrated by Ewing independently of Warburg.*

Combining the results of the theoretical considerations, which show that the area of a cyclic curve of magnetization is proportional to energy per unit volume of the iron expended on something analogous to molecular friction, and the experimental considerations which show that the area is proportional to approximately the 1.6 power of the maximum density of the magnetic cycles, gives the following relations for the hysteresis loss per cycle in a piece of iron:

$$W = \frac{AL}{10^7 \times 4\pi} \int_{+B}^{+B} H dB = \eta V B^x \times 10^{-7},$$

in which η is a numerical quantity depending on the quality of the particular sample of iron and is often called the "hysteresis constant" of the iron, and V is the volume of the iron. The numerical value of η for any sample of iron also depends upon the unit of volume adopted, and the values hereinafter given assume the volumes to be measured in cubic centimeters.

The expression is ordinarily written

$$W = \eta V B^{1.6} \times 10^{-7},$$

since the exponent x has approximately the value of 1.6; and the hysteresis loss in watts is equal to this loss per cycle multiplied by the number of cycles per second, or

$$P = \eta V f B^{1.6} \times 10^{-7},$$

where f is the frequency of the magnetic cycles.

When put in words, this is sometimes referred to as Steinmetz's law; namely, *the hysteresis loss in each unit volume of iron is equal to the product of* (1) *a hysteresis constant fixed by the quality of the iron under consideration*, (2) *the frequency of the magnetic cycles, and* (3) *approximately the* 1.6 *power of the maximum magnetic density occurring per cycle.*

107. Curves of Hysteresis, and of Current and Voltage distorted by Hysteresis. — Figure 245 illustrates cyclic hysteresis curves for the same test piece with varying degrees of satura-

* Ewing, *Magnetic Induction in Iron and Other Metals*, p. 99.

tion, when produced by alternating magnetizing currents with a wave form in which the ordinates increase continuously to a maximum and then similarly decrease to zero, as in a sine or a parabolic wave. The three cyclic curves correspond to the three exciting currents affording maximum field strengths, respectively, of two, five, and fifteen gilberts.

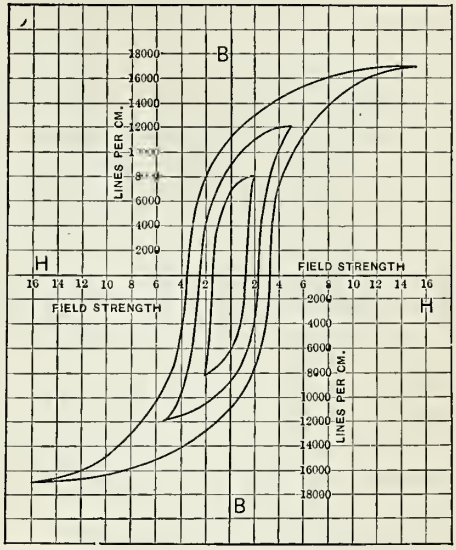


FIG. 245. — Hysteresis Curves corresponding to Three Maximum Magnetic Densities, each produced by an Alternating Magnetizing Current with a Single Maximum Value.

Figure 246 shows a similar curve resulting from a magnetizing wave which contains marked indentations, such as the current illustrated by curve *I* in Fig. 247; that is, the wave does not increase continuously to a maximum, but has ripples superposed on its form. It will be noted here that the cyclic hysteresis curve contains small auxiliary loops. The area now is proportional to the area of the main cyclic curve plus the sum of the areas of the auxiliary loops. Figure 247 shows the current wave used in the last series resolved into its harmonics, which consist of a fundamental and a fifth harmonic of considerable amplitude.

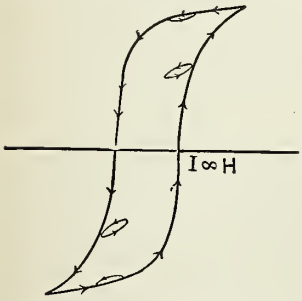


FIG. 246. — Hysteresis Curve caused by a Wave of Magnetizing Current which has more than One Maximum.

If an air space exists in series with the iron part of a magnetic circuit, its effect is to increase the exciting current which is required to produce a given maximum magnetic density; but, except so far as

there is a greater loss in the electric circuit due to the larger current, it does not increase the active component of the exciting current, inasmuch as the area of the hysteresis cycle is not altered, assuming that the iron is not changed in volume or conformation and the same maximum magnetic density is maintained. This case may be illustrated by the tilted magnetic curve shown in Fig. 248. The effect of the air space is here represented by the position of the tilted axis OY_1 with respect to the axis OY. Curve bOb' is the hysteresis loop of the iron in the magnetic circuit, and the abscissa corresponding to any ordinate represents the magnetic force required to set up in the iron the magnetic density corresponding to the particular ordinate. The abscissa of the line OY_1 corresponding to the same ordinate represents the magnetic force required to set up the same magnetic density in the air space. Consequently these two abscissas added together give the magnetic force required to set up that magnetic density in iron and air space in series. It is thus to be seen that the hysteresis loop for the magnetic circuit composed of iron and air gap in series consists of a curve like $b_1Ob'_1$, which has abscissas equal to the sums of the abscissas of the hysteresis loop of the iron in the magnetic circuit and the straight line which corresponds to the curve of magnetization of the air gap. Manifestly, the area of the hysteresis loop and the corresponding hysteresis loss is unaltered by the introduction of the air gap in the magnetic circuit, provided the

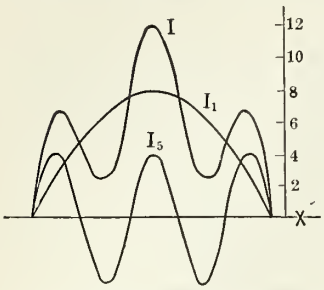


FIG. 247. — Component Harmonics of the Magnetizing Current Giving Hysteresis Loop of Fig. 246.

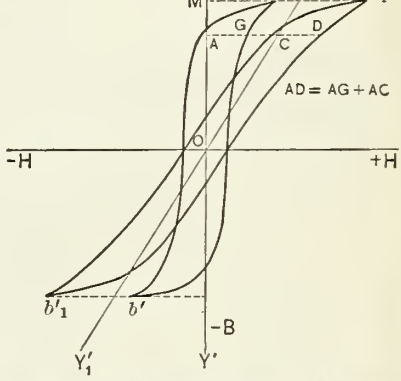


FIG. 248. — Hysteresis Cycle of a Magnetic Circuit containing an Air Space.

limits of magnetization are kept the same; but the plotted curve is skewed off to the right by the effect of the air gap, and the exciting current required to produce the maximum magnetic density is increased on account of the reluctance of the air gap.

108. Similarity between Magnetic and Dielectric Hysteresis and Friction. — All magnetic materials seem to be subject to the phenomena attendant upon a seeming molecular magnetic friction which are collectively called hysteresis. As already explained, the area of a cyclic hysteresis curve is proportional to the energy which is converted by molecular action into heat when magnetic material is carried through cycles of magnetization. In the course of extended experiments already referred to, Steinmetz demonstrated that the area of the cyclic curve for iron is approximately proportional to the 1.6 power of the maximum magnetic density within ordinary ranges of density. Similar experiments were repeated by Professor J. A. Ewing, and he therein demonstrated that the exponent 1.6 was not necessarily fixed, but that 1.6 approximately represents the correct value. In some of Ewing's experiments the results were more nearly represented by the exponent 1.55. He also showed that it may be much larger (perhaps more than 2.5) at very low densities. All experiments, however, go to show that the area of the cyclic hysteresis diagram for iron is very nearly proportional to the maximum magnetic density affected by an exponent of 1.6 within the range of densities in usual engineering practice.

Some also suppose that dielectric hysteresis is subject to a similar exponential law. When a condenser is subjected to alternating voltage, energy is absorbed in the dielectric, and this energy is proportional (according to the results of certain experimenters) to the maximum charge of the condenser, — that is, to the maximum electric stress or density of electrostatic induction in the dielectric, — raised to the 1.6 power. It is probable that the rotation of a dielectric body in an electrostatic field would result in conversion of energy due to dielectric hysteresis in a manner analogous to the conversion of energy due to magnetic hysteresis when iron is rotated in a magnetic field. It may also be expected that the losses for equal frequencies and comparatively small electrical densities should be practi-

cally similar if a given volume of dielectric was rotated in an electrostatic field, and if it was subjected to an equal frequency of equal maximum electric stress by means of an alternating electrostatic field. In the case of the rotating dielectric the increase of loss should tend to become less and less as the electric stress is increased, until with a certain value of the electric stress the hysteresis loss should become practically zero. This is analogous to the conditions that exist with respect to magnetic hysteresis, and of course can only be true in case the molecules act toward each other in a similar manner.

The foregoing conditions also have analogies in mechanical friction, which has been experimentally proved to be affected by an exponential law. Thus, for instance, a fluid friction, as that of mercury at low velocity, has been experimentally shown to vary approximately in proportion to the 1.6 power of the pressure. This points the way to an inference that the molecular actions which cause absorption and conversion of energy by friction may prove to be of the same fundamental nature and of the same order as the molecular actions which cause the absorption and conversion of energy through magnetic, and perhaps dielectric, hysteresis. Professor Ewing has suggested that experiments of his substantiate this view, but no complete demonstration of these relations has yet been disclosed. Professor Steinmetz, whose accomplishments in advancing alternating current philosophy have been varied and great, has cast doubt on the relationship of dielectric hysteresis to the other phenomena, and some of the later experiments seem to show that what was thought by some to be dielectric hysteresis is in reality only the effect of current seepage in the dielectric and is not of the nature of hysteresis.

109. The Magnetic and Dielectric Hysteresis Constants. — It is easy to compute the energy which is absorbed through the effect of magnetic or dielectric hysteresis and converted into heat thereby, provided measurements have been performed on materials similar to the samples under consideration, so that the coefficient η giving the heat developed per cycle in unit volume of the material at unit magnetic density is known. This coefficient, as already pointed out, is often called the **Hysteresis constant** of the material.

The constants of dielectric hysteresis have not been determined satisfactorily. It may be assumed that the dielectric

constant varies for different dielectric materials, but the evidence of such experience as has been had in the commercial world with condensers constructed with the ordinary dielectric substances indicates that they do not differ very widely among themselves in respect to losses per cycle, unit volume, and unit electrostatic stress that may be due to hysteresis effects.

Large numbers of experiments have been made and are being made by manufacturers of electrical machinery to determine the constants of magnetic hysteresis related to various qualities of iron and steel. Experiments on other magnetic materials except iron and steel have been few. Cobalt and nickel have been tested by Professor Ewing and others, and certain other materials have been tested; but none of these have any important significance in theory or practice at present. The range of hysteresis constants of iron is well represented in the accompanying table. The better qualities of iron are such as would prove satisfactory for use in alternating current transformers, alternating current motors, and the like.

The following table, compiled by Messrs. Swenson and Frankenfield, indicates in round figures the values of the coefficient η for various irons; but it is to be remembered that variation in the processes of manufacture or in the treatment, as well as in the composition of the metal, often make comparatively large changes in η. A small addition of nickel or silicon to soft iron intended for rolling into sheets tends to reduce the value of η and increase the electrical resistance of the iron, and such silicon iron is now generally used for the cores of transformers and the like.

TABLE OF HYSTERESIS CONSTANTS

Best Wrought Iron and Steel Sheets	.001
Good Soft Iron and Steel Sheets	.002
Ordinary Soft Iron	.003
Annealed Cast Steel	.008
Ordinary Cast Steel	.012
Ordinary Cast Iron	.016
Hard Cast Iron and Tempered Cast Steel	.025

The constants in this table are the values of η to be used in the formula with cycles per second, volume in cubic centimeters, and magnetic density in lines of force per square centimeter.

110. Measurement of the Energy absorbed by Hysteresis.— Methods of obtaining the approximate values of magnetic hysteresis constants of samples of iron are now so simple that every well-equipped electrical engineering laboratory is fitted for such work, and a large proportion of the manufacturers carry out systematic tests upon their iron. Some of the methods of measurement follow.

a. Some of the manufacturers do not make tests with special instruments, but punch a certain volume of iron sheets into transformer stampings and determine the hysteresis loss by a wattmeter when the magnetic density is at a fixed value. For this purpose the stampings are built up into a closed magnetic circuit of constant cross section. There is some uncertainty in determining the value of the magnetic density, since this depends upon the indications of a voltmeter and is affected by the form of the impressed voltage, because the maximum magnetic density depends upon the area of the voltage wave instead of upon its effective value which is indicated by the voltmeter. If two samples of iron are tested by this means, using the same exciting voltage, the results of the test are of course comparative; but exact results cannot be obtained readily unless the voltage impressed upon the exciting winding is a strict sinusoid. Figure 249 illustrates the arrangement of apparatus for this mode of testing. The method is much used in commercial testing by manufacturers of electrical apparatus who only care to obtain comparative data relating to the iron which they use. It has the advantage of testing the iron under conditions quite similar to those under which it is used in transformer cores and other such apparatus. If care is taken to impress a sinusoidal voltage wave on the coil used for exciting the test sample, the method can be used with reasonable accuracy.

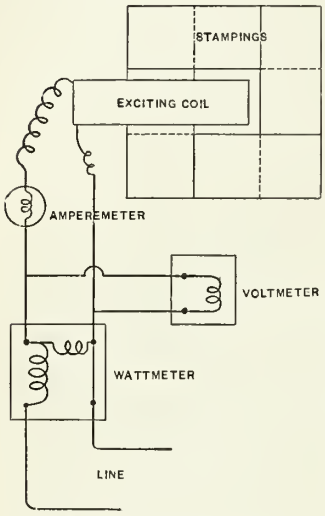


FIG. 249. — Diagram of Arrangements for approximately measuring Hysteresis Losses.

The hysteresis constant is

$$\eta = \frac{P \times 10^7}{VfB^{1.6}},$$

where P is the power shown by the wattmeter after the copper loss in the exciting coil and any loss from eddy currents in the stampings have been deducted, V is the volume of the iron in cubic centimeters, f is the frequency, and B is the maximum magnetic density in lines of force per square centimeter. The magnetic density is obtained from the formula

$$E = \frac{\sqrt{2}\,\pi n B A f}{10^8},*$$

where n is the number of turns of the exciting coil, A is the cross section of the core in square centimeters, and E is the voltage measured by the voltmeter. This is the same formula as that earlier developed in the discussion of alternator voltage. The symbol E in the formula, stands for the induced voltage set up in the exciting coil by the alternating magnetism excited by alternating current with a frequency of f periods per second flowing in the coil. This is substantially equal to the voltmeter reading, if the IR drop in the resistance of the coil is negligible, and it is usual to assign to E the voltage read on the voltmeter when testing according to this method. Caution must be used, however, to avoid appreciable IR drop in the coil because E in the formula, strictly speaking, is equal to the vector difference of the voltmeter reading and the IR drop.

The foregoing method of measurement includes the eddy current losses, and the distribution of the magnetism through the cross section of the test sample may be disturbed by the magnetic effect of the eddy currents. The magnetism set up at any point in the cross section is dependent on the resultant magnetic force at that point, and the eddy-current ampere turns may be in substantial opposition to the ampere turns of the exciting coil. The eddy currents, if of appreciable magnitude, therefore tend to reduce the magnetic density in the interior of the stampings.† Eddy current losses vary as the product of the square of the frequency, the square of the maximum magnetic density, and approximately as the reciprocal of the electrical

* Art. 11. † Art. 113.

resistance of the stampings, so that they may be calculated and subtracted from the wattmeter reading. However, as the laminations used in electrical machinery should be sufficiently thin to reduce the eddy current losses to a small fraction of the total **Iron loss**, which comprises the hysteresis and eddy current losses, the wattmeter reading is often substantially equal to the hysteresis loss alone. If the frequency of the alternating current is very low, the eddy currents are almost always negligible.

Plans have been proposed for separating the hysteresis and eddy current losses by computations based on test data. In one process, wattmeter measurements are made of the losses at two different voltages. The magnetic density is substantially proportional to the impressed voltage, since the induced voltage is equal to the vector difference of impressed voltage and IR drop in the coil, the latter being of small value by the construction of the apparatus; and the other factors of hysteresis and eddy current losses are independent of the voltage. The following expressions therefore connect the two wattmeter readings:

$$P = aE^{1.6} + bE^2,$$
$$P_1 = aE_1^{1.6} + bE_1^2,$$

where the first term in each right-hand member represents the hysteresis loss and the second term the eddy current loss. That E and E_1 vary as the magnetic density is shown by the voltage formula given earlier in the article. Hysteresis loss varies as the 1.6 power of the maximum magnetic density, and hence as $E^{1.6}$; and eddy current loss varies as the square of the maximum magnetic density, and hence as E^2. The symbols a and b are quantities depending upon the characteristics of the magnetic circuit and the frequency, and are assumed to be constant. From these equations

$$a = \frac{PE_1^2 - P_1E^2}{E^{1.6}E_1^2 - E^2E_1^{1.6}}.$$

$$b = \frac{P_1E^{1.6} - PE_1^{1.6}}{E_1^2E^{1.6} - E^2E_1^{1.6}}.$$

As P, P_1 and E, E_1 are known by the instrument readings, the values of a and b may be computed. The hysteresis losses for the two voltages, then, are

$$H = P - bE^2 \text{ and } H_1 = P_1 - bE_1^2;$$

and the eddy current loss for each voltage is

$$F = P - aE^{1.6} \text{ and } F_1 = P_1 - aE_1^{1.6}.$$

Since hysteresis losses are proportional to the 1.6 power of voltage and eddy current losses to the square of the voltage, the computed values of H, H_1, F, and F_1 ought to yield the results:

$$\frac{H}{E^{1.6}} = \frac{H_1}{E_1^{1.6}} \quad \text{and} \quad \frac{F_1}{E^2} = \frac{F_1}{E_1^2},$$

but the fact is that results of the measurements and computations will seldom bear this test. The errors of observation inherent in the measurements are large compared with the quantities computed, and the effects of the errors are magnified by the fact that observed quantities come into the computed results by differences. The consequence is that this process is not reliable and is little used.

If the frequency can be varied without altering the sine form or amplitude of the voltage wave, the iron loss for different frequencies f and f_1 may be written,

$$P = cf^{-.6} + d,$$
$$P_1 = cf_1^{-.6} + d,$$

where the first term in each right-hand member is hysteresis loss and the second is eddy current loss. The hysteresis loss varies proportionally to $f^{-.6}$ because it is proportional to the product of the number of cycles per second and $B^{1.6}$; but B varies inversely as the frequency when the voltage is maintained constant in the arrangement of apparatus under consideration, as can be seen from the formula for voltage given earlier in the Article. The part of the iron loss due to eddy currents should remain constant under the conditions here considered, because the eddy current loss is directly proportional to the product of frequency squared with magnetic density squared, and these are inversely proportional to each other in the arrangement of apparatus under consideration. The constants c and d of the equations depend upon the characteristics of the magnetic circuit and the voltage. Having observed P, P_1 and f, f_1, c and d may be computed and the values of the hysteresis loss and eddy current loss separated from each other by a

process similar to that indicated on the preceding page; but this process is also subject to the causes of inaccuracy accompanying the companion process.

b. Ewing Hysteresis Tester. — In this instrument a permanent horseshoe magnet *A*, Fig. 250, is balanced upon knife-edges and has a pointer attached to it which swings over the scale *B*. The test piece, made up of laminated iron, is rotated between the pole pieces of the magnet. If no losses were present, the tendency for the pointer to swing in one direction would be as great as in the other; but with hysteresis present, the torque upon the magnet is greater when the test piece is leaving the magnet than when approaching, and the average torque exerted on the magnet is proportional to the work done on the test piece per revolution. Since the hysteresis loss, that is, the power absorbed by the effect of hysteresis, is directly proportional to the speed, the torque is independent of the speed provided disturbing influences like eddy current losses or wind friction are not encountered. If, then, the test piece is rotated rapidly enough to give a steady deflection of the pointer on the magnet, but not to set up appreciable eddy currents or cause wind friction, the deflection of the pointer is proportional to the hysteresis loss.

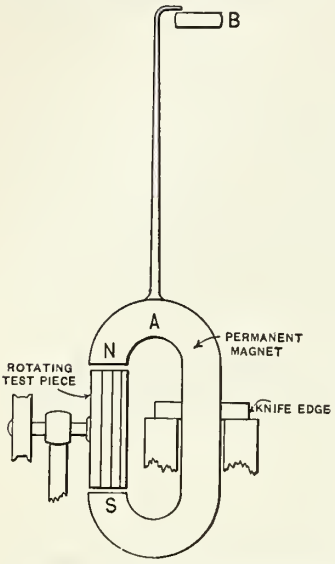


Fig. 250. — Ewing Hysteresis Tester.

Test pieces of known hysteresis constants are furnished with the instrument. The constants of unknown samples of the same length and approximately the same cross section as the standard samples can be determined by comparison with them. As the hysteresis constant is approximately the same for all values of the magnetic density within the range of ordinary commercial practice, the use of a permanent magnet in such an instrument seems justified, but the results obtained by the instrument are not very satisfactory. This is doubtless partially

due to the small size of the samples that may be used for testing.

c. The Holden Tester. — In the Holden hysteresis tester the pole pieces of an electro-magnet A revolve about a test piece B, built up from disc or ring-shaped stampings as shown in Fig. 251. The magnetic flux traversing the test piece is measured by means of a coil which surrounds the core and is cut by the flux as it revolves. This coil is attached through a commutator to a voltmeter. The density of the magnetism traversing the test piece may be determined from the observations by means of the formula

$$B = \frac{10^8 \cdot E \cdot 60}{SVA},^*$$

where E is the voltmeter reading, S the number of test coil conductors on the surface of the test piece, V the revolutions of the field magnet per minute, and A the mean cross section of the test piece in square centimeters.

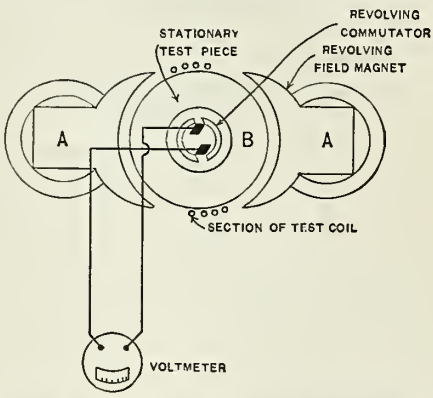


FIG. 251. — Diagram of Holden Hysteresis Tester.

The test ring is so supported that it can twist against a spring, the torsion of which can be measured. The torque acting between the test piece and the field magnet is proportional to the loss due to hysteresis and eddy currents, as in the case of the Ewing tester, and can be determined from the torsion of the spring. This instrument can be used either for comparison of different samples, by observing the deflections caused by each, other things remaining equal, or for the direct determination of hysteresis constants by computing the power absorbed by the hysteresis per cycle and unit of volume at any desired magnetic density. The apparatus has the advantage of permitting frequency and magnetic density to be varied at the will of the observer. The former is varied by altering the speed at which the field magnet is revolved by an outside motive power, and the latter is varied by changing the

* Art. 11.

exciting current fed to the field magnet. The hysteresis and eddy current losses may be separated at constant speed by obtaining the aggregate loss at two magnetic densities, the former varying as $B^{1.6}$ and the latter as B^2; or at constant magnetic density by obtaining the aggregate loss at two speeds, in which case the hysteresis loss varies as V and the eddy current loss as V^2. The separation of the two losses may thus be established by computations similar to those already explained.

d. Step by Step Method. In this method a core A, Fig. 252, is inserted in a magnetizing coil B and a measuring coil C. A ballistic galvanometer D is attached to the coil C. After the residual magnetism of the sample has been destroyed by some process, such as sending an alternating current through the exciting coil and gradually reducing it to zero, direct current is sent through the coil B, and increased by convenient increments until the highest desired excitation is reached. The exciting current is then reduced by the same steps. Upon reaching zero, circuit connections are reversed and the process repeated. Each change of the exciting current causes a change in the magnetism set up in the test piece, which in turn induces voltage in the test coil and causes a throw of the galvanometer needle, which must be read and recorded with the corresponding current value. The magnetic density set up in the test piece by the first increment of current may be computed from the relation $B = \frac{RK\beta}{2An_1}$ where R is the resistance of the galvanometer circuit, K the galvanometer constant, β its throw, A the cross section of the test piece, and n_1 the number of turns on coil C. The added magnetic density for each additional increment of current may be computed from the same relation, using the corresponding galvanometer reading. Plotting the mag-

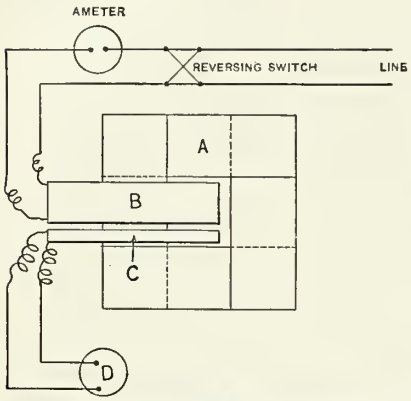


FIG. 252. Step by Step Method of Testing for Hysteresis Loss.

netic densities as ordinates with currents or magnetic force ($H = \frac{4\pi nI}{10\,l}$ where n is the number of turns on the coil B, I is the current shown by the amperemeter, and l is the average length of the magnetic circuit) as abscissas, gives the hysteresis loop, and the hysteresis loss per cycle may be estimated from the area of the loop.

It is desirable to repeat the step by step current cycle a half dozen or more times before taking readings, as otherwise the curve may not be either closed or symmetrical above and below the axis on account of the fact that the first cycle of magnetization starts at zero H and B and is therefore unsymmetrical. As the reversals proceed, the curve gradually assumes the form of one of the curves shown in Fig. 245. This method is quite laborious and liable to inaccuracies from various sources.

111. Computations of Eddy Current Losses. — In all conducting matter placed in a variable magnetic field there are induced, as already frequently referred to, certain **Eddy currents**, or Foucault currents as they are often called. The exact values of such currents cannot, as a rule, be computed, because the magnetic densities at different parts of a conductor and the rates of change of magnetism cannot be fully known, nor can the exact resistances of the paths of flow be fully determined. There are some conditions under which the computation of these eddy current losses is of value, as, for instance, in a transformer core where the character of the metal used is well known and the paths of the magnetic lines of force can be approximately predicted. Suppose the condition of a thin plate of iron in which magnetic lines of force are set up parallel to the sides of the plate and the magnetic density is uniform over the cross section. When this magnetic density passes through an alteration in strength, currents tend to circulate in paths which are parallel with the edges of the cross section and perpendicular to the lines of force. The average length of these paths and their aggregate cross section may be estimated if the dimensions of the plate are known. The voltage set up in each portion of the paths may also be calculated if the maximum magnetic density and its rate of change are known. If we are acquainted with the specific electrical resistance of the plate, we can also determine, with a considerable degree of approxi-

mation, the average resistances in the paths of the eddy currents; and we are therefore enabled to compute with reasonable accuracy the amount of power expended on account of the flow of these currents. We can do the same when the magnetic core is composed of an aggregation of cylindrical wires. In the two following discussions it is assumed that the magnetic material is so finely subdivided that the magnetic flux may be considered constant over the cross section of a lamination. This is permissible in view of the practice in modern manufacture of electrical machinery. With such fine subdivisions of the magnetic circuit, it is ordinarily sufficiently accurate to assume that $B = \mu H$, where H is the impressed field strength and μ is the permeability of the metal, or that the magnetic effect of the eddy currents is negligible.

A. Eddy Current Loss in Cylindrical Conductors. — Eddy current loss varies directly as the square of the frequency, directly as the square of the maximum magnetic density, and, if the resistance of the circuits in which the eddy currents flow is large compared with their reactance, the loss varies nearly inversely with the resistance. These relations are manifest from the fact that

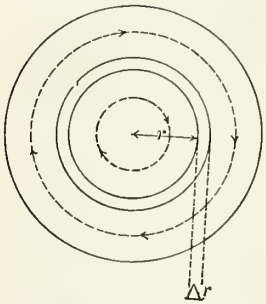


FIG. 253. — Illustration of Eddy Currents in a Cylindrical Conductor.

$$P = I^2R = \frac{E^2R}{Z^2} = \frac{E^2}{R + \frac{X^2}{R}},$$

and E is proportional to the rate of change of the magnetic linkages in the circuits. This is equally true whether the magnetic core is rotated in a fixed magnetic field or is influenced by the magnetism induced by a coil carrying alternating currents. The resistance of the eddy current circuits is dependent upon the specific resistance of the metal concerned and the volume and conformation of the metal within which the eddy currents may be induced.

Assume that the large circle, Fig. 253, represents a cross section taken perpendicular to the axis of a cylindrical wire of magnetic metal and that the magnetism passes along the wire perpendicular to the plane of the paper and of uniform mag-

netic density over the cross section. Now as the magnetism varies, cylindrical currents will be set up which are projected in Fig. 253 as concentric circles in the cross section of the wire. If we consider the path of one of these currents of radius r, the instantaneous voltage acting in the circumference having this radius is

$$e = \pi r^2 \frac{dB}{dt} \times 10^{-8};$$

which, when the magnetism varies as a sine function, becomes

$$e = \pi r^2 \frac{d(B_m \sin \alpha)}{dt} 10^{-8} = 2\pi^2 r^2 f B_m \cos \alpha \times 10^{-8},$$

since $\alpha = 2\pi ft$; and the effective voltage is

$$\Delta E = \sqrt{2}\, \pi^2 r^2 f B_m 10^{-8}.$$

The resistance of the circular paths in a unit length (centimeter) of an elementary cylinder of radius r and thickness dr is

$$R_1 = \frac{2\pi r \rho}{1 \times dr},$$

where ρ is the specific resistance of the metal in ohms per one centimeter of length and one square centimeter of cross section. The instantaneous power (watts) expended in a conductor of this resistance due to a current which is set up by voltage having the value given in the first of the foregoing formulas, considering the resistance of the eddy-current paths to be large compared with their reactance, is approximately

$$p = \frac{e^2}{R_1} = \frac{\pi r^3 (dB)^2 dr\, 10^{-16}}{2\rho (dt)^2}.$$

When the magnetism is sinusoidal, the average power (watts) expended during a period in the filamentary path or cylinder of radius r, thickness dr, and unit length (remembering the assumption that the magnetic effect of the eddy currents is negligible in comparison with the inducing magnetic field), is

$$\Delta P = \frac{(\Delta E)^2}{R_1} = \frac{\pi^3 r^3 (dr) f^2 B_m{}^2 10^{-16}}{\rho};$$

and the joules per period are

$$\Delta W = \Delta P \times T = \frac{\pi^3 r^3 f B_m^2 10^{-16} dr}{\rho}.$$

The next to the last expression gives the power expended and converted into heat by eddy currents in an elementary cylinder of unit length, thickness dr, and radius r; and the total power expended and converted into heat by eddy currents in the mass of the solid cylinder of conducting material of radius r_1 and embraced between planes one centimeter apart is equal to that expression integrated between the limits of 0 and r_1. We therefore find that the loss of power in watts which is caused by eddy currents in the unit length of the wire is

$$P = \int_0^{r_1} \frac{\pi^3 f^2 B_m^2 r^3 dr}{10^{16} \rho} = \frac{\pi^3 f^2 B_m^2 r_1^4}{4 \rho 10^{16}},$$

and the work in joules per cycle converted into heat by eddy currents is

$$W = PT = \frac{\pi^3 f B_m^2 r_1^4}{4 \rho 10^{16}},$$

$T = \frac{1}{f}$ being the length of one period.

The strength of the eddy currents may be found in a similar manner, by integrating

$$I = \sum \frac{\Delta E}{R_1} = \int_0^{r_1} \frac{\pi f B_m r dr}{\sqrt{2} 10^8 \rho} = \frac{\pi f B_m r_1^2}{2\sqrt{2} 10^8 \rho}.$$

To find the watts lost per cubic centimeter of the material at frequency f and the joules lost per cubic centimeter of the material and per cycle of the magnetism, the last two expressions but one may be divided by the area πr_1^2 of the end of the cylinder; and the watts expended per pound of material at frequency f, or the joules per pound of material and per magnetic cycle, may then be obtained by multiplying the results by the number of cubic centimeters of the material required to make a pound in weight.

The formulas show that eddy current losses in a cylindrical wire under the conditions assumed are proportional to the fourth power of the diameter of the wire. If a core of given configuration is to be made up of equal cylindrical wires, it is

manifest that the number of wires composing the core is substantially proportional to the reciprocal of their diameter squared. The total loss in a wire core of given configuration composed of equal wires is therefore proportional to the square of the diameter of the wires.

It is to be noted that the premises used in the analysis include the assumption that the magnetic lines of force are parallel to the walls of the wires, that they are uniformly distributed over the cross section of the wires, that they vary sinusoidally, and that the magnetic effects of the eddy currents may be neglected. These assumptions cannot be accepted as rigidly holding in practice, but any deviation is likely to occur in such a way as to reduce the actual losses below those computed. In particular, the magnetic effects of the eddy currents may not be negligible, but introduce a disturbing factor which reduces the power expended from $\frac{E^2}{R_1}$ to $\frac{\overline{E\cos\theta}^2}{R_1}$. This effect is generally treated as though the eddy currents shielded the interior of the conductor from the full impressed magnetic force, since the magnetic effects of the eddy currents increase with the distance from the surface of the conductor.*

B. Eddy Current Loss in Sheets of Rectangular Cross Section. — A similar integration will indicate the losses in the case of thin sheets of rectangular cross sections in which the lines of force are parallel to the sides and are uniformly distributed over the cross section.

In a sheet of thickness d let a slice be taken a distance x from the middle and of thickness dx as indicated in Fig. 254, which represents the sheet with its edge presented to the paper. This slice, taken in connection with a similar one on the other side of the center line, may be considered equivalent to a complete elementary electric circuit if the disk is thin enough so that the length required to complete the circuit at the ends is negligible.

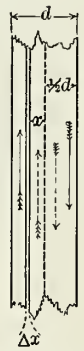


FIG. 254. — Cross Section of a Disk or Sheet of Iron much magnified, illustrating Eddy Current Circuits.

* Art. 113.

The instantaneous number of lines of force embraced by such a circuit is $2\,lxB$, where l is the length of the sheet parallel to the page. The eddy currents flow perpendicularly to the lines of force and parallel to the sides of the sheet, up on one side and down on the other as indicated by the arrows in Fig. 254.

The instantaneous voltage set up by varying magnetism in one centimeter of the elementary slice dx is

$$e = x\frac{dB}{dt} \times 10^{-8}.$$

When the magnetism varies as a sine function, this is

$$e = x\frac{d(B_m \sin \alpha)}{dt} \times 10^{-8} = 2\,\pi f x B_m \cos \alpha \times 10^{-8},$$

and the effective voltage is

$$\Delta E = \frac{\sqrt{2}\,\pi f B_m x}{10^8}.$$

The resistance in ohms of a portion of the circuit one centimeter long, one centimeter deep, and dx thick is $R_1 = \frac{\rho}{dx}$ where ρ is the specific resistance in ohms.

The power expended in the elementary slice which is dx thick, one centimeter long, and one centimeter deep (again assuming the resistance of the eddy current paths to be large compared with their reactance) is

$$\Delta P = \frac{\overline{\Delta E}^2}{R_1} = \frac{2\,\pi^2 f^2 B_m{}^2 x^2 dx}{10^{16} \times \rho};$$

and the power expended in a part of the sheet one centimeter long, one centimeter deep, and the full thickness d is

$$P = \int_{-\frac{1}{2}d}^{+\frac{1}{2}d} \frac{2\,\pi^2 f^2 B_m{}^2 x^2 dx}{10^{16}\rho} = \frac{2\,\pi^2 f^2 B_m{}^2 d^3}{4 \times 3 \times 10^{16} \times \rho}.$$

The joules per cycle converted into heat by the eddy currents are

$$W = \frac{P}{f} = \frac{2\,\pi^2 f B_m{}^2 d^3}{4 \times 3 \times 10^{16} \times \rho}.$$

The strength of the eddy currents may be found in a similar manner by integrating

$$I = \sum \frac{\Delta E}{R_1} = \int_0^{\frac{1}{2}d} \frac{\sqrt{2}\,\pi f B_m x dx}{10^8 \times \rho} = \frac{\sqrt{2}\,\pi f B_m d^2}{8 \times 10^8 \times \rho}.$$

To obtain the power expended per cubic centimeter of metal at frequency f, and the work expended per cubic centimeter of metal and per cycle of magnetism, it is only necessary to divide the expressions for P and W by d. This gives:

$$P_{cm} = \frac{\pi^2 f^2 B_m{}^2 d^2}{6 \times 10^{16} \times \rho} = S(f B_m d)^2,$$

where

$$S = \frac{\pi^2}{6 \times 10^{16} \times \rho};$$

and

$$W_{cm} = \frac{\pi^2 f B_m{}^2 d^2}{6 \times 10^{16} \times \rho}.$$

The value of ρ for soft steel and iron sheets varies from 0.9×10^{-5} to 1.25×10^{-5} with an average near 1×10^{-5} ohms.*

The foregoing computations, like those relating to the eddy current losses in wires, are founded on certain assumed premises, including the assumptions that the magnetic lines of force are parallel with the sides of the plate and uniformly distributed over its cross section, and that the magnetic effects of the eddy currents are negligible in comparison with the impressed magnetic force. Deviations from these assumptions in commercial magnetic cores are likely to cause a reduction of the actual losses below the computed values rather than an increase, and the formulas are therefore safe for use in most commercial instances. The magnetic effects of the eddy currents are often of considerable importance instead of being negligible and are therefore discussed in some detail in Art. 113.

The difference between the coefficients of r_1 and d in the foregoing formulas for the eddy current losses in wires and in

* *Smithsonian Physical Tables*, 3d ed., p. 255; Steinmetz, *Trans. Amer. Inst. Elect. Eng.* Vol. 11, p. 600; *London Electrician*, Vol. 28, p. 631; Loppé' et Bonquet, *Courants Alternatifs Industriels*, p. 273.

rectangular cross sections which are very thin compared with their length shows that it is necessary to subdivide a core more finely when the metal is in plates than when it is in wires if the losses are to be equally small, but the greater mechanical convenience during processes of manufacture and the more satisfactory mechanical stability which may be obtained by means of rectangular plates when placed in actual machinery has led to an almost universal adoption of plates for the magnetic cores of electrical machines. It is to be noted that the eddy current loss in a magnetic core of fixed dimensions is proportional to the square of the thickness of the stampings used to build up the core.

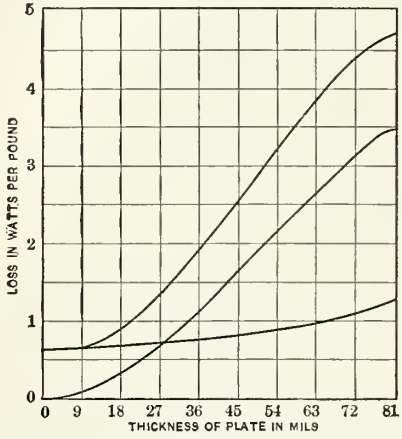


FIG. 255. — Losses due to Eddy Currents and Hysteresis in Sheet-iron Plates, produced by Sinusoidal Alternating Magnetism of 100 Cycles per Second and 4000 Lines of Force, Maximum Density.

Figure 255, following a curve calculated by Ewing, shows the relative hysteresis loss and eddy current loss in laminæ of different thicknesses when the maximum magnetic density is 4000 lines of force per square centimeter and the frequency 100 periods per second. The curve which starts at zero represents the eddy current loss, the flatter curve represents the hysteresis loss, and the upper curve is the sum of the other two and therefore represents the total iron loss. The curves take into account the magnetic effects of the eddy currents, which become important when the sheets exceed a thickness of one half a millimeter. The hysteresis curve rises slightly with the thickness of the plates on account of crowding of the lines of force by the magnetic screening effect of the eddy currents. The figure makes it plain that eddy currents may be serious sources of loss in the magnetic cores of electrical machinery, and that it is desirable that the iron used in such cores should be of high electrical resistance as well as of low hysteresis constant.

112. Cyclic Curves of Eddy Currents. — If a periodic magnetization acts upon a laminated iron core, the eddy current loss produced thereby may be represented by the area of a loop such as is shown in Fig. 256. In this curve the ordinates represent instantaneous values of the magnetic density B or of the total alternating flux ϕ in lines of force inclosed by eddy current circuits, and the abscissas represent the corresponding values of induced eddy currents i in amperes.

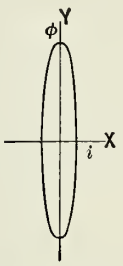


FIG. 256. — Loop having an Area Proportional to Eddy Current Loss occasioned by Sinusoidal, Varying Magnetization.

If $e = \frac{d\phi}{10^8 dt}$ is the voltage causing i to flow, it is easy to show that the area of the cyclic curve of Fig. 256 is proportional to the power expended and transformed into heat by the eddy currents: The instantaneous value of this power is

$$p = ie = i\frac{d\phi}{10^8 dt},$$

and the work expended during a time dt is

$$dW = iedt = i\frac{d\phi}{10^8}.$$

The differential of the area of the cyclic loop of Fig. 256 is

$$dA = id\phi.$$

Further, the work expended and transformed into heat by the eddy currents during one cycle of the magnetism is

$$W = \frac{1}{10^8}\int_{\phi}^{\phi} id\phi,$$

but this is equal to $\frac{A}{10^8}$.

If the abscissas are laid off on the scale of one ampere per centimeter and the ordinates on the scale of 10^8 lines of force per centimeter, the area of the curve in square centimeters is equal to the joules of eddy current loss per cycle. If other scales are used in plotting the curve, the area must be multiplied by a constant in order to arrive at the joules lost.

The cyclic curve in Fig. 256 is of the form that results when the magnetism alternates as a sine function. If the magnetic wave is irregular, the curve may lose the symmetrical regularity notable in the curve of Fig. 256, since the abscissas of the curve are proportional to the rate of change of the magnetism. Thus if the curve of magnetism has harmonics which cause it to have two maximum points per half cycle, there are two points of rate of change of magnetism and current i equal to zero, which result in subsidiary loops at the top and bottom of the cyclic curve. Or, if the maximum value of the curve of magnetism is pushed over towards $\alpha = 0°$ or $\alpha = 180°$, the time rate of change of the magnetism is decreased in one part of the period and increased in another part, so that the cyclic eddy current curve is skewed with reference to its axis.

An exciting coil which provides the impressed magnetic force in a magnetic circuit must carry an additional energy component of current on account of the eddy current loss which is approximately

$$I_1 = \frac{P}{E_1},$$

where P is the measured or calculated value of the eddy current loss in watts, and E_1 is the induced counter-voltage in the coil (substantially equal and opposite to the impressed line voltage in a highly inductive circuit as in the case of the unloaded primary coil of a transformer). The combined eddy current circuit is similar to the secondary coil of a transformer with one turn in the coil, so that approximately

$$I_2 = I_1 n,*$$

where I_2 is the effective value of the current in the combined eddy current circuit and n is the number of turns of wire in the exciting coil on the magnetic core. We then have approximately

$$R_2 = \frac{E_2}{I_2} = \frac{E_1}{n^2 I_1}.$$

In order that the cyclic curve of eddy current loss may be plotted, it is necessary that the instantaneous values of magnetism and their corresponding rates of change shall be known.

* Art. 23.

The rates of change are directly proportional to the corresponding instantaneous values of induced voltage. The actual plotting of the curve is not usually practiced, but the relationships pointed out by means of theoretical considerations are valuable.

If the abscissas of this last curve converted into terms of exciting current are added to the corresponding abscissas of the cyclic curve of hysteresis for any particular periodic wave of magnetism, a cyclic curve is obtained like that in Fig. 257, where $BTB'T'$ is the hysteresis curve and $BFB'F'$ is the combined curve. The result is a loop having an area proportional to the total iron loss in the circuit and it may suitably be denominated a **Cyclic curve of iron loss.**

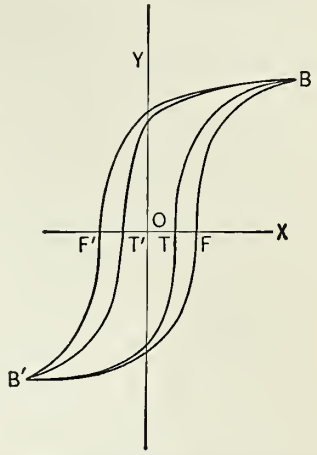


FIG. 257. — Cyclic Curve of Iron Loss.

113. Magnetic Screening due to Eddy Currents. — The law of the magnetic circuit asserts that the magnetism in any part of the circuit is equal to the net magnetomotive force in that part of the circuit divided by the magnetic reluctance thereof. The net magnetomotive force is the vector sum of all magnetic forces acting on the circuit. When a varying magnetic flux is set up in material which is an electrical conductor, the filaments of varying magnetism become surrounded by eddy currents, and the magnetic effects of the eddy currents may be quite large and even comparable to the impressed magnetic force when the material is in large masses and of high electrical conductivity. It is experimentally well known that a copper or other highly conducting plate placed in an alternating magnetic field has eddy currents set up in it to so high a degree that their magnetic effect may practically neutralize at the back of the plate the magnetic effect of the impressed magnetic field, and the space beyond the plate is practically screened from the original magnetic field. A non-magnetic copper plate has no screening effect in a magnetic field that is constant, and its screening action in an alternating magnetic field is caused by the magnetic force of the induced

eddy currents. This is in accordance with the theorem that the magnetic force of induced currents must tend to reduce the impressed magnetic flux. The induced voltage causing the flow of eddy currents is in lagging quadrature with respect to the inducing magnetic field, assuming sinusoidal functions; and if the path of the eddy currents is of very low resistance, its impedance is mostly composed of the reactance component, and the eddy currents are lagging nearly 90° behind the induced voltage. Under these circumstances, the eddy currents are nearly 180° in phase from the impressed magnetic force, and their magnetic force is therefore nearly in opposition to the impressed force. On the other hand, when the resistance of the eddy current circuit is high, as has been assumed in the computations of Art. 111, the currents are small in volume and lag little behind the induced voltage, in which case their magnetic effect is small and nearly in quadrature with the impressed magnetic force. It then has little effect on the magnetism set up.

Certain important results of magnetic screening arise in instances where the magnetic flux is constrained within a conductor in a particular manner, as in the case of the magnetism in the laminated iron core of a transformer or a dynamo armature which threads through the stampings parallel to their flat sides. A simple example is a homogeneous cylindrical wire of magnetic metal which may be considered as a part of a magnetic core of some machine, and in which the magnetism is set up parallel to the cylindrical sides of the wire by an impressed magnetic force that is uniform over the cross section of the wire. In case no other influences came into play, the magnetism set up in this wire would be uniform over its cross section, and this is indeed the case if the magnetizing force is constant; but if the magnetizing force is alternating, as when the exciting current is alternating, circular eddy currents are set up in the conductor which themselves produce magnetic force and which may be in sufficient volume to strongly influence the net magnetic field acting on the iron of the wire.

Figure 258 shows the cross section of such a wire in which the lines of force are supposed to be perpendicular to the plane of the cross section. The magnetism being induced by an alternating exciting current is itself alternating of the same frequency,

and induces eddy currents. These flow in circles concentric with the surface of the wire as illustrated by the dotted line in Fig. 258, and their volume and phase relation to the impressed magnetic force depend on the impedance of the eddy current circuits. The magnetic force which the eddy currents produce within the mass of the metal is parallel to the axis of the wire, and its magnitude varies along a radius from a maximum equal to $\sqrt{2}\,\frac{4\pi I}{10}$ gilberts at the center of the wire (in which I is the aggregate volume of eddy currents in effective amperes per centimeter of length of the wire) to a minimum of zero at the outer surface of the wire. The net vector magnetic force at any point in the cross section is equal to the difference between the vector impressed magnetic force and the vector counter magnetic force caused by eddy currents. The latter is equal to zero at the outer surface of the wire, and the net magnetic force is therefore there equal to the impressed magnetic force; but at the center of the wire the counter magnetic force may have a value comparable with the impressed force and a phase nearly in opposition to it, and the net magnetic force may there nearly disappear. The value of the eddy currents depends upon the density of the inducing magnetism, its frequency of alternation, and inversely upon the resistance of the eddy current circuits; and the phase of the eddy currents with respect to the induced voltage depends on the frequency of the alternations of the magnetism and the resistance of the eddy current circuit. This resistance depends upon the specific resistance and the dimensions of the conducting body. It therefore follows that the magnetic force of the eddy currents may be relatively large and its phase nearly in opposition to the impressed magnetic force at the center of a thick metal wire of low specific resistance, and the net magnetic force in that case may be very small along the axis of that wire although the impressed magnetic force is large. But in the case of a thin metal wire of high specific resistance the eddy currents must be more nearly in quadrature with the magnetic flux and small in volume, and the net magnetic force along the

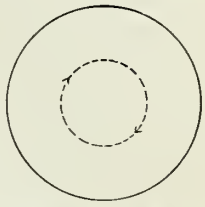


Fig. 258. — Eddy Current in Cross Section of Wire.

axis of that wire may not differ substantially from that parallel to the axis near the surface of the wire. Since the resistance of any circular path of the eddy currents of elementary width and concentric with the walls of the wire is proportional to the circumference of the circular path and therefore to the radius of the path, and the self-inductance is proportional to the area of the circle and therefore to its radius squared, it is obvious that the lag angles of the eddy currents will differ along any radius of the wire, and this will additionally cause the net magnetic force to differ in phase from the center to the circumference of the wire.

In consequence of these reactions, alternating magnetism tends to forsake the central parts of the stampings of the cores of electrical apparatus unless the stampings are thin enough or of high enough specific resistance, or both, to make the resistance of the eddy current circuits relatively large. This results in a smaller aggregate magnetic flux being set up per unit of impressed magnetic force, and gives the appearance of an increase in the magnetic reluctance of the core, unless it is rather finely laminated.

The extent of this effect in the laminated iron cores of electrical machines subjected to alternating magnetism was treated by Ewing in 1891, following a brief mathematical presentation by J. J. Thomson.*

The formulas developed by J. J. Thomson, showing the extent of magnetic screening in stampings, take into account the effect on the eddy currents of the modified distribution of the magnetism and are in vector form, and numerical computations are complicated and laborious. Professor Ewing has computed a table of the ratio of net magnetic force (H) at various depths in a plate to the magnetic force (H_0) at the surface, for iron plates of various thicknesses when the frequency of magnetic alternations is 100 cycles per second. The table is

* *London Electrician*, Vol. 28, pp. 599 and 631. See also Russell's *Alternating Currents*, Vol. I, p. 359.

TABLE

Thickness of Plate in Millimeters	Ratio of H at Middle to H_0 (at Surface)	Ratio of H_1 to H_0	Ratio of H_2 to H_0
2.	0.120	0.342	0.250
1.5	0.245	0.42	0.341
1.	0.520	0.629	0.564
0.75	0.739	0.82	0.793
0.5	0.925	0.940	0.932
0.25	0.995	0.995	0.996

here reproduced. The plotted results for three thicknesses are exhibited in Fig. 259. In this figure the ordinates represent the ratio of H to H_0, and the abscissas, measured from zero either way, represent proportional distances from the center through the thickness of the plate. It will be seen that the magnetic screening computed by the formula for a plate as thin as one half millimeter (.0197 of an inch) amounts to approximately 7.5 per cent at the center, which is not of great commercial importance in most electrical machinery. But when the thickness of the plate is 2 millimeters (.079 of an inch) the computed magnetic screening at the center of the plate is 88 per cent and the net magnetic force at the center as computed is only 12 per cent of the impressed magnetic force. The figures are made on the basis of iron with an assumed uniform magnetic permeability of 2000 units, a specific electrical resistance of 10,000 C. G. S. units ($= 10^{-5}$ ohms) and a sinusoidal magnetizing force having a frequency of 100 periods per second.

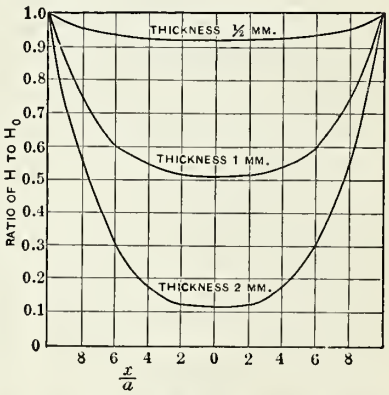


Fig. 259. — Distribution of Net Magnetic Force across Iron Plates of Various Thicknesses, when the Frequency of Magnetic Alternations is 100.

The symbol H_0 in Ewing's table and charts stands for the value of the impressed magnetic force at the surface of the plate, H_1 stands for the average value of the magnetic force

over the cross section of the plate at the instant of maximum flux, and H_2 stands for the value of H which would produce, with eddy currents absent, the maximum total magnetic flux now reached in each period. Figure 260 is a chart taken from Ewing's article, which gives the ratios of H_1 and H_2 to H_0 for various thicknesses of iron plates.

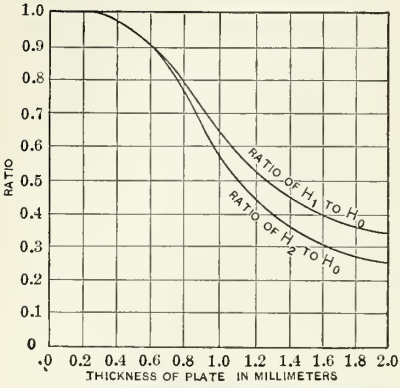


FIG. 260. — Chart showing Ratios of H_1 and H_2 to H_0 in Iron Plates of Various Thicknesses.

The formulas show that the effect of magnetic screening is to reduce the useful thickness of a plate to superficial external layers which have a thickness dependent upon $\sqrt{\frac{\rho}{\mu f}}$, in which ρ is the specific electrical resistance and μ the magnetic permeability of the material composing the plate. The formulas also show that, when the thickness of the plate is large the eddy current loss is greatly modified by the magnetic screening, and is proportional to $H_0^2\sqrt{\mu\rho f}$ which is independent of the thickness of the plate. When the thickness is small, as for iron under 2 millimeters, the heating in each plate is proportional to $\frac{H_0^2\mu^2 f^2 a^3}{\rho}$, where a is the thickness of the plate in millimeters. Since a laminated core is made up of many plates of thickness a, the total heating of a core of given size is proportional to $\frac{H_0^2\mu^2 f^2 a^2}{\rho}$. The last deduction agrees with the deductions reached in Art. 111.

The formulas do not take into account the variation of permeability with the magnetic density in iron, and the actual effect of the magnetic screening in cores of ordinary electrical apparatus is much less than the formulas indicate, because the permeability goes up with a reduction of the magnetic force when the magnetic density is as high as the usual densities of commercial practice, and the smaller net magnetic force at the center of a stamping may not be accompanied by a proportional

reduction of the magnetism set up. As a consequence the actual effects of magnetic screening are, doubtless, ordinarily of little or even negligible effect in the cores of the usual commercial electrical machinery, which, when subjected to alternating magnetism, are commonly built up of stampings not more than 20 mils thick. At rather low magnetic densities, on the other hand, the permeability decreases when the magnetic force is reduced, and the effect of magnetic screening may on this account be actually greater than the formulas indicate.

Figure 259 and the table are based on magnetic alternations of 100 periods per second. The corresponding data for any other frequency may be determined by the relation that the effect of magnetic screening changes in approximately direct proportion to the first power of the frequency.

114. Method for finding the Form of Exciting Current required to produce a Given Alternating Flux when the Curve of Magnetization is Known. — In order to indicate the form of the **Exciting Current** required to set up a given core magnetization, suppose first an electric circuit surrounding an iron core possessing a

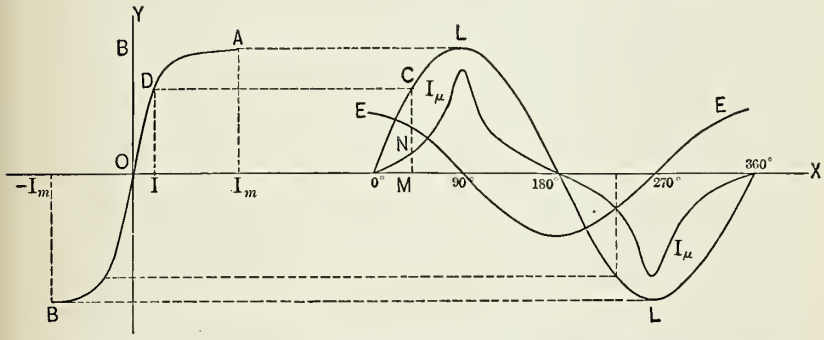


Fig. 261. — Determination of Exciting Current for a Magnetic Circuit without Hysteresis.

cyclic curve of magnetization like AOB, Fig. 261, which presupposes no appreciable hysteresis or eddy current loss. The abscissas of this curve may be scaled as amperes of exciting current, and the ordinates as magnetic density. Also suppose that the form of the wave of magnetism in the core is to be sinusoidal. The problem is to find the form of current wave that must flow under these circumstances. The curve LL in

Fig. 261 represents one period of the sinusoidal wave of magnetism in the core, and the curve AOB represents the instantaneous relations between the magnetism and the exciting current which sets it up. The ordinates of the two curves are drawn for convenience to the same scale.

To find the current required to set up the magnetism at any instant, such as corresponds to the point C on the magnetism wave, draw a horizontal line from C to D and a vertical line from D to I. The length OI evidently represents the current in the exciting coil which is required to set up the magnetism $DI = CM$. Next, from the point M, directly under C, lay off a vertical distance MN, equal to the current OI. The point N is one point on the curve of the exciting current necessary to set up the curve of magnetism LL. If a large number of points on the current curve corresponding to successive points on the magnetism wave are obtained by the same method, the resultant curve drawn through the points so determined is $I_\mu I_\mu$. The peaked form of the exciting current here indicated is notable. The peak is manifestly caused by the increasing ratio between the current and magnetism as the magnetic density becomes greater and the permeability of the core is less. If the curve of magnetization AOB were a straight line, the ratio between exciting current and magnetism would be constant, and the form of exciting current wave would then obviously be the same as the form of the magnetism wave, the scale only being changed. It is the saturation of the core which is evidenced by the crook in the curve of magnetization AOB that causes the peak in the wave of exciting current, and the degree of the peakedness depends on the degree of saturation to which the core is forced by the magnetism set up. The maximum exciting current plainly must come at the same instant as the maximum magnetism, and, since hysteresis is assumed to be negligible, the zero of current comes at the same instant as zero magnetism.

In practical cases where iron forms part of the magnetic surroundings of the electric circuit, hysteresis is not negligible, and the magnetization curve, therefore, takes the general form $AHBJA$, Fig. 262; and if the sinusoidal alternating magnetism wave LL in the core is laid out on the plot in the same manner as in Fig. 261, the exciting current $I_\mu I_\mu$ is obtained by the same

process as before; but the increasing limb of the curve of magnetization now differs from the decreasing limb, and the current wave is not symmetrical about a vertical line drawn through its maximum point. A scrutiny of the hysteresis cycle shows that the current is not zero when the magnetism is zero, but has a magnitude $ST = OJ$ when the magnetism is passing through zero in the positive direction and the corresponding value $S'T' = OH$ when the magnetism is passing through zero in the reverse direction. The current comes to zero while the magnetism has a considerable value in the decreasing quadrants between maximum and zero points. For the hysteresis cycle shown in Fig. 262, the

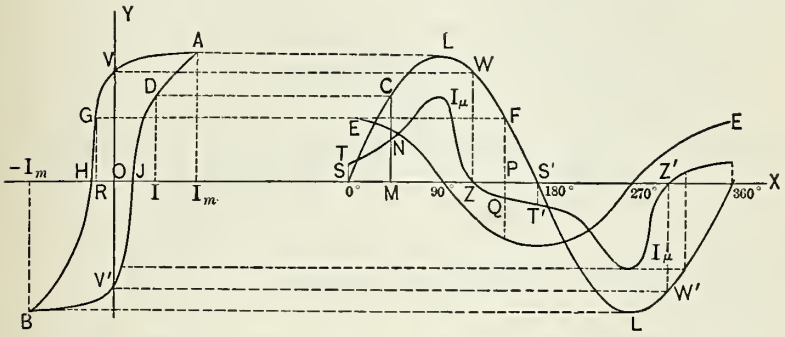


FIG. 262. — Exciting Current in a Circuit containing Hysteresis.

falling positive current comes to zero while the magnetism still has the value $ZW = OV$, and the falling negative current comes to zero while the magnetism has the value $Z'W' = OV'$. Since the point A on the hysteresis cycle represents simultaneous maxima of magnetism and current and the same is true of the point B, the maximum values of the current occur at the same instants as the maximum values of magnetism. The current wave is therefore deformed from the symmetry shown in the current curve of Fig. 261. It has its maximum points at the same instants as the cyclic curve of magnetism, as in the other instance, but the zero values of current come earlier than the zero values of magnetism, and the phase of the exciting current is advanced with respect to the wave of magnetism. The equivalent angle of this advance was called by Steinmetz the *hysteretic angle of advance*. The fact is, that the exciting current now consists of two components: an exciting component similar to the current

of Fig. 261 and in phase with the magnetism; and an energy component required to provide the power absorbed by the hysteresis loss in the core and the I^2R loss in the exciting coil. The latter component is in phase with the impressed voltage, which is in leading quadrature with the magnetism and is proportional to the rate of change of the magnetism at each instant in case the I^2R loss in the exciting coil is negligible; but if the I^2R loss is not negligible the phase of the impressed voltage is slightly retarded.

In like manner, the exciting current required to set up sinusoidal magnetism in a core when the effect due to eddy cur-

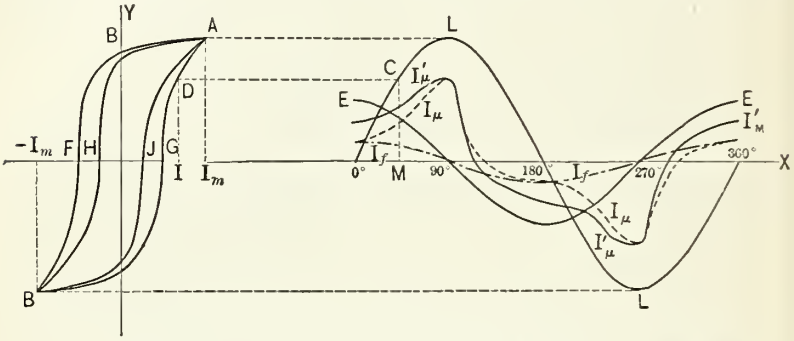


Fig. 263. Exciting Current in a Circuit containing Hysteresis and Eddy Currents.

rents is included may be determined by using the cyclic curve of the iron loss.* The construction is shown in Fig. 263, where $AFBGA$ is a cyclic curve including hysteresis and eddy currents. In this case the current is still more advanced. The component of the advanced current in phase with the voltage furnishes the power which is absorbed in the losses. Where the wave of magnetism is sinusoidal, as in Fig. 263, the eddy currents are also sinusoidal, hence the component they add to the exciting current is sinusoidal and 90° in advance of the magnetism. This element of the exciting current can be laid out to illustrate its position. In Fig. 263 the dotted curve $I_\mu I_\mu$ corresponds to the hysteresis cycle $AHBJA$, and the curve $I_f I_f$ is the portion of the exciting current required to overcome the magnetic effect of the eddy currents. The ordinates of $I'_\mu I'_\mu$ are the algebraic sums of the ordinates of these two. It will further

* Art. 112.

be shown in the chapter dealing with the transformer that the eddy currents in the core act like currents in a secondary winding of one turn.

115. Relationship between Form of Magnetism Wave and Form of Induced Voltage. — In the foregoing discussion, a sinusoidal wave of magnetism has been assumed. This may be readily produced by impressing a sinusoidal voltage between the terminals of the exciting coil even when the core is of iron, provided the resistance of the winding of the coil is small enough so that the IR drop is substantially negligible. The impressed voltage, counter voltage induced by the magnetism in the core, and the IR drop make a closed vector triangle; and if the IR drop is substantially negligible, the counter voltage must be substantially equal and opposite to the impressed voltage. It follows from the relations of the induced counter voltage and the wave of magnetism set up in the core that the latter is sinusoidal when the former is. Under the circumstances, the exciting current which is caused by the sinusoidal impressed voltage to flow through the coil inevitably takes on a form and volume which produces sinusoidal magnetism of just sufficient amount to induce a counter voltage equal to the impressed voltage.

Since the instantaneous voltage induced in a generator armature or transformer coil or any reactive coil winding is at each instant proportional to the time rate of change of the magnetism threaded through the windings (*i.e.* is proportional to $\frac{d\phi_n}{dt}$ where ϕ_n represents the magnetic linkages between the conductors and the lines of force), it follows that the magnetic linkages at each instant are proportional to $\int e dt$. From these relations it may be seen that the form of the voltage curve may be derived from the curve representing the change of magnetic linkages during a period; and *vice versa*, the form of the magnetism wave may be derived if the form of the induced voltage curve is known. For instance, if

$$\phi_n = A_1 \sin(\alpha + \beta_1) + A_2 \sin(2\alpha + \beta_2) + A_3 \sin(3\alpha + \beta_3) + \cdots A_n \sin(n\alpha + \beta_n),$$

and $e = -\frac{d\phi_n}{10^8\, dt}$,

then, since $d\alpha = 2\pi f dt$,

$$e = -\frac{2\pi f}{10^8}\left[A_1 \cos(\alpha+\beta_1) + 2A_2 \cos(2\alpha+\beta_2) + 3A_3 \cos(3\alpha+\beta_3) + \cdots + nA_n \cos(n\alpha+\beta_n)\right].$$

In other words the ratio $\frac{2\pi f}{10^8}$ is introduced into the ordinates of the curve of voltage, all the harmonics are moved one quarter period, and the curve of voltage is therefore in quadrature with the curve of the magnetic linkages, and the ordinates of each harmonic in the voltage curve are affected by a coefficient equal to the ratio of the frequency of the particular harmonic to the fundamental frequency. Every harmonic in the wave of magnetic linkages enters into the wave of induced voltage, and the ordinates of the harmonics are exaggerated in the voltage wave in comparison with the ordinates of the harmonics of the wave of magnetism, in an order proportional to the ratio of their several frequencies compared with the fundamental frequency. It is plain from this that induced voltage of sinusoidal form cannot be derived from magnetic linkages varying according to any other function. Any deviation of the magnetic wave from the sine form produces a greater deviation in the induced voltage.

In the case of a transformer or similar coil under conditions of negligible magnetic leakage, the number of magnetic linkages at each instant is equal to the number of turns in the coil multiplied by the number of lines of force in the aggregate magnetic flux at that instant; but, in the case of alternator armatures or other windings which are distributed or are affected by appreciable magnetic leakage, the number of magnetic linkages at any instant is not equal to the product of turns in the winding by instantaneous flux, but the instantaneous summations of flux embraced by the individual turns must be used as the ordinates of the magnetism wave when plotting that curve.

116. Alternating Flux caused by a Current of Predetermined Form. —The wave of current in the main wire of a circuit is the geometrical resultant of the current in all the branches of the circuit. An inductive coil placed in series with the line will have a current flow through it which is dependent for its form upon the characteristics of all parts of the circuit. Sup-

pose for convenience that the voltage impressed between the terminals of a circuit which contains an impedance coil, among other things, is of such a form that a sinusoidal current is caused to flow through the circuit. This fixes the form of the current in the winding of the impedance coil. The form of the magnetism wave set up in the core of the impedance coil by the current in its winding may be obtained by reversing the procedure set forth in Art. 114, if eddy currents are negligible. The construction is shown in Fig. 264 for a core containing hysteresis but neglecting eddy currents. Here curve *II* represents a sinusoidal current passing through the impedance coil,

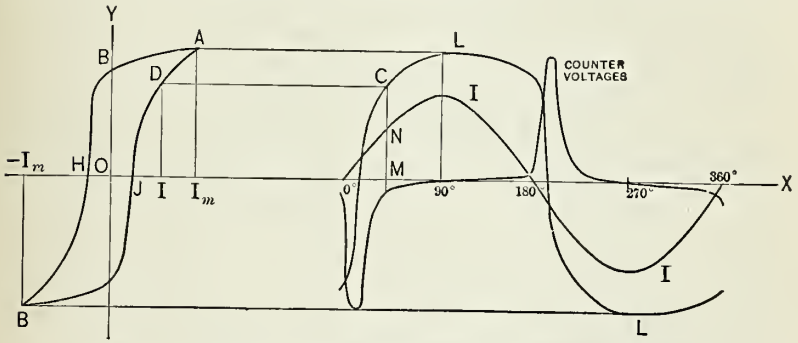


FIG. 264. — Wave of Magnetic Flux caused by Sinusoidal Current.

and *AHBJA* is the cyclic hysteresis curve. Taking any instantaneous value of the current, such as *MN*, laying it off on the axis of abscissas from *O*, as *OI*, the ordinate of the hysteresis cycle erected from *I* to the hysteresis cycle at *D* shows the magnetic density corresponding to current *MN*. The point *C* at the intersection of the horizontal line drawn from *D* with the line *MN* extended, is one point on the wave of magnetism. The curve representing the wave of magnetism may be determined for an entire cycle by continuing this process. The curve *LL* in Fig. 264 shows the wave of magnetism set up by the current *II* in the core with cyclic curve of magnetization *AHBJA*, assuming the latter to be plotted in magnetic densities for ordinates and amperes flowing in the exciting coil for abscissas.

The induced counter voltage in the coil is a very irregular curve under the circumstances here considered. It is propor-

tional at each instant to the time rate of change of the magnetism. Since the maximum of magnetism comes at the same point as the maximum of exciting current, the induced counter voltage passes through zero at the instant the exciting current passes through its maximum. It will be observed that the eddy currents in the core depend upon the induced voltage in circuits surrounding the magnetism, and the eddy current cycle cannot be assumed to be the same as one caused by sinusoidal eddy currents.

The considerations of this article are especially applicable to the study of current transformers used with amperemeters and wattmeters.

117. Effect of Iron Losses on the Apparent Resistance and Reactance of a Circuit. — The foregoing discussion of iron losses leads to a generalization of great importance.

The true electrical **Resistance** of a conductor is a quantity which depends solely on the dimensions, temperature, and character of the conductor. It is the electrical resistance which is measured by means of a Wheatstone bridge, using continuous currents in the process. It is the same whether an alternating current or a direct current flows through the circuit, though it may be masked by various effects of alternating currents.

In a perfectly insulated circuit carrying continuous currents, the causes of losses of energy lie in the passage of the currents through the resistances of the conductors composing the circuit and the currents through devices which absorb energy by presenting a counter voltage. In the case of alternating current flow, additional sources of loss of energy become evident. Energy may be absorbed by the effects of hysteresis and eddy currents. The hysteresis may be either magnetic hysteresis or dielectric hysteresis. An effect also arises from the self-induction of the conductor, which causes a concentration of the current near the surface of the conductor, which is treated in a later chapter under the name of skin effect.

Leakage effects due to imperfect insulation, and the effects of convection accompanying brush discharges and the development of a luminous corona between conductors of the circuit at very high potentials, occur in either continuous current or alternating current circuits. These effects also result in losses of energy.

The angle of lag in an alternating current circuit is obtained

from the expression $\theta = \cos^{-1}\left(\frac{P}{EI}\right)$, where P is the power measured by wattmeter and EI is the volt-amperes obtained by multiplying current by voltage. This angle θ is the angle subtended between Z and R in the right-angled triangle defining the relations of Z, R, and X, namely:

$$\text{operator } \bar{Z} = R + jX, \text{ and}$$

$$\text{tensor } \quad Z = \sqrt{R^2 + X^2}.$$

From this triangle also follows the right-angled triangle

$$I\bar{Z} = \bar{E} = IR + jIX = E_r + jE_x.$$

In this instance, the angle subtended between $\bar{E}$ and IR is the angle θ. The value of X in these triangles is determined by the frequency of the current and the characteristics of the circuit which fix its inductance and capacity, and its value is not affected by the power transferred through the circuit. The question at issue is: Does the value of R (in ohms) of the horizontal sides of these triangles correspond with the circuit resistance as measured by a Wheatstone bridge? A consideration of the relations of the sides of the triangles results in a negative answer. Since the horizontal side of the impedance triangle must be equal to $Z \cos \theta = Z\frac{P}{EI}$ and $E = ZI$, it follows that $Z \cos \theta = \frac{P}{I^2}$, which will always differ from $R = \frac{I^2R}{I^2}$ if any energy is expended in the circuit besides the I^2R loss. This points the way to the generalization of the term *resistance* in which it is referred to as either positive or negative in connection with the semicircle diagrams of Art. 70.

Since power may be either positive or negative, that is, it may either be absorbed by and expended in a circuit or be generated in and delivered by the circuit, it is obvious that $\frac{P}{I^2}$ likewise may be positive or negative, since its sign depends on P. In general, then, **Equivalent resistance** of a circuit may be defined as the ratio of power expended in or by a circuit to the square of the current flow. When an alternating current flows through a circuit, the equivalent resistance ordinarily *differs*

from the electrical resistance of the circuit as measured by a Wheatstone bridge by an amount equal to $\frac{P - I^2R}{I^2}$ where P is the wattmeter reading and R is the electrical resistance of the circuit.

When the word *resistance* is used in this text in relation to alternating current circuits, it may be taken as meaning the equivalent resistance unless it is qualified by the word *true* or the word *electrical*. The equivalent resistance is the horizontal component of the impedance triangle.

In any circuit carrying a steady current, the equivalent resistance and the true resistance are always equal unless the effects of some voltage-generating device are included within the circuit.

As a comparison of equivalent resistance with the true resistance, their values for the primary coil of a transformer may be taken when the secondary circuit of the transformer is open and therefore does not absorb any power. In the case of a certain transformer, the voltage was 1100 volts, the exciting current .2 amperes, and the wattmeter reading which gives the power expended in hysteresis and eddy currents in the core and I^2R in the winding 150 watts. From these figures, the equivalent resistance of the winding is shown to be 3750 ohms. The impedance is 5500 ohms. The electrical resistance of the conductor composing the winding was only one and a half ohms. In another instance of similar kind the voltage was 1100, the exciting current .043 amperes, and the wattmeter reading 37 watts. This gives an equivalent resistance of 20,000 ohms and an impedance of 25,600 ohms. The electrical resistance was 7 ohms. In another instance the voltage was 60 volts, the current .86 amperes, and the power 43.3 watts. This gives equivalent resistance of 58 ohms and impedance of 70 ohms. The electrical resistance of the conductor was .08 ohms.

It is obvious that the power consumption in the iron cores of these transformers enlarges the power factor by advancing the current to a phase nearer the voltage. This is partially caused by hysteresis and partially by eddy currents. The part, as stated before, caused by hysteresis was called the "hysteretic angle of advance" by Professor Steinmetz. This is an angle of advance or negative angle of lag, and is not the whole of the effect produced by the iron core, though it is often the greater part of it.

CHAPTER X

MUTUAL INDUCTION, TRANSFORMERS

118. Mutual Induction and Elementary Transformer Diagrams. — The remarkable development in the use of alternating currents for transmitting and distributing electric power is mainly due to the facility and economy with which they may be transformed from one voltage to another. The induction coils that are used for this purpose are called transformers, as already explained, and their action is due to the inductive effect which a varying current in one circuit exerts upon an adjacent circuit as well as upon the circuit itself.* Since this effect is a mutually interacting one, it is called **Mutual induction.** Evidently, if the magnetism due to a flow of current in one coil, which may be called the primary coil, see Fig. 46, passes through the turns of another coil, which may be called the secondary coil, a variation of the current in the primary coil sets up a voltage in the secondary coil. This voltage, when a sinusoidal alternating current passes through the first coil, is 90° behind the phase of the current in the primary coil, when core losses in the magnetic circuit are negligible. This is also true of self-inductive voltage, and for the same reasons.† In case the external electric circuit of the secondary coil is open and the magnetism in the magnetic circuit is sinusoidal and all passes through both coils, a phase diagram similar to Fig. 114 may be constructed to represent the phase relations of the primary current, magnetic flux, impressed voltage, and primary and secondary induced voltages. Such a diagram is exhibited in Fig. 265, in which OE represents the impressed voltage E_1; OI the current in the primary coil I_1; OF, the self-induced voltage in the primary coil E_{1m}', and OK the induced voltage of the secondary coil E_2. In this case the ratio of OF to OK is evidently equal to the ratio of the numbers of turns in the two coils, since it is

* Art. 23. † Art. 37.

assumed that all of the magnetic flux links both coils, and there is no secondary resistance drop, since no current flows. OC represents the IR drop in the primary coil, and the primary ampere-turns $n_1 I_1$ are proportional to OI. The magnetic flux is assumed to be in phase with the current. Had iron losses not been neglected the angle θ_1 would have been less, as the active component of OI would then have been required to furnish these as well as the $I_1^2 R_1$ losses. Also the angle between OI and OF would have exceeded 90° by the hysteresis angle of lead, but the magnetism itself would still remain in quadrature relation with E'_{1m}. In this and the discussion immediately following the subscript m is used to denote induced voltages caused by variable magnetic fluxes which pass through both primary and secondary coils. When the letter E is used to represent primary voltages it is given a prime when representing induced voltages, and is without the prime when representing drops of impressed voltages.

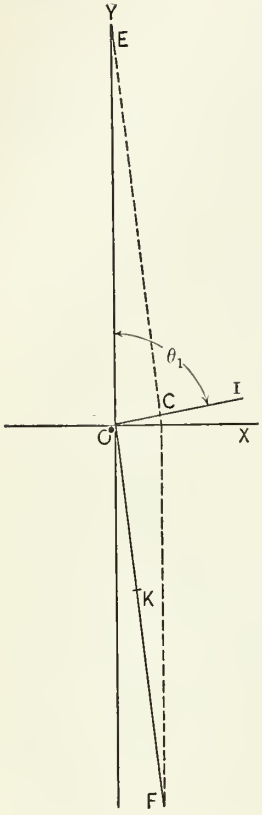


FIG. 265. — Phase Diagram of Voltage and Current in a Transformer having Negligible Iron Losses with the Secondary Circuit Open.

Thus far the effect of mutual induction is similar to that of self-induction. But when a current flows in the secondary coil under the influence of the mutually induced voltage, another effect is produced, which is not so closely related to the conditions in a self-inductive circuit. The current in the secondary coil tends to demagnetize the magnetic circuit appertaining jointly to the two coils. But the induced voltage in the primary coil must at each instant be equal to the algebraic difference between the values of the active and impressed voltages at that instant; that is, the instantaneous impressed

voltage must be equal to the algebraic sum of the corresponding instantaneous active and reactive voltages (the components of impressed voltage drop in primary resistance and reactance), the latter being equal to the self-induced voltage in scalar value but reversed in direction; or $e_1 = e_{1m} + e_{1R_1}$, where e_{1m} is the primary reactive voltage and e_{1R_1} is the active voltage which drives the primary current through the primary resistance. If the waves of voltage are sinusoidal, we may write $E_{1m} = \sqrt{E_1^2 - E_{1R_1}^2}$ or $E_1 = \sqrt{E_{1R_1}^2 + E_{1m}^2}$, in which expressions E_1 is the effective value of impressed voltage, E_{1R_1} of active voltage, and E_{1m} of reactive voltage. This is a fixed relation determined by the law that the algebraic sum of instantaneous voltages taken around a circuit must reduce to zero, and is therefore independent of the amount of current flowing in either the primary or secondary circuit. Then if the characteristics of the secondary circuit (load on the secondary) are changed so that more secondary current flows with a resultant decrease of induced voltage below the value shown by the relation $e_{1m} = e_1 - e_{1R_1}$, an increased primary current flows in order to maintain the relation; and if the characteristics of the secondary circuit are changed so as to decrease the secondary current with a resultant increase of the induced voltage as indicated by the equation $e_{1m} = e_1 - e_{1R_1}$, the primary current falls, so as to reëstablish the relation. In other words, if the magnetic flux which induces E'_{1m} is disturbed, as by introducing a magnetizing effect extraneous to the primary coil, such as is caused by current flowing in the secondary coil, the primary current will again adjust itself so as to maintain the inducing flux at such value as still to produce the relation $e_{1m} = e_1 - e_{1R_1}$. Consequently, the magneto-motive force of the primary and secondary currents of a transformer must so combine at each instant that the resultant magneto-motive force will create a wave of magnetic flux in the core, the rate of change of whose linkages with the turns of the primary coil is equal to 10^8 times the primary self-induced voltage, which voltage must in turn be at each instant equal to the difference between the impressed and active voltages at that instant, with algebraic sign reversed. In the special case where the voltages are sinusoidal, the two magneto-motive forces must combine in such a way as to create a sinusoidal

wave of flux which is advanced 90° ahead of the wave of self-induced voltage and is of a magnitude that produces a vector of self-induced voltage equal to $-(\bar{E}_1 - \bar{E}_{1R_1}) = \bar{E}_{1R_1} - \bar{E}_1$. The relation of impressed, active, and reactive voltage then is $\bar{E}_1 - \bar{E}_{1R_1} - \bar{E}_{1m} = 0$. The vector OE_{1m}' of Fig. 265 is equal to $-\bar{E}_{1m}$.

This means that when the secondary circuit is closed and a current flows under the impulsion of the secondary induced voltage, the primary current must take such magnitude and phase position as will in conjunction with the secondary ampere-turns produce the requisite magneto-motive force to act in the joint magnetic circuit. This further means that the primary ampere-turns must, at each instant, neutralize the effect of the secondary ampere-turns in the joint magnetic circuit, and must in addition supply the ampere-turns which maintain the inducing flux, *i.e.* $n_1 i_1 = n_2 i_2 + n_1 i_\mu'$, where the terms in the equation represent respectively corresponding instantaneous values of the total primary ampere-turns, the ampere-turns of the secondary circuit, and the resultant ampere-turns of the primary and secondary coils. It follows from this that $i_1 = \frac{1}{s} i_2 + i_\mu'$, in which $s = \frac{n_1}{n_2}$, n_1 and n_2 being respectively the numbers of turns of conductors comprised in the primary and secondary coils. Since flux equals magneto-motive force divided by reluctance, and the reluctance of an iron core varies with magnetic density, the exciting current i_μ' varies quite irregularly in a transformer with an iron core, even though the impressed voltage and magnetic flux vary sinusoidally. The total primary current which flows, when the secondary circuit is open, multiplied by the primary coil turns may be called the **Exciting ampere turns.**

If the magneto-motive force of either or both coils sets up flux which does not pass through the other coil, the effect is similar to the introduction of external self-inductance into the circuits of the respective coil or coils in addition to the mutual inductance, as such fluxes surround only the currents producing them. Such flux is usually termed **Magnetic leakage.** In cases where there is no magnetic leakage or other variable flux linking only the secondary coil or its external circuit, the phase of the secondary current is coincident with the phase of the induced

voltage in the secondary coil. Then if the primary current, the exciting element of the primary current, and the secondary current, are assumed to be sinusoidal, and there are no core losses in the magnetic circuit, as may be the case if the magnetic circuit includes no iron, the relation becomes $M_1 = \sqrt{M_\mu^2 + M_2^2}$, where the letters M_1, M_μ, and M_2 represent the scalar values of magneto-motive forces set up in the joint magnetic circuit by the currents. (In this early discussion of the transformer it is assumed that effects of electrostatic capacity are absent.) The vector relation is then $\bar{M}_\mu = \bar{M}_1 - \bar{M}_2$, provided no magneto-motive forces act in the magnetic circuit except those set up by the currents mentioned.

A simple diagram, using equivalent sinusoids for currents and voltages, such as is shown in Fig. 266, develops these relations graphically. Vectors $\bar{E}_1 (= OE)$ and $n_1\bar{I}_\mu (= OA)$ represent the impressed voltage and exciting ampere-turns (the latter exaggerated in length for the sake of clearness) of a transformer having negligible leakage and a non-reactive secondary circuit. The vector $n_1\bar{I}_\mu$ and the vector OB representing total primary ampere-turns may be vectors scaled to also represent either primary currents measured in amperes or magneto-motive forces measured in gilberts, while the secondary vector of ampere-turns OC, by using the same scales, equals the scalar value and position of secondary current divided by the ratio of transformation, s, or the gilberts of secondary magneto-motive force. The vector $n_1\bar{I}_\mu' (= OH)$ represents the equivalent magnetizing element of the exciting magneto-motive force and lags behind the latter by an angle equal to the angle of advance caused by the iron losses in the core. It is 90° in the lead of, and determines the direction of, the primary and secondary induced voltages, $\bar{E}_{1m}' (= OF)$ and $\bar{E}_{2m}' \left(= \frac{1}{s} OF\right)$. On the line of the latter vectors is laid out the secondary vector of magneto-motive force $n_2\bar{I}_2 (= OC)$, the secondary circuit here being assumed non-reactive. The vectors $n_2\bar{I}_2$ and $n_1\bar{I}_\mu$ must then, to conform to the results of the discussion above, form a closed vector triangle with the total magneto-motive force of the primary current $n_1\bar{I}_1$, or $n_1\bar{I}_1$ and $n_2\bar{I}_2$ must combine to furnish the resultant magneto-motive force $n_1\bar{I}_\mu$, acting in the joint magnetic circuit. On the line OB, $n_1\bar{I}_1$, a distance OG is laid off, which represents

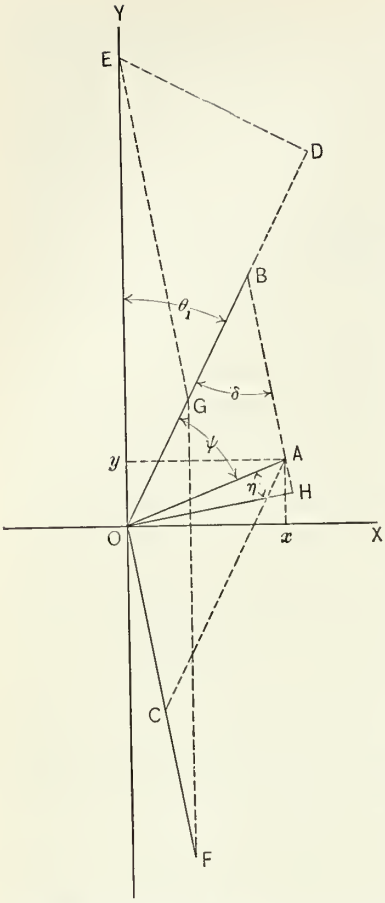


FIG. 266. — Phase Diagram of Voltages and Currents in a Transformer when a Current flows in the Secondary Circuit. The various vector quantities used in the construction are represented by lines as follows: Impressed voltage E_1 by OE; primary self-induced and secondary mutually induced voltages E'_{1m} and sE'_{2m} by OF; voltage drop in resistance of primary coil E_{1R} by OG; exciting ampere-turns $n_1 I_\mu$ and current I_μ by OA; secondary ampere-turns $n_2 I_2$ and current $\frac{1}{s} I_2$ by OC; quadrature ampere-turns $n_1 I_\mu'$ by OH; primary current and ampere-turns $n_1 I_1$ by OB.

that portion of the voltage $\bar{E}_1$ which, multiplied by the current $\bar{I}_1$, furnishes the necessary power to drive the primary current through the primary resistance, or E_{1R_1}. The voltage $\bar{E}_1$ minus this voltage $\bar{E}_{1R_1}$ must be numerically equal but opposite to the primary self-induced voltage $E_1'(=-OF)$ and also numerically equal but opposite to the mutually induced secondary voltage $\bar{E}_{2m} = -\left(\frac{n_2}{n_1} OF\right)$ multiplied by the ratio of transformation, s.

The triangle ODE of Fig. 266 corresponds to the triangle OCE in Fig. 265 and OCA in Fig. 114, *i.e.* OD, OC, and OC are the respective components of impressed voltage projected on the current, and DE, CE, and CA are the respective components in quadrature to the current; but in the case of Fig. 266, unlike the other two, only part, E_{1R_1}, of the voltage is lost by reason of power used in the primary circuit. The remaining power represented by the vector product of $\bar{E}_1\ \bar{I}_1$ is transferred to the magnetic circuit, and thence, deducting core losses, to the secondary electric circuit. Therefore, it is evident that the effect of the current in the secondary circuit has been to decrease

both the apparent impedance $Z_1' = \frac{E_1}{I_1}$ of the primary circuit, and the angle of lag, $\theta_1 = \tan^{-1}\frac{X'}{R_1'}$, where X_1' and R_1' are the apparent primary reactance and resistance. Starting with $n_1\bar{I}_1 = n_1\bar{I}_\mu$, as shown in Fig. 265, for a transformer with open secondary circuit, then closing the secondary circuit and gradually reducing its resistance until full load is reached, will cause the vectors $n_2\bar{I}_2$ and $n_1\bar{I}_1$, in a commercial, constant voltage transformer, quickly to become so extended in length that the parallelograms of magneto-motive forces and voltages will have flattened out so much that, for most practical purposes, the primary current and secondary current may be considered to differ substantially 180° in phase. Thus, in a certain transformer of 20 kw. capacity, the exciting current is about one half of an ampere, while the full load primary current is ten amperes, or I_1 is twenty times larger than I_μ, a difference which is still more marked in transformers of larger sizes. Therefore, in a transformer without appreciable magnetic leakage, and fully loaded with a non-reactive load, the apparent self-inductance almost disappears, and the primary and secondary act like a single circuit made up almost entirely of non-reactive resistance. The apparent primary impedance reduces to nearly the sum of the resistances of the primary coil and equivalent resistance of the secondary coil and the secondary external load circuit. There is very little variation in either the phase position or scalar value of $\bar{I}_\mu$ with change of load, as will be seen later.*

The various conditions and problems that arise in the operation of transformers are entered into extensively in later articles, but before taking up the special cases of sinusoidal or other periodic impressed alternating voltages, we will consider the more general case of the characteristics of mutually reactive coils as they are affected by any change of currents or voltages whatever.

119. Mutual Inductance. — Consider two adjacent coils surrounded by air and in which currents are flowing. Then at any instant the total number of linkages of lines of force with the turns of the conductors composing either one of the coils, is

* Art. 128.

the number of linkages due to the current in that coil algebraically added to the number of linkages due to the other coil which are embraced by the first coil. If the current is changed in either of the coils, an instantaneous voltage equal to $e_1' = -\frac{n_1 d\phi_1}{10^8\, dt}$ is induced in the coil under consideration, which may be called the primary coil, where n_1 is the number of turns and ϕ_1 the instantaneous average magnetic flux linked per turn of the coil. The average number of magnetic linkages per turn with the turns of the coil due to its own current is $\frac{10^8 L_1 I}{n_1}$, in which L_1 and I are respectively the self-inductance and the current in the coil.*

The number of lines of force due to the current in the first coil which pass through the second coil evidently depends upon the relative positions of the coils, but it cannot be greater than the total number of lines set up by the current in the first coil. For any two fixed coils in a medium of constant permeability, this number of lines is proportional to the current flowing in the primary coil. The voltage developed in the second coil, due to a change in the current flowing in the first, is $e_2' = -\frac{n_2 d\phi_2}{10^8\, dt}$, where n_2 and ϕ_2 are the turns in the second coil and the instantaneous average magnetic flux linked therewith per turn of that coil. The equation $e_1' = -\frac{n_1 d\phi_1}{10^8\, dt}$ may be written $e_1' = -\frac{L_1 di_1}{dt}$, when the permeability is constant.* The equation $e_2' = -\frac{n_2 d\phi_2}{10^8\, dt}$ may be similarly written $e_2' = -\frac{M di_1}{dt}$, where $10^8 M$ is the number of linkages with the turns of the second coil of lines of force which are due to the first coil when one ampere is flowing in the first coil, supposing the coils to be in air or other medium of unchanging permeability. The expression represented by M is called, by analogy, the **Mutual-inductance** or the **Coefficient of mutual induction** of the coils. If k is a coefficient numerically equal to the ratio of the reluctance of the path of the lines of force which interlink the two coils to the aggregate reluctance of the path of all the lines of force set up by the primary coil, then the value of $\phi_2 k$ is equal to ϕ_1.

* Art. 41.

If the coils are long solenoids in air, with current only in the first coil, the flux in the first coil is $\phi_1 = \frac{4\pi n_1 IA}{10\,l}$, where A is the cross section and l the length of the coil; and if the solenoids are wound one over the other so that their dimensions are practically equal, the value of k is unity, because ϕ_2 evidently becomes equal to ϕ_1, and M becomes equal to $\frac{\phi_1 n_2}{10^8 I}$. If the same current is now switched into the second coil, we have $\phi_2 = \frac{4\pi n_2 IA}{10\,l}$ and $M' = \frac{\phi_2 n_1}{10^8 I}$. But in this special case $\frac{\phi_1}{\phi_2} = \frac{n_1}{n_2}$, and hence $M = M'$. Also $L_1 = \frac{\phi_1 n_1}{10^8 I}$ and $L_2 = \frac{\phi_2 n_2}{10^8 I}$. Consequently $L_1 L_2 = M^2$, or $M = \sqrt{L_1 L_2}$.

If k is not equal to unity, it is manifest that some of the magnetic flux set up by the ampere-turns of each coil do not link with the turns of the other coil. Such lines of force are called **Magnetic leakage** or **Leakage flux**.

If the foregoing long solenoids are now separated by drawing one out of the other, the value of M continually decreases as the separation continues, because the paths of the magnetic flux set up by the two coils become differentiated. The self-inductances of the coils remain constant, so that as the coils separate, the mutual-inductance becomes less than $\sqrt{L_1 L_2}$. As the separation of the coils becomes greater, the value of M decreases towards a minimum of zero. It reaches the latter value when the coils are at an infinitely great distance apart or the axis of one is placed symmetrically but at right angles with reference to the axis of the other, so that the summation of the linkages of the magnetic flux from each coil through the turns of the other reduces to zero.

The maximum possible value of the mutual-inductance of the two coils is therefore a mean proportional between the values of their self-inductances, and the minimum value is zero. The maximum value can only be attained when all the lines of force due to one coil pass through all the turns of the other, and the value of M may therefore be written $M \leqq \sqrt{L_1 L_2}$, from which it is at once seen that the mutual-inductance of the two coils must always be small if the self-inductance of one or both of the coils is very small, while the mutual-inductance may be large if both the self-inductances are large. It is shown in a preceding par-

agraph that the value of M may be generally written $M \propto n_2 \phi_1 k$. In this expression ϕ_1 is dependent on n_1, whence it is shown that $M \propto n_1 n_2 k$. This is also to be derived from the relation $M \propto \sqrt{L_1 L_2}$, since L_1 and L_2 are respectively proportional to n_1^2 and n_2^2.*

The summation product of the number of lines of force which interlink two coils by the numbers of turns in the individual coils will not change by changing the point of reference from one coil to the other, provided the reluctance of the magnetic circuit and its shape are unchanged, and, consequently, the mutual-inductance of two coils is the same when measured from either coil. When the magnetic circuit is composed wholly or partly of iron, it is necessary to have the same distribution of lines of force in the parts of the magnetic circuit when the two measurements are made, in order that they may give equal results. When the magnetic circuit is made up partly of iron and partly of non-magnetic materials, and the coils are dissimilar in their relations to the magnetic circuit, as in the case of armature and field windings of dynamos, the condition of equal distribution of lines of force is difficult to fulfill on account of the effects caused by unequal saturation of the iron, and the value of M may be quite different when measured from the field winding and from the armature winding. This difference is due to the difference in the permeability of the magnetic circuit and the form of the magnetic paths, during the two measurements.

As shown above, the mutual-inductance is homogeneous with and therefore of the same absolute dimensions as self-inductance. Its unit is therefore the **Henry**. The Henry is 10^9 times as large as the C. G. S. absolute unit of inductance.†

If the lines of force due to one coil which enter another do not all pass through all the turns of the second coil, as may be tacitly assumed in developing the foregoing relations, the definition of mutual-inductance still holds as already given, but the summation of the linkages of lines of force with the individual turns must be taken in a manner similar to the summations required to obtain the self-inductance of a coil through all the turns of which all of the self-induced lines of force do not pass.‡

If a change occurs in the reluctance of the magnetic circuit

* Art. 41. † Art. 42. ‡ Art. 43.

common to the coils, the relative positions of the coils remaining unchanged, the mutual inductance is altered from M to $M' = M\frac{P}{P'}$, where P and P' are the reluctances of the path before and after the reluctance is changed, such as by inserting an iron core. The ratio $\frac{P}{P'}$ is dependent on the permeability of the iron in the magnetic circuit, and the value of M must therefore vary with the current in the coils in any case where iron is in the magnetic circuit, while it is independent of the value of the current when magnetic material is absent.

120. The Energy of Mutual Induction. — Assume two adjacent coils with a constant mutual-inductance, in which all of the flux passes through all the turns of both coils, such as the superimposed long solenoids described in Art. 119. In one let a current of I_1 amperes flow, and in the other a current of I_2 amperes, the two magneto-motive forces being additive in the mutual magnetic circuit. The number of linkages of lines of force due to the first coil with the turns of the second is $10^8 MI_1$, and the time rate of change in this number, when the current in the first coil changes, is $\frac{10^8 M di_1}{dt}$, where i_1 is the instantaneous value of the current in the first coil while it is changing. If the current I_1 is varied, the work due to mutual induction which is done in the second coil in changing the magnetic field against the effect of the current I_2 is obtained from the following relations, in which I_2 represents the current in the secondary coil and e_2' represents the voltage induced in the second coil by the time-rate of change $\frac{di_1}{dt}$ of the current in the first coil, and is considered positive in sign when it tends to increase I_2 :

$$e_2' = -M\frac{di_1}{dt}$$

and $$dw = I_2 e_2' dt \; ; *$$

hence $$dw = -MI_2 di_1.$$

If the current in the first coil is changed from zero to I_1 (I_2 being maintained by a battery of constant voltage), the work

* Art. 47.

done on the second coil is $W = -\int_0^{I_1} MI_2 di_1 = -MI_1I_2$, which is all abstracted from the electric circuit of the primary coil and stored in the joint magnetic field. If the current I_1 falls again to zero, this work is restored as electrical energy to the primary circuit. When M varies with the current, the work is still MI_1I_2, but M in the expression must be assigned its proper value corresponding to the maximum magnetization of the core. As the current in the first coil rises to I_1, the total work stored in the magnetic field is evidently the sum of the work due to self and mutual induction, or $\frac{LI_1^2}{2} + MI_1I_2$.

If the current is caused to vary at the same time by a source of energy in each of the coils, the following condition exists at any instant. The instantaneous voltage, which must be impressed on each coil to maintain the current flow therein, is the sum of the active voltage (iR), the reactive voltage, $\frac{Ldi}{dt}$, which is equal and opposite to the voltage of self-induction, and the reactive voltage, $\frac{Mdi}{dt}$, which is equal and opposite to the voltage of mutual-induction, whence

$$e_1 = R_1 i_1 + \frac{d}{dt}(Mi_2 + L_1 i_1), \tag{1}$$

$$e_2 = R_2 i_2 + \frac{d}{dt}(Mi_1 + L_2 i_2),$$

where e_1 and e_2 are the instantaneous impressed voltages in the two coils, i_1 and i_2 are the corresponding instantaneous currents in the two coils, R_1, R_2, L_1, L_2, are the resistances and self-inductances of the two coils, and M is their mutual inductance. By transformation, we have

$$(e_1 - i_1R_1)dt = Mdi_2 + L_1di_1, \tag{2}$$
$$(e_2 - i_2R_2)dt = Mdi_1 + L_2di_2,$$

M, L_1, and L_2 being assumed to be constant. Multiplying these equations respectively by i_1 and i_2 and adding, gives

$$(i_1e_1 + i_2e_2)dt - (i_1^2R_1 + i_2^2R_2)dt = L_1i_1di_1 + L_2i_2di_2 + M(i_1di_2 + i_2di_1). \tag{3}$$

The first term of the left-hand member of this equation represents the total work done by the impressed voltages during the interval dt, and the second term of the same member represents the work expended in heat during the same interval on account of the currents flowing through the resistances of the coils. The difference between these two terms represents the work done on the magnetic field, and this is represented by the right-hand member of the equation. The total work done on the magnetic field during any change of the currents, as from zero to I_1 and I_2 amperes, is found by integrating the right-hand member of the equation. Thus,

$$L_1\int_0^{I_1} i_1 di_1 + L_2\int_0^{I_2} i_2 di_2 + M\int_0^{I_1, I_2} (i_1 di_2 + i_2 di_1) = \frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} + MI_1 I_2. \quad (4)$$

This is all absorbed from the two electric circuits and stored in the magnetic linkages, which are set up by the rise of the currents. If the currents now fall to zero again, the same amount of energy is restored to the electric circuits as the magnetic field is restored to its initial condition. The work MI_1I_2 is equally divided between the two coils when $I_1 = I_2$.

The foregoing relates to two coils receiving their current through separate electric circuits, but if the two coils are connected in series relation in one electric circuit, so that they carry the same current, the expression for the work stored in the magnetic field as the current rises from zero to I amperes becomes

$$\frac{L_1 I^2}{2} + \frac{L_2 I^2}{2} + MI^2. \quad (5)$$

When the two coils in series are of equal self-inductances and are located relatively to each other so that there is no magnetic leakage, in which case $L_1 = L_2 = M$, the expression representing stored work becomes

$$\frac{L_1 I^2}{2} + \frac{L_2 I^2}{2} + MI^2 = 2\,MI^2 = 2\,LI^2. \quad (6)$$

Equation (3) is written on the assumption that no eddy current or hysteretic losses are involved, but in case such losses come into the operation an additional energy term representing

them may be subtracted from the first term in the left-hand member of the equation. Equation (3) is also written on the hypothesis of the magnetic fields of the two currents being additive, and if they are subtractive, the algebraic sign of one of the currents is changed. The result in equation (4) then becomes

$$\frac{L_1 I_1^2}{2} + \frac{L_2 I_2^2}{2} - M I_1 I_2;$$

and in (5) becomes

$$\frac{L_1 I^2}{2} + \frac{L_2 I^2}{2} - M I^2;$$

while (6) reduces to

$$\frac{L_1 I^2}{2} + \frac{L_2 I^2}{2} - M I^2 = 0.$$

121. Transfer of Electricity by the Effect of Mutual Induction. Coils in Series and in Parallel. — Continuing the assumption of two coils in separate electric circuits and constant L and M, suppose that no voltage is initially impressed on the second coil though its circuit is closed; then when the current in the first coil is changed, the conditions in the second coil are given from the equation

$$R_2 i_2 + \frac{d}{dt}(M i_1 + L_2 i_2) = 0.$$

Whence

$$i_2 R_2 dt = - M di_1 - L_2 di_2,$$

and

$$\int_0^T i_2 R_2 dt = - M \int_0^{I_1} di_1 - L_2 \int_0^0 di_2.$$

The reason for the two zero limits of the third term of the above equation is evident when it is remembered that the secondary current grows from zero and returns to zero during the operation. Since the last term reduces to zero, the quantity of electricity which is transferred in the second coil under the inductive influence of the first when its current changes from zero to I_1 is

$$Q_2 = \int_0^T i_2 dt = - \frac{M}{R_2} \int_0^{I_1} di_1 = - \frac{M I_1}{R_2} = - \frac{M E_1}{R_1 R_2}.$$

If the current of the first coil is now brought to its original value, we have

$$Q_2' = -\frac{M}{R_2}\int_{I_1}^{0} di_1 - \frac{L_2}{R_2}\int_{0}^{0} di_2 = \frac{MI_1}{R_2} = \frac{ME_1}{R_1R_2}.$$

The two quantities are equal and of opposite sign, so that the transfer of electricity in the secondary coil during the rise and fall of the primary current reduces to zero, provided the original and final values of the magnetism of the magnetic circuit are the same.* If the current in the first coil is a simple periodic one, a periodic current of the same frequency is set up in the second coil.

The instantaneous voltages in the two coils connected in series in an electric circuit, where the current varies, are

$$e_1 = R_1 i + L_1 \frac{di}{dt} \pm M\frac{di}{dt},$$

$$e_2 = R_2 i + L_2 \frac{di}{dt} \pm M\frac{di}{dt},$$

assuming the reluctance of the magnetic circuit to remain constant; and the instantaneous voltage across the two coils is

$$e = (R_1 + R_2)i + (L_1 + L_2 \pm 2M)\frac{di}{dt}.$$

The positive or negative algebraic sign for $M\frac{di}{dt}$ in these equations must be selected according to whether the coils are connected in the electric circuit so as to cause their magnetic effects to aid or oppose each other in the joint magnetic circuit. Reversing the connection in the electric circuit of the winding of one of the coils reverses the algebraic sign of the expression $M\frac{di}{dt}$ related to both coils.

Upon switching a constant voltage E into this circuit, the deficit of electricity transferred through the circuit during the rise of current to its final value $I = \frac{E}{R_1 + R_2}$ is

$$\frac{(L_1 + L_2 \pm 2M)I}{R_1 + R_2} \text{ coulombs.}$$

* Compare Art. 46.

The "extra current" occurring at switching out the voltage without changing the circuit resistance comprises an equal number of coulombs.

Returning now to the equation

$$e = (R_1 + R_2)i + (L_1 + L_2 \pm 2M)\frac{di}{dt},$$

assuming e to be a sinusoidal function of time, so that $e = e_m \sin \omega t$, the solution of the equation is, after a short lapse of time when the current wave has become sinusoidal,

$$i = \frac{e_m}{Z} \sin (\omega t - \theta),$$

in which $$Z = \sqrt{R^2 + [2\pi f(L_1 + L_2 \pm 2M)]^2},$$

where $R = R_1 + R_2$,

and $$\bar{Z} = R + j\,2\pi f(L_1 + L_2 \pm 2M),$$

and $$\tan \theta = \frac{2\pi f(L_1 + L_2 \pm 2M)}{R}.$$

In case $L_1 = L_2$ and there is no magnetic leakage so that $M = \sqrt{L_1 L_2} = L$, the self-inductance of either coil, the value of Z becomes either

$$Z = \sqrt{R^2 + (8\pi f L)^2}, \text{ or } Z = R,$$

and $$\tan \theta = \frac{8\pi f L}{R}, \text{ or } \tan \theta = 0°.$$

If the coils referred to in the two preceding paragraphs are of equal self-inductances and are located relative to each other so that there is no magnetic leakage, it is obvious from the voltage equation that (since M then equals $\sqrt{L_1 L_2} = L$) the reactive voltage caused by the two coils in series is either four times as great as the reactive voltage caused by one of the coils alone, uninfluenced by the other, or is zero (in the first case they act like a single coil of twice the turns of either); and, also, if the two coils are of equal self-inductances, but are located relative to each other so that M is zero, the reactive voltage caused by the two coils is twice as great as the reactive voltage caused by one coil alone.

When the number of coils in series is n, the equations of

voltage become, when the subscripts 1, 2; 1, 3; etc., represent the mutual inductances between coils of those numbers,

$$e_1 = R_1 i + L_1 \frac{di}{dt} + (\pm M_{1,2} \pm M_{1,3} \pm \cdots \pm M_{1,n}) \frac{di}{dt},$$

$$e_2 = R_2 i + L_2 \frac{di}{dt} + (\pm M_{2,1} \pm M_{2,3} \pm \cdots \pm M_{2,n}) \frac{di}{dt},$$

.

$$e_n = R_n i + L_n \frac{di}{dt} + (\pm M_{n,1} \pm M_{n,2} \pm \cdots M_{n,n-1}) \frac{di}{dt};$$

and the instantaneous voltage measured across the series of coils is

$$e = i\Sigma_1^n R + \frac{di}{dt}\Sigma_1^n L + 2\frac{di}{dt}\Sigma_{1,2}^{n-1,n} \pm M.$$

The impedance of such a circuit is

$$Z = \sqrt{(\Sigma_1^n R)^2 + [2\pi f(\Sigma_1^n L + 2\Sigma_{1,2}^{(n-1),n} \pm M)]^2},$$

and the reactance is

$$X = 2\pi f(\Sigma_1^n L + 2\Sigma_{1,2}^{(n-1),n} \pm M).$$

As far as the effects in the circuit of coils in series are concerned, the mutual inductance between the coils in the circuit merely adds to or deducts from the reactions caused by self-inductance. Mutual inductance between coils in different electric circuits, however, affords different results by causing a transfer of energy from one electric circuit to the other when the currents vary in one or both of the circuits.

The conditions are more complex when the mutually inductive circuits are connected in parallel to the same main circuit, instead of being connected in series as heretofore considered. In case of two mutually inductive coils connected in parallel, the voltage formulas are

$$e = R_1 i_1 + L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt}, \tag{1}$$

and

$$e = R_2 i_2 + L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt}. \tag{2}$$

The following equations may be derived from these by differentiation,

$$\frac{de}{dt} = R_1\frac{di_1}{dt} + L_1\frac{d^2i_1}{dt^2} \pm M\frac{d^2i_2}{dt^2}, \tag{3}$$

and

$$\frac{de}{dt} = R_2\frac{di_2}{dt} + L_2\frac{d^2i_2}{dt^2} \pm M\frac{d^2i_1}{dt^2}. \tag{4}$$

The positive sign pertains to $M\frac{di}{dt}$ when the coils are connected to the electric circuits so that their magnetic effects are cumulative, and the negative sign pertains to $M\frac{di}{dt}$ when the magnetic effects of the coils are in opposition.

The following equations in i_1 and i_2 are obtained by multiplying (1), (3), and (4) respectively by R_2, L_2, and $\mp M$, and adding to get (5), and multiplying (2), (3), and (4) respectively by R_1, $\mp M$, and L_1, and adding to get (6):

$$R_2e + (L_2 \mp M)\frac{de}{dt} = R_1R_2i_1 + (R_1L_2 + L_1R_2)\frac{di_1}{dt} + (L_1L_2 - M^2)\frac{d^2i_1}{dt^2}, \tag{5}$$

$$R_1e + (L_1 \mp M)\frac{de}{dt} = R_1R_2i_2 + (R_1L_2 + L_1R_2)\frac{di_2}{dt} + (L_1L_2 - M^2)\frac{d^2i_2}{dt^2}. \tag{6}$$

Equations (5) and (6) afford exact solutions when R_1, R_2, L_1, L_2, and M are of constant values and the voltage is a sine function of time. In this case $e = e_m \sin \omega t$, $\frac{de}{dt} = \omega e_m \cos \omega t$, and $\frac{d^2e}{dt^2} = -\omega^2 e_m \sin \omega t$. Therefore $\frac{d^2(\)}{dt^2} = -\omega^2$. Substituting these values in equations (5) and (6), and putting $R_1R_2 - \omega^2(L_1L_2 - M^2) = a$, $R_1L_2 + L_1R_2 = b$, $\frac{d(\)}{dt} = D$ and $\frac{d^2(\)}{dt^2} = D^2$ gives by integration,*

$$i_1 = \frac{[R_2 \sin \omega t + \omega(L_2 \mp M)\cos \omega t]e_m}{a + bD} + C_1\epsilon^{-m_1t} + C_2\epsilon^{-m_2t}. \tag{7}$$

$$i_2 = \frac{[R_1 \sin \omega t + \omega(L_1 \mp M)\cos \omega t]e_m}{a + bD} + C_1'\epsilon^{-m_1t} + C_2'\epsilon^{-m_2t}. \tag{8}$$

* Murray's *Differential Equations*, Chap. VI.

The exponential terms quickly become negligible after the current is started. Neglecting the exponential terms and rationalizing the right-hand member of each of these equations by multiplying numerator and denominator by $a - bD$, simplifying the expression, and factoring the resulting coefficient in the numerator gives

$$i_1 = \frac{e_m\sqrt{[aR_2+\omega^2b(L_2\mp M)]^2+\omega^2[a(L_2\mp M)-bR_2]^2}}{a^2+\omega^2b^2}\sin(\omega t-\theta_1)$$

$$= \frac{e_m\sqrt{R_2^2+\omega^2(L_2\mp M)^2}}{\sqrt{a^2+\omega^2b^2}}\sin(\omega t-\theta_1), \qquad (9)$$

$$i_2 = \frac{e_m\sqrt{[aR_1+\omega^2b(L_1\mp M)]^2+\omega^2[a(L_1\mp M)-bR_1]^2}}{a^2+\omega^2b^2}\sin(\omega t-\theta_2)$$

$$= \frac{e_m\sqrt{R_1^2+\omega^2(L_1\pm M)^2}}{\sqrt{a^2+\omega^2b^2}}\sin(\omega t-\theta_2); \qquad (10)$$

in which

$$\tan\theta_1 = \frac{\omega[a(L_2\mp M)-bR_2]}{aR_2+\omega^2b(L_2\mp M)} \text{ and } \tan\theta_2 = \frac{\omega[a(L_1\mp M)-bR_1]}{aR_1+\omega^2b(L_1\mp M)}.$$

It follows from equations (9) and (10) that

$$I_1 = \frac{\sqrt{R_2^2+\omega^2(L_2\mp M)^2}}{\sqrt{a^2+\omega^2b^2}}E,$$

$$I_2 = \frac{\sqrt{R_1^2+\omega^2(L_1\mp M)^2}}{\sqrt{a^2+\omega^2b^2}}E;$$

$$\frac{I_1}{I_2} = \sqrt{\frac{R_2^2+\omega^2(L_2\mp M)^2}{R_1^2+\omega^2(L_1\mp M)^2}}$$

and

$$Z_1 = \frac{E}{I_1} = \frac{\sqrt{a^2+\omega^2b^2}}{\sqrt{R_2^2+\omega^2(L_2\mp M)^2}},$$

$$\overline{Z}_1 = \frac{a}{\sqrt{R_2^2+\omega^2(L_2\mp M)^2}} + j\frac{\omega b}{\sqrt{R_2^2+\omega^2(L_2\mp M)^2}}.$$

$$Z_2 = \frac{E}{I_2} = \frac{\sqrt{a^2+\omega^2b^2}}{\sqrt{R_1^2+\omega^2(L_1\mp M)^2}},$$

$$\overline{Z}_2 = \frac{a}{\sqrt{R_1^2+\omega^2(L_1\mp M)^2}} + j\frac{\omega b}{\sqrt{R_1^2+\omega^2(L_1\mp M)^2}}.$$

When the coils are superposed so that there is no magnetic leakage, are connected in the electric circuit so that their magnetic effects are cumulative, and the resistances and self-inductances are equal, the foregoing expressions reduce to

$$\tan \theta_1 = \tan \theta_2 = \frac{2\,\omega L}{R},$$

$$I_1 = I_2 = \frac{E}{\sqrt{R^2 + (2\,\omega L)^2}},$$

$$Z_1 = Z_2 = \sqrt{R^2 + (2\,\omega L)^2};$$

since the named conditions make $R_1 = R_2$, $L_1 = L_2 = M$, and satisfy the uppermost of the algebraic signs associated with $M\frac{di}{dt}$ and M. It will be observed that the mutual inductance has, under these circumstances, the effect of doubling the reactance and increasing the angle of lag, compared with the values $\left(\omega L \text{ and } \tan^{-1}\frac{\omega L}{R}\right)$ for corresponding resistances and self-inductances with M zero.

It is also to be observed from the equations that when the coils are connected in the electric circuit so as to make their magnetic effects cumulative, the effect of the mutual inductance is always to increase the impedance of the coils. The impedance is a maximum when $M = \sqrt{L_1 L_2}$; that is, when magnetic leakage is negligible.

When the coils are superposed as before, but connected in the electric circuit so that their magnetic effects are in opposition, R_1 being equal to R_2 and L_1, L_2, and M being equal to each other, the equations reduce to

$$\tan \theta_1 = \tan \theta_2 = 0,$$

$$I_1 = I_2 = \frac{E}{R},$$

$$Z_1 = Z_2 = R.$$

The last condition satisfies the lower one of the algebraic signs associated with $M\frac{di}{dt}$ and M. The mutual inductance under these circumstances neutralizes the effects of self-inductance and reduces the reactance of the coils to zero.

When the magnetic effects of the coils are in opposition, the effect of their mutual inductance is always to reduce the impedance and angle of lag. The maximum influence of the mutual inductance occurs when $M = \sqrt{L_1 L_2}$, that is, when there is no magnetic leakage.

The formulas representing the reactions of coils in parallel may be extended to any number of coils larger than two, by the same processes as are exhibited in the foregoing. If either branch circuit has electrostatic capacity in series with its resistance and self-inductance, a term representing condenser voltage must be added to the voltage equations corresponding to equations (1) and (2). When the capacity measured from one coil to another is appreciable, it may be treated approximately by considering it as a condenser composing a separate branch in parallel with the coils; but an exact solution of this case requires the use of equations which represent the effects of distributed resistance, inductance, and capacity, and have the character of those developed in a later chapter.

122. Ratio of Transformation in a Transformer.—The formulas of Art. 119 show that the voltage developed in a secondary coil is at any instant

$$e_2' = -\frac{d(Mi_1)}{dt}.$$

If the current wave is a sinusoid, this becomes

$$e_2' = -\frac{d(Mi_{1m}\sin \omega t)}{dt}.$$

If the conditions require that M be treated as a variable dependent upon the varying permeability of an iron core, this equation is practically insolvable. For a close approximation to practical conditions, however, it is sufficient to assume M as having a constant value which depends upon the iron of the core and the maximum magnetic density used in the transformer. The equation then becomes $e_2' = -\frac{Mi_{1m}\cos \alpha d\alpha}{dt}$. The maximum value of the voltage is then $e_{2m}' = 2\pi f M i_{1m}$, where f is the frequency of the current wave, since $d\alpha/dt = \omega$. The effective value of the secondary voltage is therefore evidently $E_2' = -2\pi f M I_1$, where I_1 is the effective primary current. If

the secondary circuit is open, the following equations may be written

$$I_1 = \frac{E_1}{\sqrt{R_1^2 + 4\pi^2 f^2 L_1^2}},$$

while $-E_1' = 2\pi f L_1 I_1 = \frac{\sqrt{2}\pi f n_1 \Phi}{10^8}$, where $-E_1'$ is numerically equal but opposite to the total induced primary counter-voltage. (This includes the voltage induced by both the leakage and joint magnetic flux.) E_1 has been used to represent the impressed voltage, and Φ the maximum number of lines of force in the cycle. If the resistance of the primary winding is considered negligible, the former equation becomes

$$I_1 = \frac{E_1}{2\pi f L_1}$$

and

$$E_1 = 2\pi f L_1 I_1 = -E_1';$$

and, in this case, if the primary and secondary coils are so completely superposed that there is no magnetic leakage, the value of M becomes $M = \sqrt{L_1 L_2}$; whence

$$\frac{E_1}{E_2} = \frac{2\pi f L_1 I_1}{2\pi f M I_1} = \frac{L_1}{\sqrt{L_1 L_2}}, \text{ or } \frac{E_1}{E_2} = \frac{\sqrt{L_1}}{\sqrt{L_2}}.$$

But $\frac{\sqrt{L_1}}{\sqrt{L_2}} = \frac{n_1}{n_2}$, since the magnetic circuit of the two coils is assumed to be identical, that is, $M = \sqrt{L_1 L_2}$, and consequently there is no magnetic leakage. Therefore

$$\frac{E_1}{E_2} = \frac{n_1}{n_2} = s.$$

In other words, if the active voltage in the primary circuit may be considered negligible when compared with the impressed voltage, and there is no leakage of magnetic lines, the ratio of the impressed voltage to the voltage induced in the secondary winding *is equal to the ratio of the number of turns of wire in the two coils.* The ratio of the primary voltage and the secondary voltage of a transformer is commonly called the **Ratio of transformation.** The ratio of transformation of well-designed transformers intended for use on constant voltage circuits is

practically equal to $\frac{n_1}{n_2}$ when the secondary circuit is open, showing that the assumption that the active voltage and magnetic leakage are negligible in most commercial transformers, when the secondary circuit is open, is entirely allowable. An example will show this in a striking manner. In a certain transformer of 22.5 kilowatts capacity the electrical resistance of the primary winding is practically 1 ohm and the inductance is 9.1 henrys. At a frequency of 60 cycles per second and a voltage of 2000 volts, the square of the value of the reactance is nearly 12,000,000. In another transformer of 11.25 kilowatts capacity, designed for 2400 volts primary voltage, the value of the electrical resistance of the primary winding is 6.45 ohms (when squared 41.6), and the square of the value of the reactance is 10,000,000. In three other transformers designed for a voltage of 1000 volts and respectively of 7.5, 4.5, and 1.5 kilowatts capacity, the electrical resistances of the primary windings are 1.16, 2.15, and 8.90 ohms, while, at a frequency of 60 cycles per second, the squares of the values of the reactances are respectively 25,000,000, 31,000,000, and 100,000,000; and in a transformer of .5 kilowatt capacity, the electrical resistance of the primary winding is 25 ohms and the square of the value of the reactance is 400,000,000. In each of these cases, which represent common practice in the construction of transformers, the value of R_1^2 is entirely negligible when compared with $4\pi^2 f^2 L_1^2$.

When the secondary circuit of a transformer without magnetic leakage is closed, neither R_1 nor R_2 can be neglected,* and then the ratio of transformation evidently is decreased when the secondary voltage is higher than the primary, and is increased when the secondary voltage is lower than the primary on account of voltage due to the current flowing through R_1 and R_2 (that is, for a given impressed primary voltage the secondary terminal voltage is decreased).

123. Magnetic Leakage in Transformers. — The primary and secondary coils in the discussion of Art. 122 are supposed to be so sandwiched together that magnetic leakage is negligible when there is no current in the secondary coil. This is not necessarily the case. A case when magnetic leakage is always

* Art. 127.

present is shown in Fig. 267. From the figure it is evident that even if no current flows in the secondary winding, the counter-voltage in the primary winding will be greater per turn of wire than the voltage induced per turn in the secondary winding; hence, even if the self-induced or counter-voltage in the primary winding is practically equal and opposite to the impressed voltage, the ratio of transformation will be changed by this relative decrease of the secondary induced voltage. If a current flows in the secondary winding, the self-induction due to magnetic leakage in the secondary (lines of force linking with the secondary coil, but not linking with the primary coil) will still further reduce the active secondary voltage, and

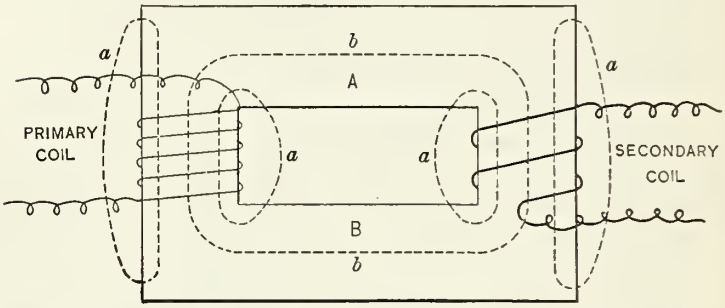


FIG. 267.—Diagram for showing Transformer Leakage. *a*, *a*, leakage flux; *b*, *b*, mutual flux.

the ratio of transformation will be further changed. The effect of magnetic leakage in altering the ratio of transformation (decreasing the proportional voltage induced in the secondary winding by decreasing the magnetic flux passing through it) was early shown by an experiment reported by Professor Ryan.* In the experiment recorded by him, the primary and secondary coils were wound on opposite sides of a laminated iron ring with much the same arrangement as indicated in Fig. 267. The number of turns in the primary and secondary windings were respectively 500 and 155, or $\frac{n_1}{n_2} = 3.2$. When a voltage of 75.6 volts was impressed upon the primary winding with the secondary circuit open, a voltage of only 16.4 volts

*Some experiments upon Alternating Current Apparatus, *Trans. Amer. Inst. E. E.*, Vol. 7, p. 324.

was induced in the secondary winding, or $\frac{E_1}{E_2} = 4.6$. The whole difference in the two ratios was due to magnetic leakage, and the magnitude of the difference shows that M was much less than $\sqrt{L_1 L_2}$. The magnetic leakage was, in fact, nearly 30 per cent; that is, the number of lines of force that linked with the primary winding, but not with the secondary winding, was 30 per cent of the total magnetic flux set up in the magnetic circuit.

The change in the ratio of transformation, just spoken of, occurred when the secondary circuit was open. When current is permitted to flow in the secondary circuit of such an arrangement, the effect is much more striking, for under such conditions the magneto-motive force of the secondary winding opposes that of the primary winding, and there is a strong tendency to force magnetic flux across the air space surrounding the coils. That is, because of the counter magneto-motive force of the secondary winding, the difference of magnetic potential between the points A and B in Fig. 267 may be very great. When the secondary current is zero, the maximum magneto-motive force tending to send flux from A to B through leakage paths is measured by $m = \Phi P$, where Φ is the maximum flux in the core and P is the reluctance in the iron circuit connecting those points through the secondary coil. When a current flows in the secondary winding, the maximum magneto-motive force between A and B is $m = \Phi P + \frac{\sqrt{2} \times 4\pi n_2 \mathrm{I}_2}{10}$, where the second term on the right represents the part of the primary magneto-motive force equal and opposite to the back or counter magneto-motive force of the secondary winding, which in commercial transformers, except for small secondary loads, is many times larger than the first term. The values of Φ and P usually change slightly when current flows in the secondary coil.

The effect of magnetic leakage on the action of a transformer is analogous to the effect which would be produced on a transformer without leakage by inserting coils having inductive reactance, *i.e.* **Reactance** or **Impedance coils,** in the primary and secondary circuits outside of the transformer, as is illustrated in Fig. 268. These coils would have such self-inductances as to increase the self-inductances of the primary and secondary circuits in the

ratio of a: 100, where a is the magnetic leakage in per cent. Since leakage causes a proportional increase in the apparent self-inductance of the primary and secondary circuits, it causes an equivalent lag of the currents in the two circuits. The reactance of the windings of a transformer which is caused by the leakage flux may, for convenience, be called the **Leakage reactance**, to distinguish it from the reactance caused by the total of magnetic linkages. The part of the self-inductance, which is due to the leakage flux may also, for convenience, be called the **Leakage inductance**, to distinguish it from the total self-inductance of the windings.

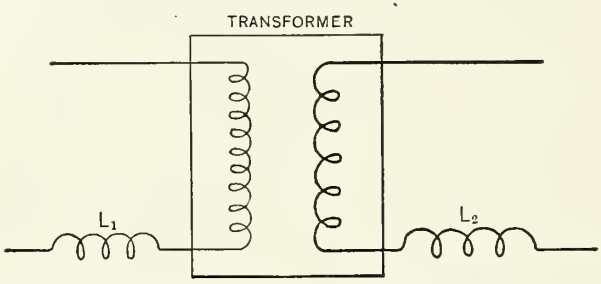


FIG. 268. — Reactance Coils placed to act like Transformer Leakage.

The effect of magnetic leakage in transformers which transform at constant primary voltage — **Constant voltage transformers** — is evidently to interfere with the regulation of the secondary voltage, so that in such transformers the leakage paths are made of as high reluctance as possible. While **Constant current transformers**, that is, those which transform from constant voltage in the primary circuit to constant current in the secondary circuit, depend upon leakage for regulation.* Induction motors are affected in much the same way as constant potential transformers, with which, in principle, they have much in common.†

124. Diagram of Transformer with Magnetic Leakage. Equivalent Circuit. — The effect of magnetic leakage, as discussed in the preceding article, may be shown by a simple diagram, such as Fig. 269. In this diagram, OC represents the vector magneto-motive force caused by the current $\bar{I}_2$, which flows in the secondary circuit when voltage $\bar{E}_1$ (OE) is impressed on the

* Art. 143. † Art. 192.

primary winding, as in Fig. 266, and this magneto-motive force is just balanced by the magneto-motive force produced

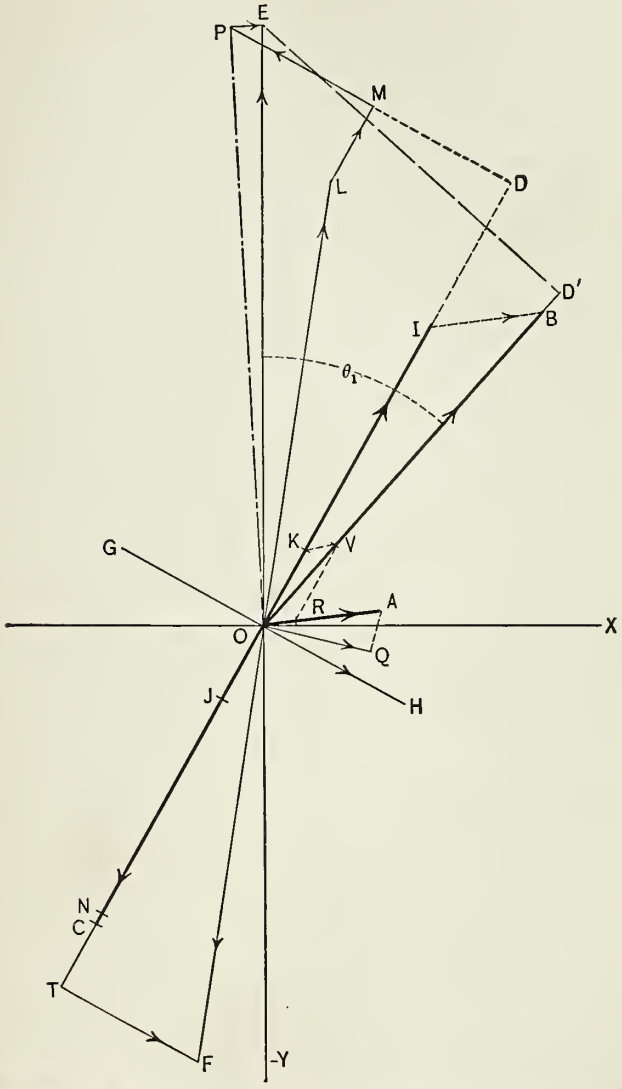


FIG. 269. — Diagram showing the Relations of Currents and Voltages in a Transformer with Large Magnetic Leakage.

by the component $\bar{I}_1'$ (*OI* of the diagram) of the primary current. These magneto-motive forces are equal and opposite to

each other as far as the joint magnetic path through the primary and secondary coils are concerned, the mutual flux being produced by the magneto-motive force of primary component OA; but the magneto-motive forces of $\bar{I}_2$ and $\bar{I}'_1$ act in parallel on the paths of leakage flux, such as the leakage path from A to B in Fig. 267, and set up leakage fluxes which create the reactance-induced voltages E_{1L_1} and E_{2L_2} (OH and OG of the diagram). Except for the effect of the magnetizing current I_μ on $\bar{E}_{1L_1}$, these reactance-induced voltages are opposite to each other. They lag substantially 90° behind their respective inducing magneto-motive forces, since the reluctance of each leakage path is mostly outside of the magnetic material and there is therefore no appreciable iron loss angle of advance of the current.

For complete accuracy, the voltage induced by the primary leakage flux should be taken in quadrature with the total primary current, OB, and it is not exactly opposite to the voltage induced by secondary leakage flux. It may be considered as comprising two components, of which one is OH, in lagging quadrature with OI and due to the leakage flux caused by the OI component of primary current, and the other is in lagging quadrature with OA and is due to the leakage flux caused by the exciting component OA of the primary current. The first of these components is opposite to OG. For simplicity, the second of the components is neglected in the figure, since it is very small in commercial transformers because the exciting current is small. If the highest accuracy is desired, both components may be introduced in the diagram and treated in a manner corresponding to the treatment of OH in the following discussion. The reactance-induced voltages OH and OG are not necessarily of equal numerical values per turn in the windings, because the reluctance of the leakage paths about the primary and secondary coils may not be the same.

The diagram, Fig. 269, is drawn with all vectors reduced to primary equivalents by multiplying secondary voltages and dividing secondary currents by the ratio of the turns in the primary and secondary windings. The counter-induced voltage in the primary winding is the vector sum of $\bar{E}'_{1m}$, the voltage induced by the mutual flux threading both the primary and secondary windings, and represented by OF in the diagram, and $\bar{E}_{1L_1}$, the voltage induced by the primary leakage flux and

represented by OH in the diagram. The secondary induced voltage $\bar{E}_{2m}$ is represented by OF in the diagram. The voltage drops in the secondary circuit when current flows therein are: $OJ = NT$, which is the drop of voltage $\bar{E}_{2R_2}$ in the resistance of the secondary winding when current OC flows; $TF = -OG$, which is the drop of voltage $\bar{E}_{2L_2}$ required to overcome the secondary leakage reactance; and ON, which is the drop of voltage through the load, *i.e.* is the terminal voltage $\bar{E}_2$.

The voltage drops in the primary circuit are: $OL = -OF$, which is the component $-\bar{E}_{1m}$ of impressed voltage required to overcome the counter voltage set up in the primary winding by the mutual flux which threads both windings; $LM + PE = OK + OR = OV$, which is the drop in the resistance of the primary winding caused by the flow of primary current OB; $MP = -OH$, which is the component of impressed voltage $-\bar{E}_{1L_1}$, required to overcome the primary leakage reactance. The impressed voltage is $OE = OL + LM + MP + PE$. The primary angle of lag is θ_1. The angle between the secondary current and secondary induced voltage is angle TOF, but the angle of lag of the secondary current with respect to the terminal voltage is zero in the diagram; that is, the external secondary circuit is assumed to be non-reactive. The vector OD' is the total voltage representing power expended in the primary and secondary circuits, and $D'E$ is the total reactive voltage in primary and secondary circuits.

The active secondary voltage OT is $\bar{E}_{2R_2} + E_{2R} = \bar{E}_{2(R_2+R)}$ which is required to send the secondary current I_2 through the given resistance of the secondary coil and external circuit. The value of $\bar{E}_{2(R_2+R)}$ is determined from the vector diagram comprising itself and the induced voltages $\bar{E}_{2m}$ and $\bar{E}_{2L_2}$, the relation being $\bar{E}_{2(R_2+R)} = \bar{E}_{2m} - \bar{E}_{2L_2}$. The voltage used in the secondary coil resistance is $\bar{E}_{2R_2}$, and $\bar{E}_{2R}$ is the drop in the load. The primary winding, being cut by the same flux that sets up $\bar{E}_{2m}$, has a counter-voltage $\bar{E}_{1m}$ set up in it, in phase with and equal to $s\bar{E}_{2m}$, where s is the ratio of the number of primary turns to secondary turns. The impressed primary voltage $\bar{E}_1$ must have a component equal and opposite to this $\bar{E}_{1m}$. The impressed voltage $\bar{E}_1$ must also furnish the voltage $\bar{E}_{1R_1}$ required to drive the element of the primary current $\bar{I}_1'$, which has a magneto-motive force equal and opposite to that of $\bar{I}_2$, through the primary coil

resistance. It must also have a component $-\bar{E}_{1L_1}$ equal and opposite to the primary leakage voltage $\bar{E}_{1L}$. These three voltages are closed, as indicated in Fig. 269, by the vector voltage represented by OP, forming the vector polygon $OLMPO$. But $\bar{I}_\mu$, the exciting current, joins vectorially with $\bar{I}_1'$, in making up the total primary current $\bar{I}_1$, and $\bar{I}_\mu$ is in advance of $\bar{E}_{1m}$ by an angle of 90° plus the iron loss angle of advance. Its component OQ is in quadrature with OF and the angle AOQ is the iron loss angle of advance. The voltage required in phase with $\bar{I}_\mu$ to drive it through the resistance of the primary circuit is $\bar{E}_c$ $(= OR)$, and this must therefore be vectorially added to the voltage represented by OP. The primary voltage drops from the vector polygon $OLMPEO$, which has the resultant OE equal to E_1. Instead of separating the voltage drop due to the resistance of the primary coil into two components E_{1R_1} and E_c, its combined value in phase with $\bar{I}_1$, the total primary current, might have been used as was done in Fig. 266. The primary and secondary phase angles are θ_1 and θ_2, the latter in this case being zero as the load assumed is nonreactive.

The triangle ODP is especially worthy of notice. The side OD represents the voltage that would be necessary to drive $\bar{I}_1'$ through the resistance of the primary coil R_1 plus the voltage necessary to drive the secondary current I_2 divided by s through the resistances of the secondary coil R_2 and external load resistance R multiplied by s^2. Dividing I_2 by s gives a quotient equal to I_1', and therefore $\bar{I}_1'^2 (R_2 + R)s^2 = \bar{I}_2^2(R_2 + R)$. For this reason $\frac{\bar{I}_2}{s}$, $R_2 s^2$, Rs^2, and $E_{2m}s$ are sometimes called the primary equivalents of the secondary quantities. For like reasons secondary reactances and impedances must be multiplied by s^2 to reduce to primary equivalents; thus, $\frac{\bar{E}_{2m}}{\bar{I}_2} = \frac{\bar{E}_{1m}}{s^2\bar{I}_1'} = \sqrt{(R_2+R)^2+(x_2+x)^2}$, where x_2 and x are the reactances of the secondary coil and load respectively, or $\frac{\bar{E}_{1m}}{\bar{I}'} = s^2\sqrt{(R_2+R)^2+(x^2+x)^2}$. The side OD of the triangle ODP is thus an active component of the voltage represented by OP, while the side DP, 90° therefrom, is reactive

and equal to the primary leakage voltage drop E_{1L} plus the equivalent secondary leakage voltage drop sE_{2L_2}.

Such a triangle of voltages can also be formed if the secondary winding is replaced by a similar coil having s times as many turns and a resistance equal to s^2R_2, connected in series with and magnetically opposed to the primary coil and in series with a load circuit in which the resistance equals s^2R. Such an arrangement is diagramed in Fig. 270. Then OP (Fig. 269) is the

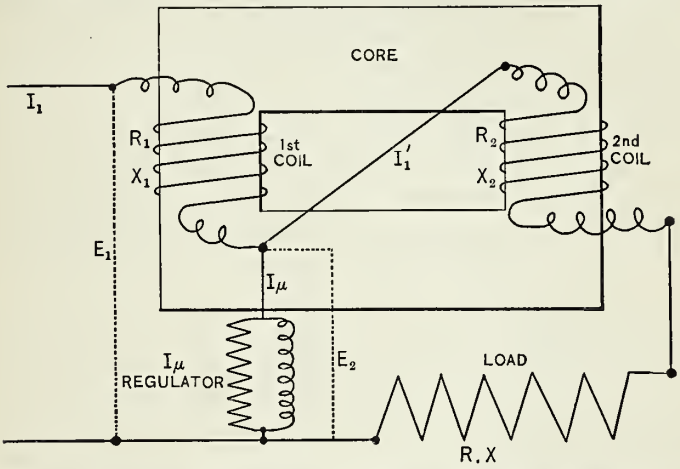


FIG. 270. — A Conventional Arrangement of Coils to absorb the Same Power and produce the Same Reactances as a Transformer of Equivalent Impedance.

voltage measured across the three parts, *i.e.* the first coil, the second coil, and the load; OD is the active voltage, and DP is the leakage reactive voltage, as in the case of the transformer discussed above. Under this arrangement there is no mutual flux in the core, but only the leakage flux. To make the substitution complete, a current equal in scalar value and phase to $\bar{I}_\mu$ must be shunted around the second coil and the external load. The magneto-motive force of this current, when it flows through the first coil to the dividing point, sends the same magnetic flux through the core as $\bar{I}_\mu$ sets up in the transformer, and by which the voltages $\bar{E}_{1m}$ and $\bar{E}_{2m}$ are induced (Fig. 269). This current also combines with the impressed voltage to furnish the power absorbed by the iron losses. The component of the impressed voltage required to drive $\bar{I}_\mu$ through the primary resistance is then $\bar{E}_c$; this combined vectorially with the voltage

represented by OP equals the impressed voltage $\bar{E}_1$, which is required to make the currents $\bar{I}_1$ and $\frac{1}{s}\bar{I}_2$ flow respectively through the first and second coils, analogously to the transformer. The counter-voltages set up in the two coils by the mutual flux which threads both coils exactly counterbalance each other, and have no effect on either the voltage or current of the supply wires or the load, so for purposes of analogy they may be considered to be absent. The combination of resistance and reactance necessary to shunt the current $\bar{I}_\mu$ around the second coil and load is indicated in Fig. 270 by the part marked I_μ Regulator.

In this manner two coils in series opposition have been substituted for the transformer reduced to terms of unity transformation, in which the equivalent voltages for the transformer are impressed on the system, driving equivalent currents at the same angles of lag, and expending the same power in each coil and in the load.

If the secondary load of a transformer is reactive, an inspection of Fig. 269 will show that θ_2 and θ_1 tend to increase, while with a load containing electrostatic capacity these angles tend to lessen, or to reverse in sign if the capacity is sufficiently large.*

The leakage fluxes and internal losses of voltage indicated by Fig. 269 are, for the purpose of obtaining a clear diagram, made much greater in proportion to the impressed and mutual induced voltages than would be found in commercial constant voltage transformers. However, the leakage fluxes used for the diagram are not necessarily excessive for conditions in induction motors and constant current transformers, to which the diagram is also applicable. Likewise, the exciting current I_μ has been given a large value in Fig. 269, for the same reasons. It is usually so small in constant voltage transformers that the voltage drop OP may be considered numerically equal to and of the same phase as OE (Fig. 269) for the solution of such problems as usually arise in commercial circumstances. However, when the load current considered is a small fraction of the full load current or the transformer is special, so that in either case the drop due to the exciting current is material, the exact triangle representing apparent primary impedances and resultant

* Art. 131.

active and reactive voltages is $OD'E$. In this triangle, when scaled as a triangle of voltages, the side in quadrature to the current $\bar{I}_1 = OB$ must be composed of the quadrature projection of the drops of primary voltage that are due to those induced in the primary and secondary coils and load and is equal to $D'E$. It will be here remembered that OH, when fully expressed, is at right angles to OB. The active voltage OD' in phase with $\bar{I}_1$ must be composed of the projection of the active drops of voltage caused by the work done in driving the current through the primary and secondary coils and the load. When the triangle $OD'E$ is scaled to represent impedances, *i.e.* voltages divided by currents, the line OE is the total apparent impedance, OD', the apparent resistance, and $D'E$, the apparent reactance offered by the two windings and the load of the transformer. The power input is of course $E_1 I_1 \cos \theta_1 = OE \times OB \times \cos \angle EOD'$ and the output $E_2 I_2 \cos \theta_2 = OC \times ON$.

125. Transformer Exciting Current. — In the case of an ideal transformer, that is, one without losses or magnetic leakage, the lag of the primary current, when the secondary circuit is open, is 90° with respect to the impressed voltage; for R_1 and $\cos \theta_1$ are assumed to be zero. Since the induced secondary voltage lags behind the magnetism, which is in phase with the primary current when the secondary circuit is open, by an angle of 90°, the phases of the primary impressed voltage and the secondary induced voltage are exactly 180° apart; that is, *they are in exact opposition.* The current in the primary circuit of an ideal transformer, when the secondary is open, is all reactive (*i.e.* wattless), and of a magnitude which depends only upon the total inductance L' of the primary coil.

On the other hand, the losses due to hysteresis and eddy currents in the iron core and to resistance in the primary coil are by no means negligible in commercial transformers, but are of such a magnitude as to decrease the lag of the primary current until the power factor of the primary circuit is ordinarily between 50 per cent and 70 per cent when there is no current in the secondary circuit; but the magnetism in the core remains nearly in phase with the reactive component of the primary current and is almost 90° behind the phase of the primary voltage, so that *the primary impressed and secondary*

induced voltages are still almost in opposition. The current which flows in the primary circuit when the secondary circuit is open may therefore be considered as composed of two components, one of which supplies the power required to make up the transformer losses, and the other of which serves simply for setting up the magnetism, and is therefore wattless. The primary current which flows when the secondary circuit is open is sometimes called the leakage current, open circuit current, or magnetizing current. In this volume, however, the more satisfactory term, **Exciting current**, is used, the term magnetizing current being reserved to apply only to the magnetizing (*i.e.* reactive) component of the exciting current.

It is desirable to look a little farther into the effect upon the cyclic curve of iron loss which is produced by substituting a sinusoidal exciting current for the irregular current which actually flows, as has been done in the transformer diagram of Figs. 265, 266, and 269, and others following. In Fig. 271, ΦΦ is a

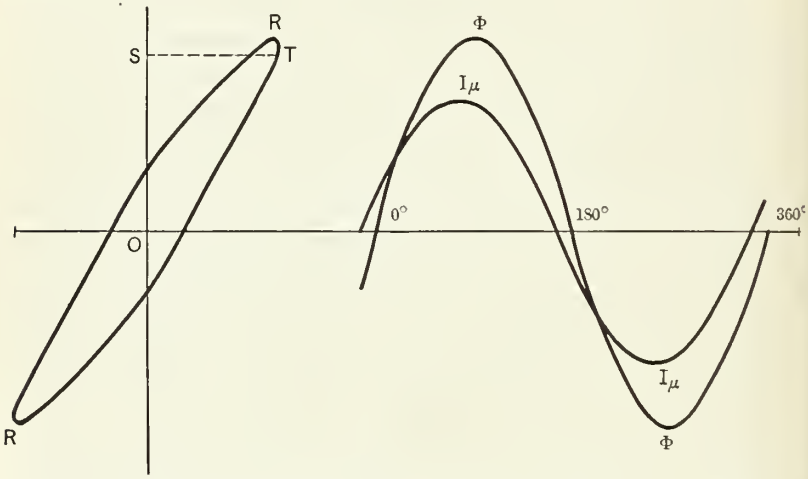


Fig. 271. — Conventional Cyclic Curve of Iron Loss, constructed from the Curve of Flux in a Transformer Core and the Equivalent Sinusoid of Exciting Current.

sinusoidal curve of magnetic flux which is set up in a transformer core, and $I_\mu I_\mu$ is the sinusoidal curve of exciting current, leading the curve ΦΦ by an iron loss angle of 18°. From these curves the cyclic curve is constructed as heretofore explained,*

* Art. 114.

with its ordinates and abscissas equal respectively to magnetic flux and exciting current. The shape of the cyclic curve (RR) is not that of a true cyclic curve of iron loss, as may be seen by the fact that the maximum value of the exciting current ST does not occur at the point of maximum flux, a condition which could only occur in fact if the eddy current part of the iron loss was relatively greater than is found in commercial transformers. But if curve $I_\mu I_\mu$ is the equivalent sinusoid of the actual irregular exciting current, the area of RR must be equal to the area inclosed by the true cyclic curve of iron loss. It is thus seen that the substitution of an equivalent sinusoid for the actual curve of exciting current can usually be made without involving serious error.

126. The Effects of Variable Reluctance, of Hysteresis, and of Eddy Currents on the Form of the Primary Current Wave. — In the preceding discussions it has been assumed that the reluctance of the magnetic circuit of a transformer can be taken at an average constant value which is practically equal to the value when the current is at its maximum point. The low maximum value of the magnetic density which is used in commercial transformers as ordinarily constructed makes this assumption allowable, though it is by no means exact; and if the magnetic density is pushed above the bend in the curve of magnetization, the influence of the lowered permeability of the iron becomes marked. The curve OM in Fig. 272 may be taken to represent the curve of magnetization of iron in a transformer core, plotted with volts induced in the primary windings as ordinates and exciting ampere-

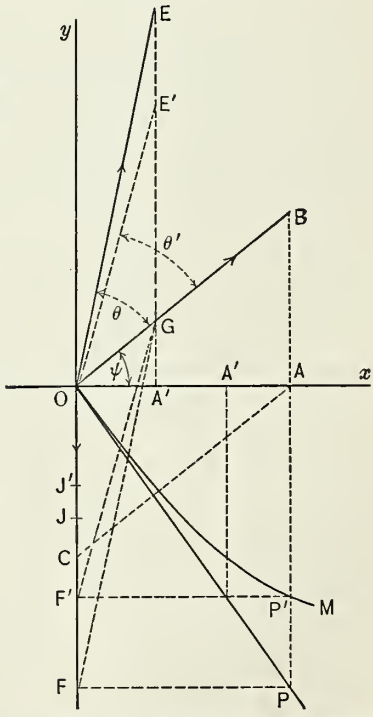


FIG. 272. — Diagram showing the Effect of Variable Permeability, with Iron Loss Negligible.

turns as abscissas, supposing the effect of hysteresis, eddy currents, and magnetic leakage to be negligible, and line OP may be taken to represent a corresponding hypothetical curve of magnetization, assuming the effect of saturation to be negligible, *i.e.* the reluctance to be constant. Vectors OB and OC are the primary and secondary ampere-turns respectively for a given load. Then when the magnetizing ampere-turns equal $n_1\bar{I}_\mu$, represented by OA, the induced voltages in the primary and secondary coils are reduced by the effect of saturation from OF and OJ, which would be reached with a constant reluctance, to OF' and OJ'. And, conversely, if the induced voltages are to be OF' and OJ', OA ampere-turns are required to bring the magnetism up to the requisite value when the curve of magnetization is OM, while OA' ampere-turns would be sufficient if the magnetic circuits were not influenced by saturation.

The construction shows that the saturation of the iron makes necessary more turns of wire on the primary and secondary coils in order that a given output may be obtained. Since, in this case, the permeability varies through each period with the magnetizing ampere-turns, there is a periodic variation of θ, and the primary current wave is distorted from the form of the primary impressed voltage wave. This is shown clearly in Fig. 261 of Art. 114 and is there described.

When the transformer has its secondary circuit open, the drop of voltage due to the exciting current flowing through the primary winding is usually negligible, so that the primary impressed voltage at each instant is proportional to the tangent of the curve of magnetism; and is 90° in advance of the magnetization when the latter is sinusoidal. The tangent relations between the curves of flux and voltage * must exist as long as I_1R_1 is negligible. The effect on the output of impressing excessive voltages upon the primary winding of a transformer is indicated clearly in Fig. 272. Thus, there is but slight rise of magnetic flux in the curve OM beyond the point M (near saturation) for increases of exciting current. If, therefore, OE is much increased, the exciting current must increase excessively. It is thus seen that if transformers, or other apparatus depending upon counter-induced voltage, are subjected to voltages that demand rise of core flux beyond the point

* Art. 115.

of saturation, the exciting currents demanded may reach dangerous or destructive proportions. Another important reason for having transformers designed for low flux densities in the core is because an excessive exciting current such as spoken of above will cause a material resistance drop in the primary, while high core reluctance will cause relatively high leakage flux, thus tending to interfere seriously with the regulation.

As a very simple example showing that the primary induced voltage is always equal and opposite to the impressed voltage when I_1R_1 is negligible, and that the exciting current varies in such a way as to furnish this induced voltage in a transformer with the secondary circuit open, we may consider an inductance coil of negligible electrical resistance with a sinusoidal voltage E_1 of, say, 100 volts impressed upon it. Suppose the frequency is 60 periods per second, and the self-inductance L_1 is .01 of a henry, then $2\pi fL_1 = 3.77$ ohms is the impedance. The current I_1 flowing is $\frac{E_1}{2\pi fL_1} = 26.5$ amperes with a lag angle of 90°. The induced voltage E_{1m} is $2\pi fL_1I_1 = 100$ volts, which is equal and opposite to the impressed voltage. Suppose the self-inductance L_1 changes to $L_1' = .005$; then the current becomes $I' = 53$ amperes; but the induced voltage E_{1m} is $2\pi fL_1'I_1' = 100$ volts, which is the same value as before. It is seen, in this case of negligible resistance, that whatever value the self-inductance may have, the exciting current takes such a magnitude that the counter-voltage of self-induction is numerically equal to the impressed voltage. Since the self-inductance of the electric circuit is inversely proportional to the reluctance of the magnetic circuit, when the reluctance changes, the exciting current changes, so that, as before, the counter-voltage equals the impressed voltage. Figure 262, described in Art. 114, shows that when core losses and copper losses are present, the exciting current is advanced in phase so that it is less than 90° behind the impressed voltage, and that the summation of the products of the instantaneous values of impressed voltage and current is no longer equal to zero for the inductance coil, but becomes equal to the sum of the core losses and the I^2R losses.

The effect of hysteresis in the core of a transformer is to distort the form of the primary current wave to a still more marked

degree than would occur from the effects of magnetic saturation without hysteresis, and the higher the maximum magnetic density is carried, the greater the distortion becomes. The ordinates of the primary current wave are at each instant proportional to the difference between the corresponding ordinates of the wave of primary impressed voltage and the wave of induced counter-voltage. The latter is, in an ordinary transformer, practically similar to the form of the secondary voltage wave. With the primary impressed voltage sinusoidal and the reluctance of the magnetic circuit uniform, the primary current wave would be sinusoidal. With a variable reluctance, but no hysteresis, the current wave becomes peaked, but remains symmetrical; but when hysteresis is taken into account, the symmetrical form is lost. This is illustrated in Figs. 261 and 262.*

The effects which eddy currents in the core have upon the exciting current are shown in Fig. 257, which is described in Art. 112, and in Fig. 263, described in Art. 114. The instantaneous value of the portion of the exciting current which is required to make up the losses due to eddy currents at any instant is equal to the corresponding instantaneous value of the eddy current loss divided by the instantaneous value of the primary voltage. The total core loss may be shown by plotting a cycle of the core flux having abscissas equal to the arithmetical sum of the corresponding abscissas of the hysteresis and eddy cycles (Fig. 263). The total transformer exciting current may be plotted from this as shown by I' in Fig. 263.

The eddy current loss is similar to that which would be produced by a closed secondary circuit of one turn of appropriate electrical resistance, and is unlike hysteresis loss, which is determined by the magnetic quality of the magnetic core. However, eddy current loss causes an increase in the angle of advance of the exciting current, as also does current in the secondary coil.† Moreover, the loss caused by the exciting current in the resistance of the primary winding — usually very small — causes a still further advance of the exciting current.‡ The phase of the exciting current is thus caused to be less than 90° behind the impressed voltage on account of (*a*) the hysteresis loss, (*b*) the eddy current loss, and (*c*) the resistance loss in the primary coil caused by the flow of exciting current.

* Art. 114. † Art. 126. ‡ Art. 123 *a*.

The angle by which the exciting current is advanced from lagging quadrature with respect to the voltage is called in this volume the **Excitation Angle.** Some inaccurately use the term hysteretic angle of advance to express this relation, instead of confining the use of the latter term to the advance caused by hysteresis alone.

If the eddy current cycle has a large area, its effect may cause an advancement of the instantaneous maximum point in the exciting current. This would be the case, referring to Fig. 263, if the current curve I_f had a maximum greater than the magnetizing current curve I_μ.

127. Forms of the Primary Current Waves as affected by Current in the Secondary Winding. — When the secondary circuit is closed, the form of the primary current is changed by the effect of the secondary current. In Fig. 273, curve e_1 represents the primary impressed voltage; $n_1 i_\mu$ represents the exciting current times the primary turns. Now, if by closing the secondary circuit through resistance free of reactance, a current $\bar{I}_2$ is caused to flow and the instantaneous values of its ampere-turns are represented by the curve $n_2 i_2$ — the effects of magnetic leakage are here supposed to be negligible — the effect of the current flowing in the secondary circuit is to cause a corresponding increase in the current flowing in the primary circuit.* The ampere-turns of this increase of the primary current may be represented by curve $n_1 i_1'$. The total primary wave of ampere-turns is represented by curve $n_1 i_1$, and is the sum of $n_1 i_\mu$ (the exciting ampere-turns) and $n_1 i_1'$. The pri-

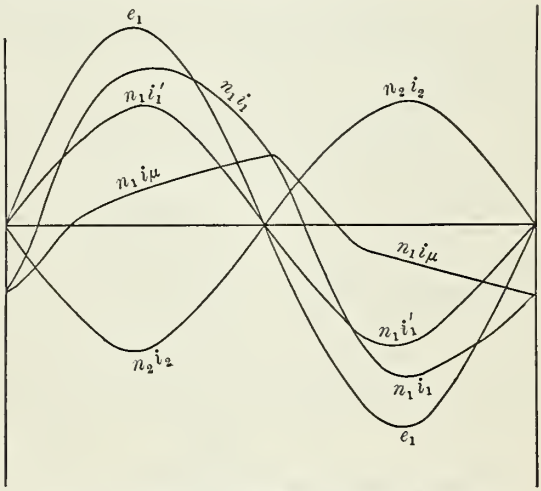


FIG. 273. — Curves showing Effect on Primary Current of a Transformer caused by Current in a Non-reactive Secondary Circuit.

* Art. 118.

mary current and its components may be directly shown from this curve by a single change of scale. It is thus shown by the figure that the secondary current, when in phase with the secondary voltage, tends to reduce the distortion and lag of the primary current.

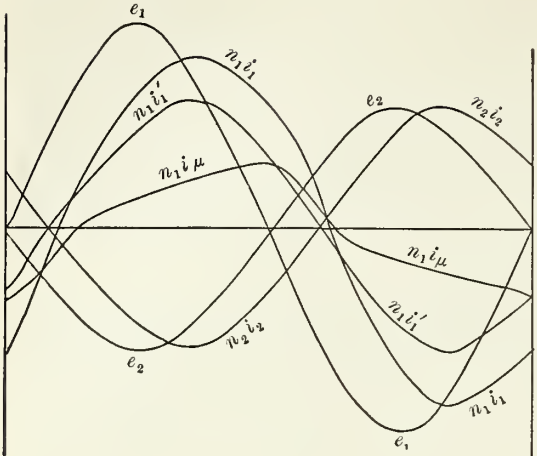


FIG. 274. — Curves showing Effect on Primary Current of a Transformer caused by Current in an Inductive Secondary Circuit.

If the secondary circuit is inductive, the effect is altered so that the lag of the primary current is larger, as shown in Fig. 274. In this case $n_2 i_2$ lags behind e_2; and $n_1 i_1'$ also lags behind e_1. The sum of $n_1 i_1'$ and $n_1 i_\mu$, which gives $n_1 i_1$, is therefore in a lagging position with respect to e_1, the angle of lag being largely determined by the angle of lag of i_2 with respect to e_2 where $n_1 i_\mu$ is small in comparison with $n_2 i_2$.

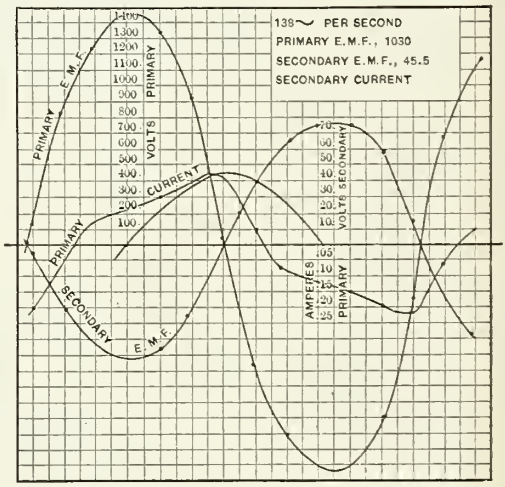


FIG. 275. — Experimentally Determined Curves of Current in a Transformer with Open Secondary Circuit.

If the secondary current leads the secondary induced voltage, the curve corresponding to $n_2 i_2$ leads e_2, and $n_1 i_1$ (which is the sum of $n_1 i_1'$ and $n_1 i_\mu$) also leads e_1 when $n_1 i_\mu$ is small in comparison with $n_2 i_2$.

Figures 275 and 276 show transformer curves experimentally observed by Professor Ryan in 1889,* using sinusoidal impressed voltage. These show a striking resemblance to the hypothetical curves built up from the loss cycles. The unmarked half sinusoid in Fig. 275 is a half cycle of the magnetic wave, which is in lagging quadrature with the primary impressed voltage.

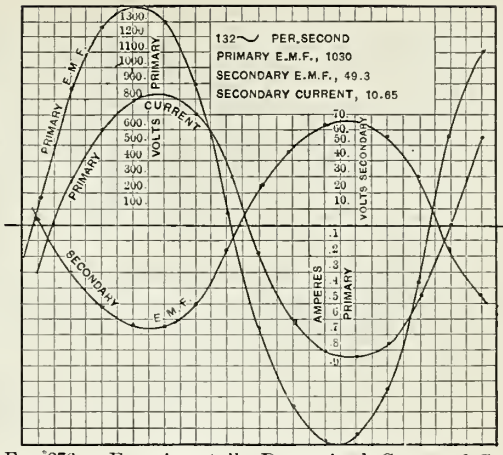


FIG. 276. — Experimentally Determined Curves of Current in a Transformer with slightly Inductive Full Load in Secondary Circuit.

128. Core Magnetization. — The maximum magnetic density during a period, in the iron core of a transformer, is dependent upon the maximum value of the magnetizing component of the exciting current in the primary circuit, and is equal to

$$\Phi = \frac{4\pi n_1 i_{\mu_m}}{10\,P} = 1.257\,\frac{n_1 i_{\mu_m}}{P};$$

where i_{μ_m} is the maximum value of the magnetizing current and P is the reluctance of the magnetic circuit, in gilberts, at the time that the current is maximum.

Assume, for a moment, that the transformer has a constant reluctance, no magnetic leakage, and neither iron nor copper loss, which is, of course, an ideal case. Then,

$$\Phi = \frac{4\sqrt{2}\pi n_1 I_1}{10\,P} = 1.777\,\frac{n_1 I_1}{P},$$

when the secondary circuit is open. Closing the secondary circuit so that a current may flow in it under the impulse of the induced secondary voltage materially changes the conditions. We will neglect hysteresis and eddy current losses in the iron core and resistance losses in the windings, and first assume that the secondary circuit is without reactance in which

* *Trans. Amer. Inst. E. E.*, Vol. 7, p. 1.

case the secondary current I_2 will be in unison with the induced or secondary voltage E_2. This current has its own magnetizing effect on the magnetic circuit. If i_1 and i_2 are the primary and secondary currents at any instant, the total magnetomotive force in the circuit at that instant is

$$\frac{4\pi(n_1 i_1 + n_2 i_2)}{10} = \Phi_i P; \tag{1}$$

where Φ_i is the instantaneous value of the magnetic flux, and P is the assumed constant value of the reluctance of the magnetic circuit. From this is found

$$i_1 = \left(\frac{10\,P\Phi_i}{4\pi n_1}\right) - \left(\frac{n_2 i_2}{n_1}\right). \tag{2}$$

If a sinusoidal voltage is impressed upon the primary winding, which under the assumed conditions results in sinusoidal waves of magnetism and current, the instantaneous primary current is

$$i_1 = \frac{10\,P\Phi}{4\pi n_1}\sin\alpha - \frac{n_2\sqrt{2}\,I_2}{n_1}\cos\alpha; \tag{3}$$

but

$$\frac{10\,P\Phi}{4\pi n_1} = \sqrt{2}\,I_\mu,$$

whence

$$i_1 = \sqrt{2}\left(I_\mu \sin\alpha - \frac{n_2}{n_1} I_2 \cos\alpha\right), \tag{4}$$

where α is measured from the zero instant of the cycle of flux, which in this case is in unison with I_μ since losses are assumed to be absent. Whence

$$n_1 i_1 = \sqrt{2}(n_1 I_\mu \sin\alpha - n_2 I_2 \cos\alpha),$$

and

$$\begin{aligned} n_1^2 I_1^2 = \frac{2}{\pi}\int_0^\pi (&n_1^2 I_\mu^2 \sin^2\alpha \\ &- 2\,n_1 n_2 I_\mu I_2 \sin\alpha\cos\alpha \\ &+ n_2^2 I_2^2 \cos^2\alpha)\,d\alpha, \end{aligned} \tag{4}$$

where I_1 is the effective value of the primary current when the secondary current is equal to I_2; and I_μ, as before, is the wattless primary current when the secondary circuit is open. Performing the integration gives

$$n_1^2 I_1^2 = n_1^2 I_\mu^2 + n_2^2 I_2^2. \tag{5}$$

Remembering that I_μ and I_2 have, in this case, 90° difference of phase, the three terms of this formula may be represented by the three sides of a right-angled triangle, as in the triangle formed by the points OAB in Fig. 272, in which excitation losses are assumed absent. The current I_1 amperes, which flows in the primary circuit when the secondary circuit is loaded, is in advance of the current I_μ by an angle ψ, the tangent of which is shown by the figure to be $\tan \psi = \frac{n_2 I_2}{n_1 I_\mu}$. As I_μ is here a wattless magnetizing component of the primary current, losses being absent by assumption, it is directly dependent on the reluctance of the magnetic circuit and the counter-voltage in the primary coil. As the reluctance is usually small, I_μ is small.

When the transformer has an iron core, the exciting current is no longer sinusoidal or in exact leading quadrature with the secondary induced voltage, but is distorted and is advanced by the amount of the excitation angle.* Call this angle η, and consider it the advance of the equivalent sinusoid substituted for the irregular curve of exciting current. This substitution can be made with sufficient accuracy for this discussion. The magnetism remains, as before, 90° ahead of the induced voltage although reluctance varies, so that the following relations obtain:

$$\frac{10 P\Phi}{4\pi n_1} \sin \alpha = \sqrt{2}\, I_\mu \sin (\alpha + \eta), \tag{6}$$

I_μ being the primary exciting current.

Substituting, as before, in (3) gives

$$\begin{aligned} n_1^2 I_1^2 &= \frac{2}{\pi} \int_0^\pi (n_1^2 I_\mu^2 \sin^2 (\alpha + \eta) \\ &\quad - 2\, n_1 n_2 I_\mu I_2 \sin (\alpha + \eta) \cos \alpha \\ &\quad + n_2^2 I_2^2 \cos^2 \alpha)\, d\alpha, \end{aligned}$$

or,

$$\begin{aligned} n_1^2 I_1^2 &= n_1^2 I_\mu^2 + n_2^2 I_2^2 - 2\, n_1 n_2 I_\mu I_2 \sin \eta \\ &= n_1^2 I_\mu^2 + n_2^2 I_2^2 - 2\, n_1 n_2 I_\mu I_2 \cos (90° - \eta). \end{aligned} \tag{7}$$

This is the formula of a triangle where $90° - \eta$ is the angle included between the sides representing $n_1 I_\mu$ and $-n_2 I_2$, such as

* Art. 125.

OA and AB of the triangle OAB of Fig. 266, in which the effect of excitation losses is shown.

129. The Circle Diagram for Non-reactive Secondary Circuit, and Active Voltage Locus. — In a constant voltage transformer the load is zero when the resistance of the secondary circuit is infinite, and the load increases as the secondary resistance decreases through the range of normal currents for which the transformer is designed. Then, if leakage reactance is considered constant for all loads, as may be done without appreciable error when the frequency of the exciting current is constant, its relative effect in causing voltage drop will increase with increase of load. These relations may be neatly shown by a construction similar to that discussed before.* Thus, in Fig. 277, OC represents the impressed voltage of a transformer in which, for convenience, a unity ratio of transformation is assumed. The right-angled triangle OGC is then constructed on OC. The side GC is made equal to the drop of impressed voltage due to the reactive voltage CF of the primary coil and FG of the secondary coil or $-GC = \bar{E}_L = \bar{E}_{1L_1} + \bar{E}_{2L_2}$. The side OG is made equal to the sum of JG, the voltage expended in the resistance of the primary coil R_1 on account of the flow of the current I_1', which is equal and opposite to the secondary current I_2, the voltage KJ, which is equal and opposite to the voltage expended in the secondary coil resistance R_2, and the voltage OK, which is equal and opposite to the

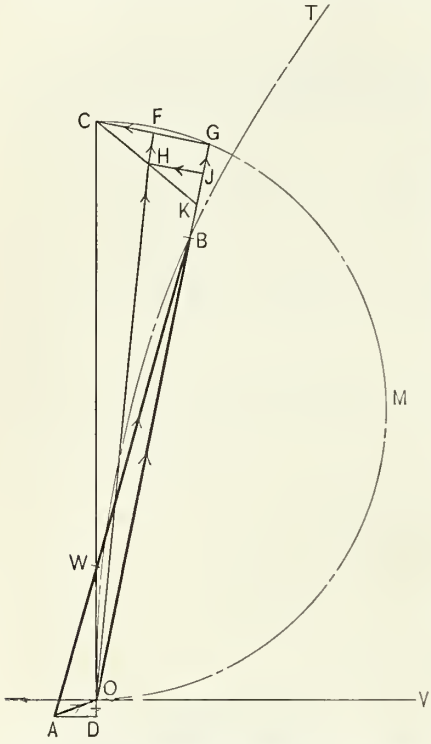


FIG. 277. — Transformer Diagram for a Transformer containing Leakage Reactance and Non-reactive Secondary External Circuit.

*Art. 85.

voltage expended in the load resistance R. In obtaining these voltage drops the exciting component of the primary current is neglected for simplicity, as its effect is ordinarily small. Therefore

$$OG = \bar{E}_{1R_1} = -\bar{I}_2 (R_1 + R_2 + R) = \bar{I}_1' (R_1 + R_2 + R),$$

the ratio of transformation being taken as unity. The sides OG and GC thus form the right-angled triangle of active and reactive voltages in the transformer.

Since GC increases with the current and angle OGC must remain a right angle for any current,* the locus of the point G, which moves as the current changes, is the arc of a circle with OC as its diameter. In a well-made commercial transformer with its primary and secondary coils divided into parts and sandwiched together to avoid leakage, the line GC is relatively very short even at full load, so that the point G travels only a short distance from C as the secondary current is increased from zero to the full load current. The point G would be at the point C for all loads if the load and coil reactances were zero.

The secondary terminal voltage $KO = RI_2$, the voltage in the resistance $JK = \bar{E}_{2R_2}$, or the reactance voltage $FG = \bar{E}_{2L_2}$ of the secondary coil may readily be obtained for any ratio of transformation by merely using a scale equal to $\frac{1}{s}$ times that used in the original construction and reversing the secondary circuit voltage or current arrow shown in the figure. The total induced voltage due to the mutual magnetism of the primary and secondary windings is $HO = \bar{E}_{2m} = \bar{E}'_{1m} = -\bar{E}_{1m}$, the ratio of transformation being unity. The voltage drop in the primary coil resistance, due to the current component opposite to the secondary current, is $JG = HF = \bar{E}_{1R_1}$, and the voltage drop by reason of the primary coil reactance is $FC = E_{1L_1}$.

The component of primary current used for excitation is indicated by the line AO, and is composed of an active component DO, in phase with OC and a quadrature component AD in phase with the core magnetism. The component DO is of such value that when multiplied by OC, the product equals the excitation losses. The exciting current is assumed for the purpose of the diagram to be constant in scalar value and phase position, which

* Art. 85.

is sufficiently accurate for most practical cases. Its assumed constant value is usually taken as the amperes flowing in the primary circuit when the secondary circuit is open, *i.e.* the **Open circuit current.** On the other hand, the current $\bar{I}_2$ of the secondary coil and its opposing element $\bar{I}'_1$ of the primary vary directly with the load or inversely with the impedance. For the present we will continue to consider the load non-reactive and the leakage reactances constant. Then we have a case equivalent to a circuit having variable resistance and fixed inductive reactance. In such a circuit, as was proved earlier,* the locus of the current vector is a semicircle, located in the first trigonometrical quadrant, with its diameter on the X-axis, one end being at the origin. The diameter of this semicircle was shown to be equal to the impressed voltage divided by the reactance. In Fig. 277, OC may correctly be used as the voltage absorbed in the primary, secondary, and load impedances, by the flow of the secondary current, or its primary opposing component, as heretofore explained. Then a semicircle with its center on OV extended, as shown at OBT, with a diameter equal to OC divided by the combined leakage reactances $\left(\frac{GC}{I'_1}\right)$, that is, equal to $\frac{E_1}{X_{1L_1} + X_{2L_2}}$, is the locus of the end of OB $(=I'_1)$ or of the secondary current vector reversed. In this case, X, the reactance of the external secondary load is assumed to be zero, that is, the load is non-reactive. When the primary triangle of voltages is OGC, the secondary current is BO, which is in phase with the active element of the voltage, $JO = E_{2(R_2+R)}$. The primary current is then the vector $AB = AO + OB$. The secondary current can be read directly from the diagram for any ratio of transformation by using a scale with s times as many amperes to the inch as that used in the construction.

If the impressed voltage, kilowatt capacity, exciting current as a percentage of full load current, core losses in percentage of kilowatt capacity, and the resistances and reactances of the coils and load of a transformer are known, the construction, such as shown in Fig. 277, can at once be made. From this the primary and secondary currents are scaled off on AB and BO in amperes. The angle of lag of the primary

* Art. 70, Case (1).

current is the angle BWC. The angle of lag of the secondary current is the angle between the terminal voltage $KO = \bar{E}_2$ and the current $BO = \bar{I}_2$, which angle is zero under the circumstances here assumed. The per cent regulation for the given load is one hundred times the ratio of $OC - OK$ to OK, assuming unity ratio of transformation. The various reactance and resistance drops are measured along GC and OG respectively. The power W_i received from the mains is OC times the projection of AB thereupon, or $W_i = E_1 I_1 \cos \theta_1$. The power absorbed in the load is KO times BO or $W_o = E_2 I_2$, which, though obtained in the diagram for a transformer with ratio of transformation (s) equal to unity is also true without change when s has any other value, since the primary equivalent voltage is divided by, and the current multiplied by, s to reduce to secondary values. The total power losses in the transformer are: exciting current losses $DO \times OC$; the primary coil copper loss due to current OB, $JG \times OB$; and the secondary copper loss, $JK \times BO$. The sum of these losses subtracted from the power input equals the power output. The commercial efficiency of the transformer is the total output divided by the total input or $\eta = \frac{W_o}{W_i}$.

Changing the position of B to various points on its locus OBT and moving G along the semicircle OMC so that OG is kept in the same direction as OB, permits the determination of the entire transformer performance for any value of the secondary current or fraction of the rated capacity of the transformer.

When the reactive voltage GC is very short, as in the case of a well-made commercial transformer supplying, for example, an incandescent lamp circuit, the line $OKJG$ almost coincides with the line OC. In this case the locus OBT has such a large diameter that it is almost a vertical straight line from no load to full load; then, if OA, for the particular character of the calculation in question, may be considered of negligible length, KO, AB, and BO may be considered without serious error to vary with the load up and down the line OC, and the calculation of the transformer regulation resolves itself into the simple problem of three resistance drops in series. The three resistances causing these drops are the fixed resistances of the primary and secondary coils and the variable resistance of the load.

In the foregoing discussion AO was considered constant. Though approximately true this is not exactly the case, as the magnetizing current must be proportional to the total mutually induced voltage HO, and leading it by an angle of 90° plus the excitation angle of advance. An approximately accurate correction to calculations can be made, when necessary, by giving AO, for each load, such a length that it will have the same ratio to OH that the open circuit current has to OC, and also turning it to such a position that it maintains the constant angular relation to OH. Inasmuch as the construction of the length OH is dependent upon the resistance drop HF, the length and phase of HF must be corrected if great accuracy is desired so as to include the drop of voltage due to excitation losses. The diagram presupposes, also, that the reluctance of the core is constant for all loads, and that the leakage paths are free from iron losses. These assumptions for commercial transformers introduce no great error. The further assumption that the voltages and currents are sinusoidal may, under some circumstances, require correction, as when resonance is present for some of the harmonics. The effects of irregular waves of voltage and current will be treated later. For ordinary transformer calculations it is usual to assume that the voltage and currents are sinusoidal.

130. Circle Diagram where the Transformer Load contains Constant Reactance and Variable Resistance. — Suppose now, without changing the unity ratio of transformation or other assumptions, that the load contains inductive reactance X, then the construction shown in Fig. 277 may be conveniently modified as in Fig. 278. Here the component of the impressed voltage OC, used in overcoming the total reactive voltage, is QC, in which CF and FG are the primary and secondary coil reactive voltages respectively and GQ the load reactive voltage transferred to the primary circuit. The total active voltage in phase with the component of the primary current which is equal and opposite to the current in the secondary coil is OQ. This is expended in the resistance of the coils and load. The secondary terminal voltage is KO, and is composed of KK', which is equal and opposite to the reactive voltage of the load, and $K'O$, which is the voltage drop in the load resistance. The secondary angle of lag θ_2 is the angle $K'OK$. The dia-

gram is similar in construction to that of Fig. 277, and its further description and use may be obtained by the preceding discussion concerning the latter.

Electrostatic capacity in the load tends to neutralize the self-inductance of the primary and secondary coils.* Thus, if capacity is introduced into the secondary circuit of such value that the capacity voltage equals the inductive voltage GQ in Fig. 278, the reactive voltage remaining is CG, and the construction becomes like Fig. 277. If the capacity increases, the point G moves toward C, reaching it when the capacity voltage equals the inductive voltage CQ. Because of this increase of capacity, the diameter of the current locus OBT changes from the value $\frac{E_1}{X_{1L_1} + X_{2L_2} + X_L}$ to the value $\frac{E_1}{X_{1L_1} + X_{2L_2} + X_L - X_c}$, where X_c is the capacity reactance, and the diameter becomes equal to $\frac{E}{0} = \infty$ when the capacity reactance in the denominator is equal to the sum of the inductive reactances. For each increase in capacity a locus corresponding to OBT must be drawn with a diameter larger than the last, until the capacity balances the self-inductance, when the diameter becomes infinite so that the arc of the circle coincides with OC. When the capacity voltage exceeds CQ, the system acts as one having resistance and capacity alone, and the current takes a leading angle determined by the excess of capacity reactance; then the point Q has a

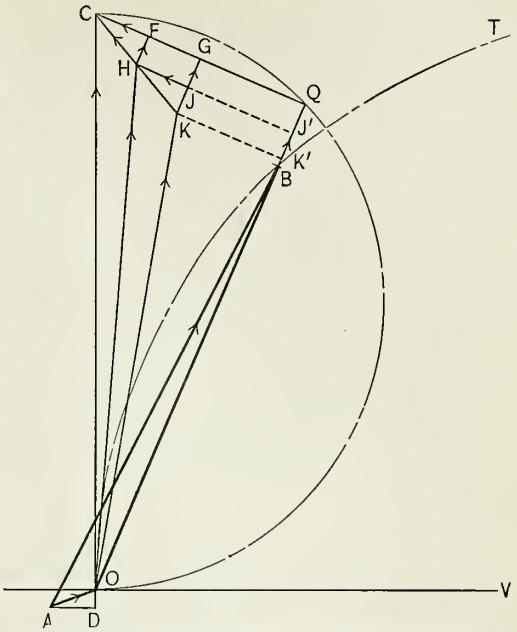


Fig. 278. — Transformer Diagram showing the Effect of Inductive Reactance in the Secondary External Circuit.

* Arts. 81 and 82.

locus on the left of OC, as shown in Fig. 279. Thus the current locus is a semicircle to the left of OC, such as $OB_1'T''$, and has a diameter equal to the impressed voltage divided by the net excess of capacity reactance. This is in accord with the theorem earlier proved * for a circuit having constant capacity-reactance and variable resistance. Otherwise the diagram is essentially as in the case of an inductively reactive circuit, except that the primary current is somewhat smaller and the angle of lead somewhat less than would have been the current and lag angle for an equal net amount of inductive reactance instead of capacity reactance. This is because the exciting component of the current, which need move its vector position only slightly under ordinary conditions and is here considered stationary, is in a different phase position in the two instances with relation to the remaining component of the primary current.

Figure 279 shows the conditions assumed above when the secondary current is of the same value as in Figs. 277 and 278. The lettering is similar in the three figures. In Fig. 279 is shown the impressed voltage OC; the active voltage locus for inductive and capacity reactances, which is the circle $OMCN$; the current locus OBT where the net predominance of inductive voltage is CQ, as in Fig. 278; the current locus $OB_1'T''$ where the net predominance of capacity voltage is CQ_1'; and the current locus OCT'' when inductive and capacity voltages neutralize each other or are both negligible.

To determine the lengths and phase positions of the lines representing the vectors of mutually induced primary voltage and the secondary terminal voltage, take, for example, the internal and load reactive voltages of Fig. 278, which are reproduced in the line CQ of Fig. 279, and suppose a net excess of capacity reactive voltage is represented by the line CQ_1'. To give this excess, the total capacity voltage must then be equal to CQ_1' plus a voltage equal and opposite to the inductive voltage. The point Q_1' is fixed by the length $Q_1'C$ and the locus of the active voltage ONC; therefore, the total capacity voltage may be laid out from Q_1' to Q', the length of CQ' being equal to CQ but its direction being in quadrature with the current OB_1' and therefore in line with $Q_1'C$. The part of CQ' represented

* Art. 70, Case (1).

by $G'Q'$ neutralizes the effect on the transformer of the load inductance. The triangle of voltages at the secondary terminals is then $OK_1'K'$ and the impressed voltage OC may be considered to be applied in the following components: OK_1', which

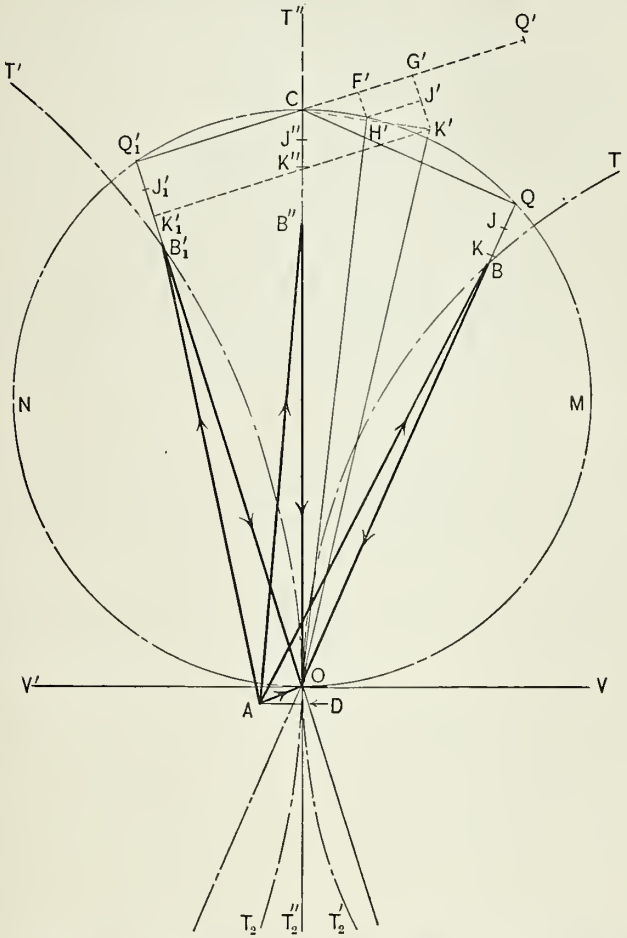


FIG. 279. — Transformer Locus Diagram showing the Effects of Capacity or Inductance Predominant or Equal.

is equal and opposite to the secondary active voltage; $K_1'K'$ which is equal and opposite to the secondary capacity voltage in excess of the inductive voltage of the load; $K'J'$, the drop equal and opposite to the drop in the secondary coil resistance; $J'H'$, the drop equal to the secondary leakage voltage; $H'F'$, the

drop in the primary coil resistance; and $F'C$, the drop equal and opposite to the primary leakage voltage. The total primary mutually induced voltage is $H'O$ and the total secondary terminal voltage is $K'O$. The coil resistance drops may be transferred from $K'J'G'$ to the active voltage line at $K_1'J_1'Q_1'$, and the diagram then becomes similar to those previously discussed. The secondary angle of lead is the angle $K'OK_1'$ and the primary angle of lead is the angle between the lines CO and $B_1'A$ extended.

When inductive and capacity reactances neutralize, as when the current locus is OCT'', the line $Q_1'Q'$ swings to a horizontal position and $Q_1'C = CQ'$; the coil impedance triangle $CG'K'$ therefore swings down so that its side CG' is horizontal. It is thus seen from Fig. 279 that for a given angle between the current and impressed voltage the value of $H'O$ is greater when the capacity reactance is in excess compared with its value when inductive reactance is in excess, but for commercial constant voltage transformers operating with ordinary power factors and normal secondary currents, the change of $H'O$, the mutually induced primary voltage, in either length or position is not great. When neither inductance nor capacity is present, $H'O$ swings into the line OC, and in practice it usually remains close thereto.

The circular arcs OT_2, OT_2', and OT_2'' are the circular loci that would be made by the secondary currents had they been drawn outward from O instead of inward as shown in Fig. 279. When these arcs are used, a phase instead of a vector diagram of the currents results.

Prob. 1. Given a 60-cycle transformer of 100 kilowatts rated capacity, in which the primary voltage is 2200 volts, the ratio of transformation, $\frac{n_1}{n_2}$, is 20, the copper loss at full load 2 per cent of the rated capacity and equally divided between the primary and secondary coils, the iron loss 1 per cent of the transformer rated capacity, the exciting current (assumed constant) 2 amperes, and the drop of voltage at full load due to magnetic leakage $1\frac{1}{2}$ per cent of the impressed voltage when current is being delivered to a non-reactive secondary circuit: Draw for this transformer a large transformer diagram similar to Fig. 277.

(NOTE. — For convenience use secondary quantities in terms of primary equivalents in making the diagram. After its completion read off the correct secondary quantities by changing the scale. The point G for full load may be found by laying out GC so that OG is $1\frac{1}{2}$ per cent less in length than OC. Since OG is the active component of OC it is proportional to the terminal voltage plus the 2 per cent allowed for copper losses, and KG is therefore $\frac{.02}{1.02}$ per cent of OG, while KO is the primary equivalent of the secondary terminal voltage E_2 at the full load of 100 kilowatts. Divide 100,000 (the load in watts) by the number giving the volts represented by the scale length of KO, and the value of amperes in the full load secondary current, $-I_1'$, is obtained, expressed in primary equivalents. Then the total primary equivalent leakage reactance X_L is approximately the volts CG divided by I_1', and the diameter of the current locus is $\frac{E_1}{X_L}$. Make OD of a length which represents in amperes that number which when multiplied by the volts represented by OC gives one per cent of the transformer's rated capacity. This fixes the phase angle of OA under the assumption that the magnetizing element of the current is constant in value and at right angles to OC.)

Assuming the exciting current constant in length and phase position, find the primary and secondary currents, the secondary terminal voltage, the primary and secondary angles of lag and the regulation (ratio of primary voltage minus secondary terminal voltage in terms of the primary to primary voltage) for full load and for 25 per cent, 50 per cent, 75 per cent, and 125 per cent of full load.

Prob. 2. Assume the same specifications as in Prob. 1 for the transformer and conditions there specified, except that the power factor of the load has become 0.9 on account of the introduction of self-inductance. Draw the diagram and compute the currents, voltages, angles of lag, and regulation as required in Prob. 1. Also give the amount (in henrys) of self-inductance introduced into the load.

Prob. 3. Assume the same specifications as in Prob. 1 except that the secondary circuit contains a net capacity reactance, so that the power factor of the load is .9, with current leading. Draw the diagrams and compute the currents, voltages, angles of lag, and regulation as required in Prob. 1. Find the amount (in microfarads) of capacity in the secondary circuit.

131. Short Circuits. — As explained in the preceding articles, the point G in a transformer diagram, such as Fig. 277, does

not, for the ordinary working power factors and loads used in commercial, constant voltage transformers, move far from the point C, within the practicable ranges of continuous load. If, however, for any reason, the terminals of the secondary winding are connected together — short-circuited — through a negligible resistance, the secondary terminal voltage KO becomes zero and G swings around the semicircle until K lies on O. This means that the primary and secondary currents may become enormous compared with the maximum currents for which the transformer was designed and will be approximately equal to $\frac{E_1}{Z_1}$, where E_1 is the impressed voltage, and Z_1 is the combined impedance of the primary and secondary coils — the latter reduced to primary equivalents.

The conditions are represented in Fig. 280, where the current locus is $OBB'T$, the voltage locus $OMCN$, the rated full load

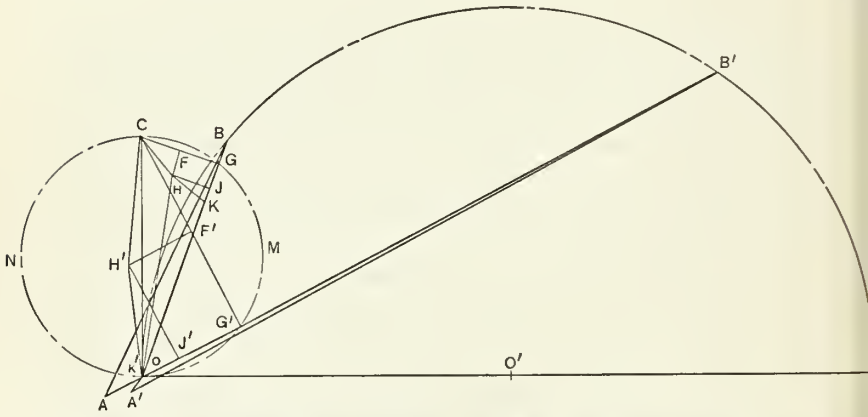


FIG. 280. — Transformer Diagram showing Conditions when the Secondary Terminals are Short-circuited.

secondary current BO, the primary and secondary impedance voltage triangles HFC and KJH respectively, the diagram being lettered to correspond with Fig. 277. All secondary quantities are shown, as heretofore, as primary equivalents; and, as before, the coil impedances are exaggerated for the sake of clearness in the diagram.

When the secondary terminals are short-circuited, G moves to G' and the entire impressed voltage OC is expended in

driving current through the coil impedances as shown by the triangles $H'F'C$ and $K'J'H'$. The secondary current is $B'O$, which is several times larger than BO. The mutually induced primary voltage here becomes $H'O$, which is out of phase with HO and is much shorter. Since $H'O$ must be proportional to the magnetic flux which is mutual to the primary and secondary windings, this means that the mutual flux in the core decreases with the increase of magnetic leakage and resistance drop caused by the extraordinarily large currents and the accompanying large magneto-motive forces impressed on the leakage paths. Exciting current AO can no longer be considered constant but must be changed by reducing its vertical (active) component in a ratio approximately equal to $\frac{(OH')^{1.6}}{(OH)^{1.6}}$, and, supposing that the core reluctance remains approximately constant, reducing its horizontal (reactive) component in the ratio of $\frac{(OH')}{(OH)}$. It is then $A'O$ and the primary current is $A'B'$.

For the sake of making a clear diagram of small dimensions, the leakage and copper drops have been greatly exaggerated in Figs. 277, 278, 279, and 280, as compared with the values they would have in large, well-designed, commercial transformers intended for constant voltage working. In such transformers the radius of the semicircle $OBB'T$ would be many times larger than is shown in Figs. 277, 278, 279, and 280, and the point B would move farther along the semicircle towards T in case of short circuit, so that OB' may become enormously larger in proportion to the full load current than is shown in Fig. 280.

The coils of a transformer are usually wound close together, as seen in Fig. 46, or are sandwiched together in sections, to prevent undue magnetic leakage, and are bound and supported only by insulating materials. When a short circuit occurs, the forces tending to repel the primary and secondary coils one from the other, by reason of the large currents and excessive leakage fluxes (proportional to $F'C$ and $G'F'$ in Fig. 280), may become very great. The primary and secondary currents being substantially in opposition, the forces set up between them, by virtue of their magnetic leakage fluxes, are of repulsion. Also, the leakage fluxes being proportional to the currents in primary and secondary coils, this repulsion is proportional to

the product of the currents, and as the secondary and primary currents of a transformer approximately vary together, the mechanical force exerted between the coils is closely proportional to the square of the current flowing in either winding.

The mechanical strains thus set up in a small transformer which becomes short-circuited may not exceed the strength of the insulating supports of the coils. In the case of large transformers the short-circuit currents become larger in proportion to the rated full load currents, because of the proportionately lower resistances of the windings of the larger machines, and the fluxes are larger, hence, the electro-magnetic stresses may become very great if short circuits occur. For transformers, of similar physical form but different sizes, the stresses referred to may increase in proportion to the squares of the rated loads; so that of two transformers, one of which has a capacity of 10 kilowatts and the other of 1000 kilowatts, the repulsion strain on conductors of the second may be in the neighborhood of 10,000 times as great as in the first. The result is that the windings of very large transformers are sometimes torn to pieces, as by an explosion, when the secondary circuits are accidentally short-circuited.

The burning out of the coils, due to the excessive I^2R losses, when a short circuit is maintained, will occur as in any other electrical machine or circuit, but injury from repulsion may occur before the circuit is broken by a burn-out or by the action of an automatic circuit breaker. The computation of the electro-magnetic stresses has been attempted and may be accomplished to some degree of approximation.*

132. Locus of Current in a Transformer when the Load Reactance is varied and the Resistance is kept Constant. Relation of Power Factor to Capacity. — In the discussions just preceding, the load and coil reactances of a constant voltage transformer were considered to be constant while the resistance varied. It is also possible to have a load in which the resistance is approximately constant while the power factor varies because of changing inductive or capacity reactance, as when the load comprises a synchronous motor operating at constant torque but with various field excitations.† Consider a case of the latter kind and assume the transformer to have negligible mag-

* *Trans. Amer. Inst. E. E.*, Vol. 30. † Art. 169.

netic leakage. Then let OC in Fig. 281 be the impressed voltage. The voltage locus is $OMCN$, as before. Consider first that the load and coil reactances are zero. Then the voltages of the primary and secondary coils induced by the mutual flux lie on the line OC, if the slight deviation therefrom due to

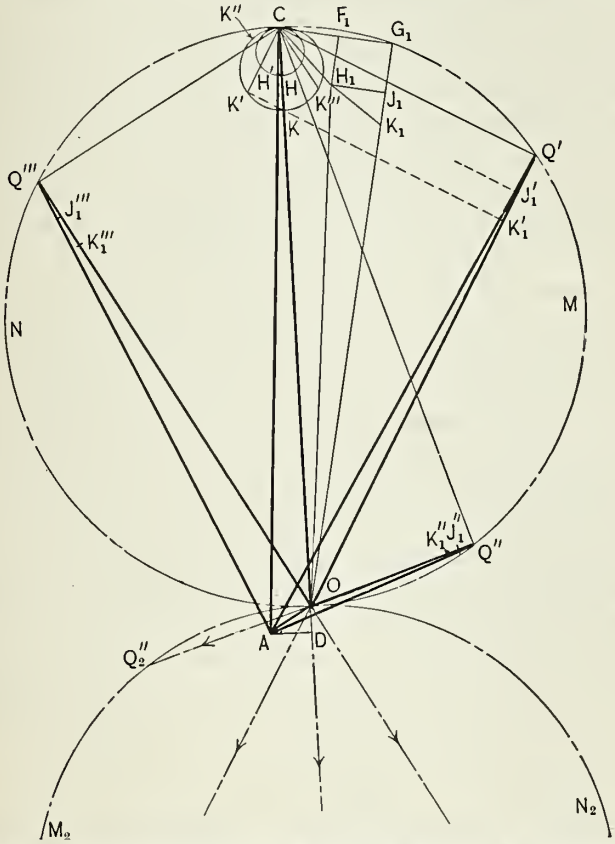


FIG. 281. — Transformer Diagram when the Load Reactance is Varied and the Resistance is maintained Constant.

the small element of impressed voltage in phase with the exciting current is neglected. The line OC represents the impressed voltage on the primary coil; HC is the primary voltage drop in the primary coil resistance; KH is the primary voltage drop due to the secondary coil resistance (the ratio of transformation is taken as unity for convenience); OK is the primary voltage drop due to the resistance of the load; HO

is the mutually induced voltage. In this case, for ease in construction, the secondary current is scaled so that it equals OC in length. It will coincide with that line because there is no appreciable reactance present.

Now if positive or negative reactance is introduced so that the apex of the right-angled triangle of voltages takes any position on the locus $OMCN$ as Q', Q'', or Q''', the ends of the lines representing the current vectors for the various conditions will also fall on the locus $OMCN$; for it was proved in an earlier chapter that when a circuit contains variable reactance and constant resistance, the locus of the current is a circle having a diameter lying in the line of the impressed voltage and equal to the impressed voltage divided by the resistance, or $\frac{E}{R}$.* This is the maximum current and is represented by OC in Fig. 281. The diameter of the circular locus is equal to this current. The secondary current for a certain positive reactance is then $Q'O$ and the primary current for the same reactance is AQ', where AO is the exciting current. The line $CH'K'$ being drawn perpendicular to CQ' and $K'K'_1$ being parallel thereto, the mutually induced voltage, if there is no magnetic leakage in the coils, but when the reactance voltage of the load is CQ', is $H'O$, and the primary and secondary coil resistance drops are respectively $J'_1Q' = H'C$ and $K'_1J'_1 = K'H'$. The total active primary voltage is $OQ' = K'C + OK'_1$, and the secondary terminal voltage is K'_1O. It will be noted from the construction that as Q moves along its locus by reason of a change of reactance, K moves around a circle having a diameter CK, and H describes a circle with a diameter CH. The smaller circle is evidently the locus of the mutually induced voltage HO and of the resistance voltage drop HC of the primary coil, and the larger circle is the locus of the resistance drop KC of the primary and secondary coils combined.

When magnetic leakage is present, the diagram for a non-inductive load is as in Fig. 277. In Fig. 281 the coil voltage drops are similarly shown for a non-reactive load where magnetic leakage is present, by the lines F_1C, H_1F_1, J_1H_1, and K_1J_1, and the secondary current is then on the line G_1O. Then when reactance is introduced into the load and Q moves on

* Art. 70, Case 2.

the locus $OMCN$, the figure $CG_1K_1H_1$ swings around and remaining symmetrical diminishes in size when the current decreases, as did the line CK when leakage was absent. When Q reaches O, the secondary current becomes zero and only the exciting current remains in the primary circuit, for then the reactive voltage equals the impressed voltage minus the small component required to drive the exciting current through the primary coil. As this requires infinite reactance it is equivalent to a transformer with the secondary circuit open. The right-hand semicircle $OQ'C$ is the current locus for varying net inductance and $OQ'''C$ for varying net capacity.

The broken arc of a circle M_2ON_2 is of the circular locus that would be formed if the secondary currents for varying inductance were drawn outward from O instead of inward.

Prob. 1. Given a 50-kilowatt 60-cycle transformer transforming from 2200 to 220 volts, in which the primary and secondary coil resistances are .8 and .008 ohm respectively, the core losses 100 watts (assumed constant), the exciting current .3 of an ampere, and the magnetic leakage negligible. When the load resistance is maintained such that the output is equal to 50 kilowatts, the primary voltage being 2200 volts, what are the primary and secondary currents for power factors of 1, .8, .5, and .2, for both positive and negative secondary angles of lag?

133. The Use of Equivalent Impedances in Solving Transformer Problems. Vector Formulas. — It is possible to substitute impedance coils for the transformer which will duplicate the transformer diagrams in Figs. 277 to 281. Thus, in Fig. 282, Z_1, Z_2,

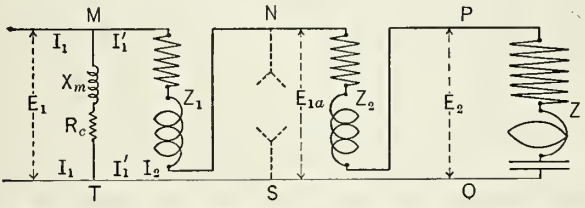


FIG. 282. — Combination of Impedances Equivalent to the Impedances of a Transformer.

and Z represent three sets of impedances connected in series which are equal respectively to the impedances of the primary coil, $Z_1 = \sqrt{R_1^2 + X_{1L}^2}$, the secondary coil, $Z_2 = \sqrt{R_2^2 + X_{2L}^2}$, and the load, $Z = \sqrt{R^2 + X^2}$, of a transformer after its secon-

dary quantities have been reduced to primary equivalents. A branch, which carries a current equal to the exciting current, is connected across the main circuit at MT. The arm R_e is of such resistance that an active current flows which when multiplied by the impressed voltage, E_1, gives a product equal to the no load losses (see DO, Fig. 277); while the arm X_m is such a reactance as will permit the flow of the proper magnetizing element (AD, Fig. 277) of the exciting current; the two combine to give the total exciting current I_μ(AO, Fig. 277). Under this arrangement the impressed voltage and current of the series of impedances are E_1, I_1; the load voltage and current are E_2, I_2; and the angles of lag are θ_1, θ_2. These and component voltages and currents are as shown in Figs. 277 to 281; the particular figure and the character of the loci which apply depending upon the dimensions and character of the various impedances.

In order that the combination may be a complete reproduction of conditions of a transformer the exciting current path should shunt the second impedance and load at NS. The current flowing in it would then be proportional to the voltage induced in the primary coil by the mutual flux (HO, Fig. 277 and other figures) and its vector would make a constant angle therewith, instead of being proportional to and at a fixed angle with the impressed voltage.

The first arrangement is sufficiently accurate for ordinary calculations and is a less complicated arrangement. It is noticed that by the substitution of impedances the solution of transformer problems becomes in principle equivalent to the simple solutions of circuits dealt with quite extensively in an earlier chapter.*

The vector equations necessary for determining the most important characteristics of the transformer may be readily written from Fig. 282, thus: Considering the first arrangement of impedances illustrated in Fig. 282, the total impedance offered to the impressed voltage by the main circuit $MNPQ$ is

$$\bar{Z}_0 = \bar{Z}_1 + \bar{Z}_2 + \bar{Z},$$
$$\bar{Z}_0 = (R_1 + R_2 + R) + j(X_{1L} + X_{2L} \pm X);$$

where Z_1, R_1, and X_{1L} are the impedance, resistance, and leak-

* Art. 86.

age reactance respectively of the first coil, Z_2, R_2, and X_{2L} of the second coil, and Z, R, and X of the load; X is negative when capacity reactance predominates in the load. Let the first term on the right equal R_t and the second equal $\pm jX_t$. The admittance of the circuit is then

$$\bar{Y}_0 = \frac{R_t}{R_t^2 + X_t^2} \mp j \frac{X_t}{R_t^2 + X_t^2},$$

and the current flowing through it, when the voltage E_1 is impressed, is

$$\bar{I}_1' = E_1 \bar{Y}_0 = \frac{E_1 R_t}{R_t^2 + X_t^2} \mp j \frac{E_1 X_t}{R_t^2 + X_t^2}.$$

The two terms in the right-hand member of this equation represent respectively the active and reactive components of $\bar{I}_1'$. This current represents the secondary current in a transformer having a unity ratio of transformation and the resistances and reactances of the windings are as denoted in Fig. 282, the vector angle being measured from the impressed voltage $\bar{E}_1$.

The total primary current flowing is

$$\bar{I}_1 = \bar{I}_1' + \bar{I}_\mu\,; \quad \bar{I}_\mu = I_c - jI_m,$$

where I_μ, I_c, and I_m are the total exciting current and its active and reactive components respectively when referred to the impressed voltage E_1 as the initial vector direction. Then, calling $\frac{E_1 R_t}{R_t^2 + X_t^2} = I_{R_t}$ and $\frac{E_1 X_t}{R_t^2 + X_t^2} = I_{X_t}$, there results

$$\bar{I}_1 = (I_{R_t} + I_c) - j(I_m \pm I_{X_t}).$$

The angle of lag between E_1, the impressed voltage, and I_1, the total current that would flow in the primary winding of the equivalent transformer, is

$$\theta_1 = \tan^{-1} \frac{I_m \pm I_{X_t}}{I_{R_t} + I_c}.$$

The total voltage across the second coil and load, which is equal and opposite to the total mutually induced primary and secondary coil voltages in the equivalent transformer of unity ratio of transformation, is

$$\begin{aligned} \bar{E}_1' &= I_1' \bar{Z}_2 + I_1' \bar{Z} \\ &= I_1'(R_2 + R) + jI_1'(X_{2L} \pm X). \end{aligned}$$

The voltage across the load is

$$E_2 = I_1'\overline{Z} = I_1'R \pm jI_1'X,$$

and the angle of lag of I_1' with respect to E_2 is

$$\theta_2 = \tan^{-1}\frac{\pm X}{R},$$

which is the secondary angle of lag in the equivalent transformer.

The power input is $E_1I_1 \cos\theta_1$; the power output, $E_2I_2 \cos\theta_2$; the excitation losses are $I_\mu^2R_c$; the coil resistance losses due to I_1' are $I_1'^2R_1$ and $I_1'^2R_2$. The regulation for any load, neglecting the small drop due to the exciting current, is $\frac{E_1 - E_2}{E_1}$, where E_2 is the voltage across an impedance which is equivalent to that load. The commercial efficiency, or the ratio of the output to the input for any load, can be readily calculated from the data obtained by the formulas above.

If any difficulty attaches to a clear understanding of the formulas just given, it is well to lay out impedance and admittance triangles for the equivalent impedances and admittances (Fig. 283) exactly as is done in Chapter VI, though reference to Figs. 277 and 281 may serve as well, as they represent voltage and current diagrams of the same general character. After the data required for a transformer are obtained by the use of equivalent impedances, which requires the reduction of the secondary quantities to primary equivalents, the secondary currents and voltages may be stated in their correct values by multiplying and dividing respectively by the ratio of transformation s.

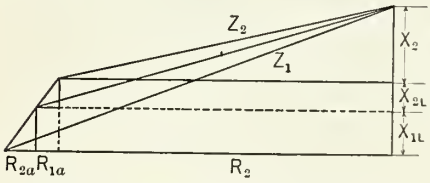


FIG. 283. — Impedance Diagram obtained from Equivalent Transformer Impedances.

Prob. 1. A 500-kilowatt, 60-cycle, 5500-volt (nominal primary voltage) transformer having a ratio of transformation of 20 to 1 develops its full capacity with a voltage of 230 volts at the secondary terminals. At full load the copper losses in the primary and secondary coils are for each $\frac{1}{2}$ per cent of the transformer's output. The drop in voltage at full load due to mag-

netic leakage is 1 per cent of the no load voltage. The iron losses are 1 per cent of the full load output, and the exciting current is 2 amperes. The load reactance causes the current to lead the secondary terminal voltage by an angle of 30°. Find by the method given in this article the primary voltage, primary current, and primary angle of lag, and the secondary current. Find, also, the primary current, and the secondary voltage and current when the rated voltage (5500 volts) is impressed on the primary winding, the power factor of the load being as before. Give the regulation and efficiency in each case. Answer the same questions when the load is reduced to one half rated capacity (250 kilowatts), assuming that the iron losses and exciting current do not change.

134. The Effect of Harmonics in the Waves of Voltage and Current upon the Operation of a Transformer. — If the primary voltage impressed on a constant voltage transformer in a single phase circuit contains harmonics, the harmonics are transmitted into the voltage of the secondary circuit, since the form of the wave of magnetism in the core is determined by the form of the impressed voltage wave after the small part that is absorbed in copper losses has been deducted, and the mutually induced secondary voltage is at each instant proportional to $-\frac{d\phi}{dt}$.

If the secondary circuit contains considerable capacity reactance and comparatively small resistance, the higher harmonics of the secondary voltage cause exaggerated current harmonics.* The magneto-motive forces of these secondary current harmonics would affect the magnetic flux of the core and they must be offset by opposing magneto-motive forces in the primary winding, which results in harmonics of current flowing in the primary circuit that are equivalent to those in the secondary circuit.

Thus, irregular voltages and currents in the circuits to which a transformer is attached act very much as would be the case if the transformer was replaced by the equivalent network discussed in the previous article.

As the exciting current of a transformer is quite irregular in form, it contains harmonics of relatively large amplitude. Therefore, in a circuit containing a large number of unloaded or lightly loaded transformers, conditions may arise under

* Art. 69.

which the resultant flow of irregular current may prove troublesome. The principal harmonic in the exciting current of higher frequency than the fundamental when the impressed voltage is sinusoidal is the third, although the fifth also may be observed. The conditions may be understood from Fig. 263, which shows a hysteresis and eddy current cycle and the current wave required to produce a sinusoidal magnetic flux. The quadrature component of current is peaked as a result of the increased reluctance caused by partial saturation of the iron toward the top of the flux wave, which occurs at even the relatively low magnetic densities used in transformers. This symmetrical peaking of the quadrature current component indicates a prominent third harmonic therein.

No matter what the primary voltage wave may be like, the secondary voltage wave must have practically the same form, the exciting current and magnetization curves varying to meet this requirement. When the voltage curve is peaked, the magnetization curve is flattened and does not reach so high a maximum as for an equivalent sine curve; and conversely, when the voltage curve is flat, the magnetization curve will be peaked, as has been hereinafter proved in Art. 157. Since the iron losses of a transformer depend upon the maximum value of the magnetic flux in the core, it is to be expected that a peaked voltage curve will cause minimum iron losses. Roessler has shown that the iron losses of a transformer depend upon the ratio of the average value of the voltage wave to its effective value, *i.e.* upon the **Form Factor**. Steinmetz cites a case where the losses in a 200-kilowatt transformer were 13 per cent less when worked on a peaked curve than when a sine curve was impressed. The change in full load efficiency is, however, very slight in ordinary commercial practice, although the "all day" efficiency may be considerably affected by the form of the voltage wave impressed on a transformer which runs for any considerable part of the day very lightly loaded. Generally speaking, constant voltage transformers have the best efficiencies when operated on circuits having peaked voltage waves. Experience shows that alternating current arc lights give the best results when worked on a flat-topped curve, while a sine curve is most convenient for the operation of synchronous motors and gives the best efficiency in the operation of induction motors.

The last statement is shown later to be theoretically correct.* Since a transformer may be required to supply any one of these kinds of loads, an approximate sine wave probably gives the best general results.

135. Effects of Changes of Frequency and Voltage. — The formulas giving the voltages developed in the coils of an unloaded transformer,

$$E_1 = 2\pi f L_1 I_\mu = \frac{\sqrt{2}\,\pi \Phi f n_1}{1\ \ 0^8} \text{ and } E_2 = \frac{n_2}{n_1} E_1, \dagger$$

show that if the maximum value of the magnetic flux in the core of a transformer is fixed, then for a given impressed voltage the number of turns in the coils must vary inversely with the frequency. To design transformers for all frequencies with a fixed value of the maximum magnetic flux is not an economical plan, however, since the core loss per cubic centimeter of iron depends upon the frequency. If a certain core loss is determined upon as being that which will give the most satisfactory general results in the case of a transformer of given size and for a given duty, then the magnetic density in the core must be designed to be smaller as the frequency is made larger. This may be seen from the fact that the eddy current loss varies as the square of the frequency, and the hysteresis loss as the first power of the frequency, other things being equal. Since the former is not uncommonly between one tenth and one fourth of the latter, this makes the total core losses vary somewhere between the first and second powers of the frequency, if the maximum magnetic density is designed to be the same for all transformers. The eddy current loss also varies as the square of the maximum magnetic density, and hysteresis loss varies with some degree of approximation within the limits of transformer practice, as the 1.6th power of the maximum density. Consequently, the total losses vary as some power of the maximum density between the 1.6th and the second. It is thus shown that the core losses will not be greatly changed if the maximum magnetic density, within reasonable limits, varies inversely with the frequency. Now it is evident from the formula that, if the turns and voltage remain unchanged, when the frequency changes, the maximum flux

* Art. 198. † Art. 122.

will change inversely; hence a transformer should give nearly the same efficiency for currents of different frequencies reasonably near that for which it was designed, only a small rate of increase of the iron losses occurring as the frequency decreases; and the number of turns in the coils may be nearly the same in a series of transformers of equal capacities designed for the same voltages but different frequencies. This statement is made under the supposition that the lower frequencies do not call for a maximum magnetic flux which exceeds the saturation point of the magnetic core.

Changing the frequency alters the quadrature component of the exciting current in a proportion which depends upon the saturation of the core, and transformers built of poor material, or which operate with their cores near the point of saturation when used on normal frequency, may give unsatisfactory service and overheat on lower frequencies than their normal, on account of an excessive magnetic density and an excessive exciting current. Transformers under these circumstances may have poor regulation due to large copper drop and increased magnetic leakage. This is seen from the voltage formula, where Φf must be constant while E_1 is constant. It will be seen also from the formula that raising E_1 above the rated value tends to cause a greater maximum flux Φ, so that even if the insulation will stand an increased voltage, the increase cannot go beyond the point where the core becomes saturated, without causing excessive core losses and an excessive flow of exciting current.

Since the rates of variation of the hysteresis and eddy current losses with the frequency and with the reciprocal of the maximum magnetic density are not exactly the same, it is to be expected that the efficiency of a transformer, so far as it is affected by the iron losses, will vary to some degree with the frequency.

Since the flux density varies inversely as the frequency of the impressed voltage in any particular transformer, as is shown by the foregoing formula, and the eddy current loss is proportional on the one hand to the square of the frequency and on the other hand to the square of the flux density, a variation of the frequency leaves the eddy current loss substantially unaffected, provided the paths for the flow of the eddy currents do not change. In the case of the hysteresis loss, different consid-

erations arise, for this loss is proportional to $f\Phi^{1.6}$, while Φ in any particular transformer is proportional to $1/f$. The consequence is that the hysteresis loss in a particular transformer, when the frequency is varied, changes proportionally with $1/f^{.6}$. In first class modern transformers, the eddy current loss is usually much smaller than the hysteresis loss, and the efficiency of a particular transformer increases appreciably as the frequency of the impressed voltage is increased.

Transformers of equal outputs but for different frequencies may be built with windings having like numbers of turns but with the cross sections of the cores inversely proportional to the frequency. The magnetic density would then be the same in all the transformers. This is unsatisfactory, however, since it makes low frequency transformers large and expensive; and since the weight of iron in a core must vary more rapidly than the cross section, this method of construction also causes comparatively large core losses in low frequency transformers and gives them comparatively low efficiency.

The conclusion is therefore derived that the core and windings of a well-designed and well-constructed transformer will be satisfactory in performance for small variations in frequency, but that transformers designed for high and low frequencies respectively should be made with such numbers of turns, weights of iron, and magnetic densities as will give in each case the efficiency and regulation sought. With the modern high grade alloy steels the hysteresis is so much less than in common steels and irons that it is possible to carry the magnetic density well toward the point of saturation for all frequencies, and this is now standard practice.

The modern standard frequencies in this country are 60 cycles per second for lighting and small power purposes, 25 cycles per second for alternating current railroad and large power units, and sometimes 40 cycles per second for combined lighting and power circuits.

136. Transformer Iron and Steel. — Early experience showed that the core losses in some transformers increased to a very considerable extent during the first few months of operation. The increase in losses was found to be due to increased hysteresis loss per cycle, and was originally ascribed by Ewing to magnetic fatigue of the iron. Later it was conclusively proved by

experiments and the records of transformer manufacturers to be caused by the continuous condition of heightened temperature at which the iron is operated. This effect of **Ageing** seems to have the greatest effect upon poor qualities of iron, hastily and imperfectly annealed, and the least effect upon the best grades of iron which have been annealed with great care. The conditions under which the annealing of transformer iron is performed, especially with reference to temperature and duration of the process, have much to do with the extent of the ageing effect, and by proper annealing it can be rendered very small in cores made of proper qualities of iron. The iron formerly and now much used for transformer cores is a very mild steel made by either the bessemer or open-hearth process, though puddled iron sheets were formerly used to some extent. But the magnetic material most approved at the present day is an alloy of silicon with mild steel. This is steel containing from $2\frac{1}{2}$ to 4 or more per cent of silicon, called silicon steel. Such steels are harder and more difficult to make and work, so that they are more expensive than ordinary soft steel. But these steels are apparently little affected by ageing— which sometimes increases the hysteresis losses in ordinary steels as much as 100 and 150 per cent after the metal has been in service in a transformer core for a few months. Further, these new steels have a lower hysteretic constant, and transformer cores made of some of them waste from 35 to 45 per cent less power in hysteresis than do the ordinary steels. They also possess higher specific resistances, and this reduces eddy current losses. Thus, for the same efficiency and duty a transformer can be built having a core of the best transformer silicon steel at a much less weight per kilowatt capacity, than when any one of most of the standard soft steels is used. This is because the magnetic density can be increased from 45 to 65 per cent, without increasing the loss due to hysteresis, which results in a consequent reduction in the cross section of the core and a resultant lessening of the length of wire per turn.* A fair value for the hysteretic constant for good soft transformer steel is .0021 and for silicon steel .00094 in terms of watts per pound and per cycle.* In Fig. 284 the curves A and A' are plotted from the hysteresis equation † and show the hysteresis

* *Trans. A. I. E. E.*, Vol. 28, p. 439; also Vol. 30. † Art. 106.

loss in watts per cubic centimeter and per pound of metal when the frequency of magnetic cycles is 100 per second and the

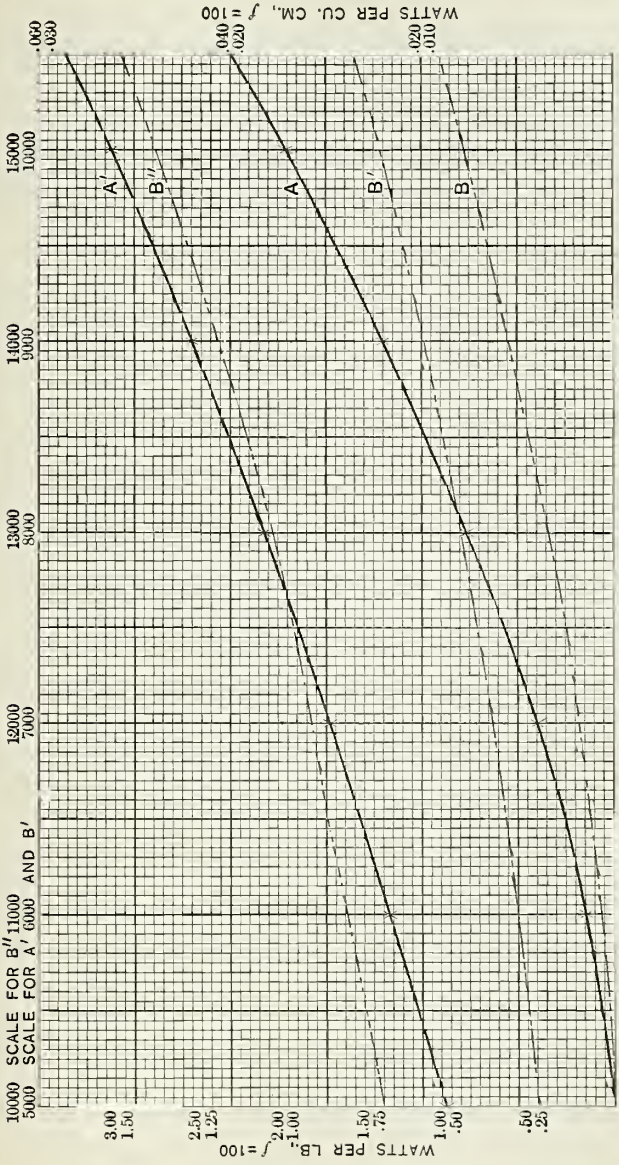


FIG. 284. — Curves of Hysteresis Losses in Transformer Irons. A, A' is for Soft Steel; B, B', B'' is for Silicon Steel.

hysteresis constant is $\eta = .0021$. Curve A is plotted for maximum flux densities from 0 to 5500 lines of force per square

centimeter, and curve A' is plotted for maximum flux densities from 5000 to 10,500; while the curves B, B', and B'' are plotted for $\eta = .00094$, the maximum flux densities being 0 to 5500 lines of force per square centimeter for curve B, 5000 to 10,500 lines of force per square centimeter for curve B', and 10,000 to 15,500 for curve B''.

It must be remembered that the character of steels, especially of silicon or other alloyed steels, is apt to be quite variable. Therefore, in designing magnetic apparatus, the exact kind of material to be used should be tested and its qualities as thus determined used in making calculations. As seen from the hysteresis equation just referred to, it is evident that the curves of Fig. 284 may be used for any other hysteresis constant and frequency by multiplying the vertical scale of losses by the ratio $\frac{\eta' f'}{\eta f}$, where ηf is the product of the hysteresis constant and frequency taken in calculating the curves referred to, and $\eta' f'$ is the product of the values of those quantities for which the data are sought.

The eddy current losses are dependent both upon the thickness of the sheets and the specific electrical resistance of the steel. The thickness of laminations quite commonly used for 60-cycle transformers is from 14 to 15 mils, or more exactly No. 29 sheet-iron gauge, which has a thickness of 14.1 mils, while for 25 cycles No. 26 gauge sheets, having a thickness of 18.7 mils, do not permit the eddy current loss to exceed economical limits. The specific resistance of iron varies quite widely, and this is especially true of the alloy steels, which may have a much greater resistance than the soft steels and irons. In Fig. 285 curves A, B, and C are the calculated eddy current losses in plates of 10, 15, and 20 mils thickness respectively for various maximum magnetic densities, a frequency of 100 periods per second, and a specific resistance of 1×10^{-5}, which is a fair value for good transformer steel.* The curve shown in three sections, D, D', D'', is for a plate 15 mils in thickness and having a specific resistance three times as great as for the first curves. It represents some of the harder alloys. The curve can readily be used for other frequencies and thicknesses of laminations, as will be seen by an inspection of the eddy current formula.†

* Art. 111. † *Ibid.*

The total core loss in transformers varies more or less, depending upon the characteristics desired, being from about 1.1 to

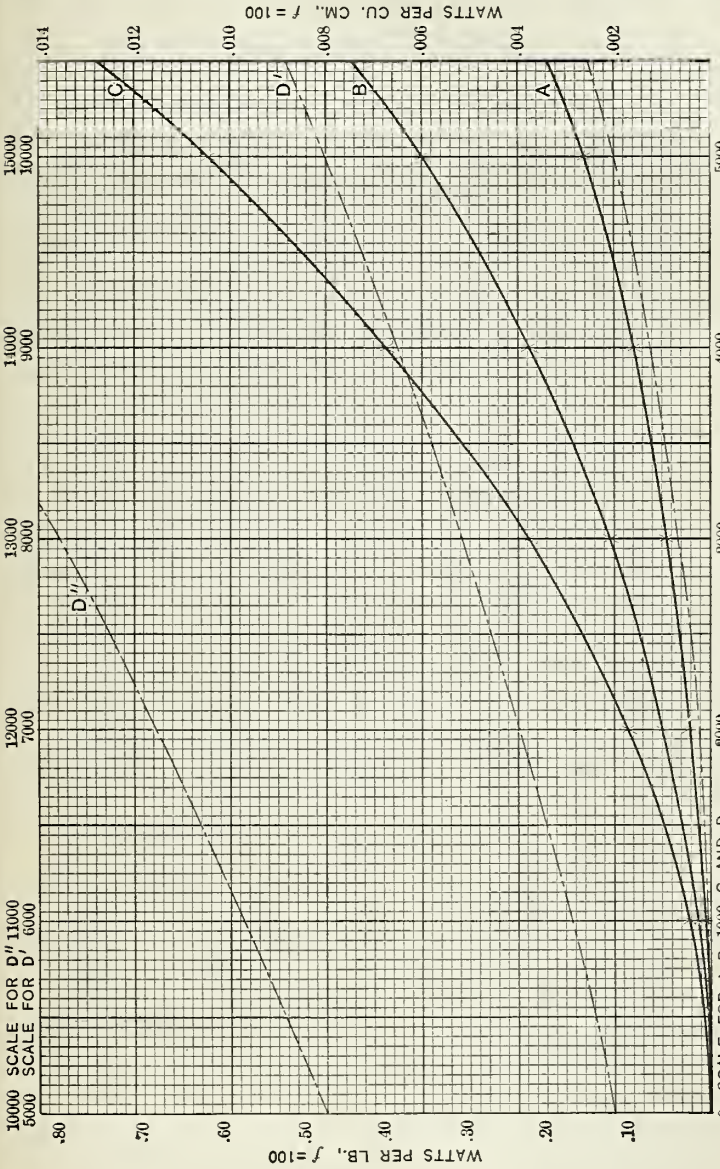


Fig. 285. — Curves of Eddy Current Losses.

3 per cent in small transformers, such as those under 5 kilowatts capacity, and decreasing to about $\frac{3}{4}$ of a per cent in those of 25 kilo-

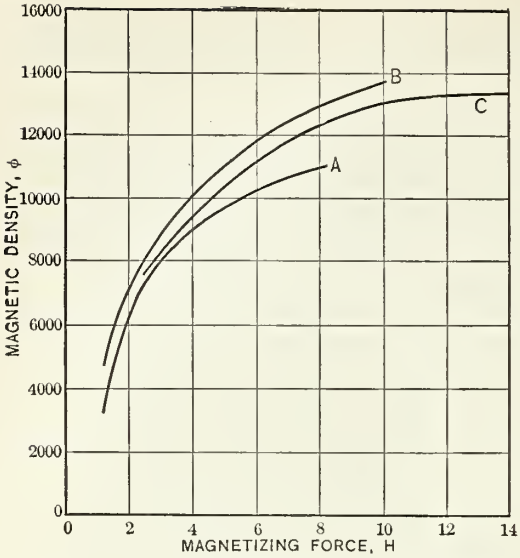


FIG. 286. — Curves of Magnetization of Transformer Steels.

watts capacity, to $\frac{2}{3}$ of a per cent in those of 50 kilowatts capacity, and to a smaller percentage as the capacity still further increases. The eddy currents seldom comprise over 10 per cent of the total core losses.

As the best grades of low carbon steels and wrought irons are used for transformer cores, the ratio between the maximum magnetic density ϕ and the field strength H is high. Curve B* in Fig. 286 is from a good grade of transformer steel, and curve A represents a soft steel of poorer grade. The silicon steels do not ordinarily have quite such high curves as the best transformer carbon steels. The relation between curves B and C fairly indicates the difference. The permeability of B in Fig. 286 is shown by B in Fig. 287. Curve A is a poorer grade of soft iron and C represents the permeability of

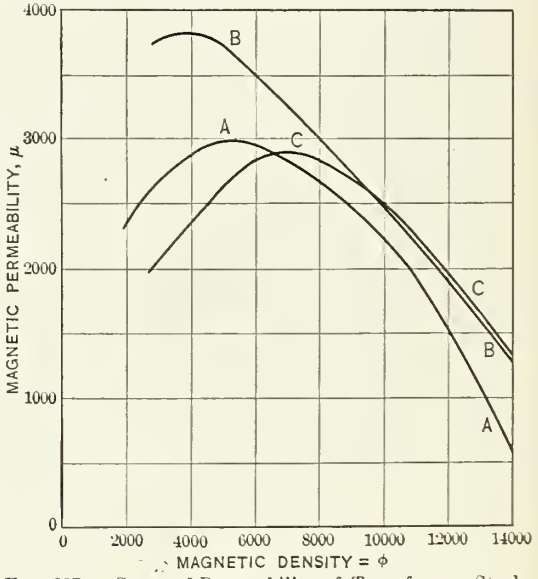


FIG. 287. — Curve of Permeability of Transformer Steels.

* *Smithsonian Tables*, pp. 274 and 275.

a good grade of silicon steel. The silicon steels take somewhat more exciting current than the best grades of soft iron and soft carbon steels, on account of their lower permeability.

The maximum magnetic densities to which transformer cores are worked is dependent somewhat upon the frequencies for which the transformers are designed. For 60 cycles the densities vary from 6000 to 10,000 lines of force per square centimeter, or slightly higher, with a tendency toward 10,000 when the highest grade transformer core metal is used. For 25 cycles they vary from nine or ten to twelve or more thousand lines of force per square centimeter. The usual silicon steels cannot be run too high in magnetic density or the exciting current will be larger than desired. In silicon steels a maximum magnetic density of 10,000 lines of force per square centimeter is apt to be about as near the point of saturation as it is wise to go, while economy of material dictates as high a density as possible, so that in modern transformers, using the new metals, the density will not ordinarily be far from that figure. With ordinary steels the greater iron losses tend to make lower densities more desirable.

In addition to silicon there are other alloying materials used with iron for the purpose of reducing the hysteresis coefficient below figures common in ordinary soft carbon steels. Among these are nickel,* tin, and arsenic.† The new pure irons recently produced have also unusually good magnetic qualities. The state of the art in the manufacture of iron and steel is rapidly advancing, and it is probable that present practice will be materially modified as a result.

137. Efficiencies of Transformers. — The average working efficiency of a transformer is by no means equal to its full load efficiency. In the case of dynamos placed in a central station it is usual to divide the generating units so that the plant operating during any part of the day will be well loaded. In the same way the capacities of stationary motors may ordinarily be chosen so that the motors seldom operate at very small partial loads. The way that transformers are usually operated, however, makes it quite difficult to keep a uniform load on them, and, in fact, for a considerable portion of the day they may have their secondary circuits open. When this is the case,

* Art. 109. † Electromet. Ind., Vol. VII.

the iron losses of transformers are of much greater influence on their **All-day efficiency** than the copper losses, and it is desirable to reduce the iron losses to a minimum. For instance, suppose a transformer of 5000 watts output at normal full load has iron losses of 125 watts, and copper losses at full load of 100 watts. Then the full load efficiency of the transformer is 95.7 per cent, the half load efficiency is 94.3 per cent, and the quarter load efficiency is 90.4 per cent. Supposing that the daily output of the transformer is equivalent to 25,000 watts for one hour (25,000 watt-hours), obtained by full load operation for five hours and open-circuit operation for the remaining 19 hours, then the losses in the iron core are equivalent to 3000 watts for one hour, and in the copper to 500 watts for one hour (neglecting no-load copper losses), or a total of 3500 watts for one hour. The all-day efficiency is then 87.7 per cent. To increase this all-day efficiency, it is evidently necessary to decrease the iron losses. To do this for a fixed frequency requires a decrease in the amount of iron in the magnetic circuit or a decrease of the maximum magnetic density. Either process calls for an increase in the windings and consequently in the copper losses. Suppose that decreasing the core losses to 75 watts makes it necessary to increase the full load copper losses to 150 watts; then, other things being equal, the efficiencies become, full load, 95.7 per cent; half load, 95.7 per cent; quarter load, 93.7 per cent; and the all-day efficiency, on the assumption made above, is 90.7 per cent. There is a saving by the latter construction of 950 watt-hours in twenty-four hours, and in one year of 365 days the saving is nearly 350 kilowatt-hours. If one kilowatt-hour is worth 4 cents to the central station, the difference in the amount of energy wasted each year by the two transformers has a value of nearly fourteen dollars, which is a substantial part of the difference in the original cost of the two transformers. If the average load of the transformer were less than that assumed, which is frequently the case for small transformers in practice, the iron losses would have a still greater influence on the all-day efficiency.

A still greater decrease in the iron losses with its attendant increase of copper losses would evidently raise the all-day efficiency to a higher point. Here, however, is met the question of regulation, which will not satisfactorily admit of too great a copper loss

at full load on account of the attendant drop of secondary voltage, but this difficulty may be met by increasing the cross section of the copper. This alternative causes an increase in the cost of the transformer, but a transformer with small losses and good regulation is worth more to the central station than one with large losses or poor regulation. The advantage of decreasing the iron losses, which is thus shown, led Swinburne some years ago to advocate and build transformers with a cyclindrical iron wire core under the windings, but without closed iron magnetic circuits. A diagram of this transformer, called the Hedgehog, which was somewhat extensively used in England at one time is shown in Fig. 288.* Decreasing the amount of iron in the magnetic circuit decreases the core losses but at the same time increases the reluctance, and therefore increases the exciting current. This is a decided disadvantage if carried to excess. While the true magnetizing current is in quadrature with the mutually induced voltage of the transformer, yet it does result in a continuous I^2R loss in the conductors leading to the transformer and in the primary winding of the transformer itself. It also causes an extra demand for current from the dynamos supplying the circuit, so that extra generators may have to be operated at periods of light load in order to supply the quadrature currents. In other words, the power factor of the system as a whole is decreased, with an accompanying loss of plant efficiency. Finally, a large magnetic reluctance causes a considerable magnetic leakage and consequent increase in the secondary drop of a transformer, and therefore impairs its regulation.

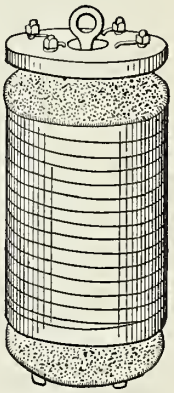


FIG. 288. — Hedgehog Transformer.

Without entering into a controversy regarding the exact point at which a high reluctance in the magnetic circuit of a transformer causes more disadvantage than is counterbalanced by the decreased iron losses brought about by decreasing the amount of iron, the examples may serve to show the necessity of carefully designing the magnetic circuit to fit the conditions if they can be predetermined.

*For test on transformer of this type see *Trans. A. I. E. E.*, Vol. 10, p. 497; *Elec. World*, Vol. 22, p. 357.

The commercial efficiency of a transformer is equal to the watts delivered to the external secondary circuit by the transformer divided by the watts absorbed by the primary coil. It may be written

$$\eta = \frac{P_2}{P_1} = \frac{P_2}{P_2 + L},$$

where P_1 and P_2 are the power absorbed and delivered respectively by the primary and secondary coils, and L represents the total losses of power in the transformer. The losses are made up of the I^2R losses in the primary and secondary coils, eddy currents in the windings caused by leakage flux, and the losses due to hysteresis and eddy currents in the core. The I^2R loss in the secondary winding is directly proportional to the square of the load (secondary output), assuming the secondary terminal voltage to be constant, while the I^2R loss in the primary winding is nearly proportional to the square of the load, though it contains a small approximately constant term due to the exciting current. The rated capacities of very large transformers are not uncommonly expressed in kilovolt-amperes instead of kilowatts, since the I^2R heating effect is dependent upon the current, which depends not only upon the load but also inversely upon the power factor when the voltage is constant; but ordinarily the rating of small transformers is in kilowatts capacity, the assumption being that the power factor for which they are specified is approximately unity. The efficiency will obviously have different values for different loads even when power factor and primary impressed voltage are kept constant; and also if the load (*i.e.* the output in kilowatts) is kept constant, the efficiency will have different values for different power factors.

The hysteresis and eddy current losses vary from .5 to 1.5 per cent of the full load capacity in transformers of 500 or more kilovolt-amperes full load rating, and the percentage increases as the capacity decreases. In a transformer of 100 kilovolt-amperes capacity the losses may be only a little higher than the minimum given above, or they may be as much as one-half or more above the maximum figure given above. In transformers of from 5 to 100 kilovolt-amperes capacity of a given make and type, the iron losses usually increase gradually with decreasing rated capacities until at the small size named they are

from 1 to 2.5 per cent or more of the rated capacity. In very small transformers they rise to from 3 to 5 per cent of the rated capacity. The loss in watts per pound of iron varies from less than $\frac{1}{2}$ a watt per pound for the best grades of alloy steels, when a low magnetic density is used, to as high as 2 or 3 watts per pound where poorer grades of metal are used and the magnetic density used is higher.

The copper losses at full load are apt to be from 25 to 100 per cent greater than the iron losses, though the tendency in modern designing is to keep these losses down, for by increasing the weight of the core and decreasing the number of the coil turns, the IR drop and the IX drop are reduced, and the regulation is improved. The range of variation in watts lost per pound of copper for full load current in commercial transformers is in the neighborhood of from slightly less than one to six or more watts per pound. For transformers to go into lighting service or for duties where there are short full load peaks, however, the copper losses may to advantage be made relatively still higher, with an accompanying reduction of the iron losses; for then the copper losses varying with the load do not have so much influence on the all-day efficiency and the reduction of the fixed iron losses results in better all-day efficiency. This is not always desirable, however, as there is then a higher load on the circuit during the peak when all power available in the circuits may be most valuable. In any event, the copper losses must not exceed reasonably low values, or the regulation of the transformer becomes too poor to afford satisfactory conditions for electric lighting where incandescent lamps are used.

The United States Government specifies the iron and copper losses shown in the table on page 520 for small step-down lighting transformers designed for a frequency of 60 cycles per second, 2200 volts primary voltage, and ratios of transformation of 10 and 20 to one.

The losses shown are low, and the relatively low iron losses compared with the copper losses result in the maximum commercial efficiencies occurring at loads less than the full loads, with resulting high all-day efficiencies compared with those which would be obtained were the same total losses at full load equally divided between the copper and the core.

U. S. TABLE OF IRON AND COPPER LOSSES

Kva. Rated Capacity	Iron Losses in Watts	Full Load Copper Losses	Total Full Load Losses in Per Cent of Capacity
1	21	26	4.70
2.5	33	55	3.52
5	45	93	2.76
10	82	160	2.42
20	135	280	2.08
30	175	374	1.83
40	210	470	1.70
50	255	580	1.67
75	400	800	1.60
100	560	1000	1.56

The commercial efficiency of transformers at full load ranges from 95 to 98 per cent or better in machines of from 10 kilowatts to the large sizes of several thousands of kilovolt-amperes capacity. In smaller sizes the full load efficiency becomes materially smaller and at one or two kilowatts capacity it may not exceed 92 or 93 per cent. Commercial transformers designed for the standard lighting frequency of 60 cycles per second are apt to have a slightly higher efficiency than those designed for 25 cycles per second, and likewise transformers designed for 2200 volts in the high voltage coil (whether primary or secondary) usually have a slightly higher efficiency than those designed for higher voltages. In Fig. 289 the curves *A* and *B* represent the full load efficiencies of a standard line of 500 Kva. transformers designed for voltages in the high voltage coil of 2200, 6600, 11,000, 16,500, 22,000, 33,000, and 44,000 volts respectively. Curve *A* is for transformers designed to operate on 60 cycles per second; and *B* for those operating on 25 cycles per second. The curve in Fig. 290 is the commercial efficiency curve of one of these 500 Kva. transformers designed for 2200 volts in the high voltage coil and a frequency of 60 cycles per second. The curve is typical of good efficiency curves for transformers of this size. It is noted in this figure that the efficiency at $\frac{1}{4}$ load is only about 2 per cent less than that at full load, and in this regard it is not far from the performance of other well-designed transformers, except those of very small capacities, — five kilowatts

or less capacities, — which show relatively lower efficiencies at light loads.

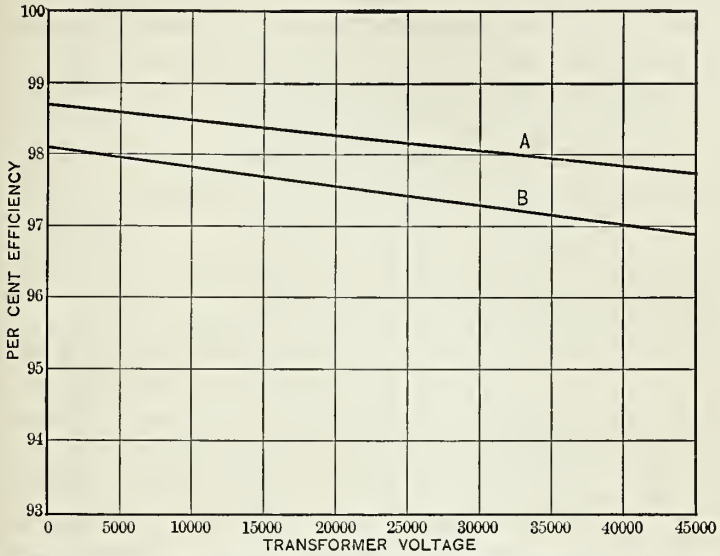


FIG. 289. — Curves showing the Full Load Commercial Efficiencies of the Same Size and Style of Transformers when designed for Various Voltages and for Frequencies of 60 and 25 Cycles respectively.

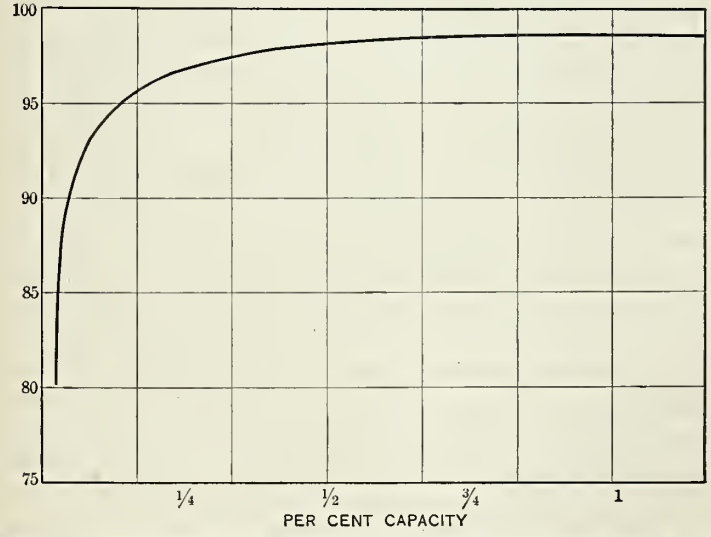


FIG. 290. — Curve of Commercial Efficiency of a 500-Kva., 60-cycle, 2200-volt (High Voltage Coil) Transformer.

The weight efficiency, that is, the weight of transformers without their containing cases or tanks per kilowatt of rated capacity, is from 7 to 20 pounds for transformers of 500 Kva. capacity and over. This increases by 50 to 100 per cent as the size decreases to 50 Kva. or thereabouts, and from there to 10 Kva. capacity the weight increases by 10 to 25 per cent further. For smaller sizes, especially those less than 5 Kva., the complete weight per kilo-volt-ampere of rated capacity including oil and cases, may be as much as eight or more times that of transformers of 500 Kva. capacity. Consequently, large transformers are very much less expensive to build than small ones per unit of output. The weight of a transformer is dependent not only on the design, but also upon the frequency and voltage of the circuit with which it is intended to be used; for low frequencies require a larger total magnetic flux to induce a given voltage in a given number of turns in the windings and hence they require larger cores or greater windings; very high voltages require more insulation space to be allowed in association with the windings; and very low voltages require the use of inconveniently thick conductors or conductors composed of multiples of parallel elements. The ratio of weight of copper to weight of iron varies from about 50 to 80 per cent. when the design follows the usual practice.

The regulation of transformers is of the highest importance, especially where carbon filament incandescent lamps are attached to the secondary circuit, as such lamps are liable to serious injury when subjected to varying voltages. A closely constant voltage for what is termed constant voltage apparatus is always desirable, though the percentage permissible variation is dependent somewhat upon the character and duty of the load. The accepted general definition of "regulation" of a machine or apparatus in regard to some characteristic quantity (such as terminal voltage, current, or speed) is "the ratio of the deviation of that quantity from its normal value at rated load to the normal rated load value,"* and for a constant voltage transformer it is specifically: "the ratio of the rise of secondary terminal voltage from rated non-inductive load to no load (at constant primary impressed terminal voltage) to the secondary

*Amer. Inst. of Elect. Eng., Standardization Rules, No. 187, *Trans.*, 1907, p. 1809.

terminal voltage at rated load." * The regulation is dependent upon the drop of primary voltage due to the leakage reactance and the resistance in the primary and secondary coils of a transformer, as may be seen by a study of the internal reactions of transformers given earlier.† The total rise of the secondary terminal voltage when the load changes from non-reactive full load to no load, and the primary impressed voltage is maintained constant, is

$$E_d = [(E + I_2R)^2 + I_2^2X^2]^{\frac{1}{2}} - E,$$

where I_2 is the full load secondary current, R is the sum of the primary and secondary coil resistances, the former reduced to the secondary equivalent, and X is the sum of the leakage reactances of the primary and secondary coils, the former reduced to the secondary equivalent, and E is the secondary terminal voltage at rated full load. The drop due to the primary exciting current is neglected here. This formula becomes evident from a study of the loci of the earlier article to which reference has just been made.

The regulation with a load of unity power factor (as called for in the foregoing definition) varies from less than 1 per cent to about 2 per cent in transformers of 10 Kva. capacity and greater, and may increase to 3 or 4 per cent in the very small sizes. The regulation obtained when the power factor of the load is other than unity is impaired if the load is inductive, but improves with a load exhibiting capacity reactance until the effect of the leakage reactance of the coils is neutralized and then with further increase of capacity reactance it becomes poorer. With inductive reactance in the load, the regulation is always poorer than when the load is of unity power factor. As inductive loads must ordinarily be reckoned with in commercial electric circuits, it is common for buyers of transformers to specify the regulation, not only for loads having unity power factor, but also for loads in which the power factor is .9, .8, and sometimes down to .6. If the constants of a transformer are known, it is easy to compute the regulation for any power factor of the load by constructing transformer diagrams or using the transformer formulas.‡

* *Ibid.*, No. 197, p. 1810. † Art. 123. ‡ Arts. 117–118.

138. Polyphase Transformers. — A saving in the amount of material used, and therefore a reduction of the cost of construction and an improvement of the operating efficiency, may be effected for polyphase transformation by combining the magnetic circuits of the individual transformers in the several phases in a manner which is analogous to that in which polyphase electric circuits are combined into common circuits.

Figure 291 represents a quarter-phase transformer with a combined magnetic circuit. Since the phases of the magnetic fluxes in the two halves of the transformer are 90° apart (the phases of the impressed voltages being 90° apart and here assumed sinusoidal), the resultant magnetism in the middle tongue is their vector sum and is $\sqrt{2}$ times as great as that in either of the cores under the windings. Therefore this central tongue should have $\sqrt{2}$ times as great a cross section as the remainder of the magnetic circuit. There is a saving of iron in the combined transformer, as compared with two independent transformers, which is equal to $(\sqrt{2}-1)$ times the weight of the central tongue. This saving is further emphasized by the advantage of the small space occupied and the convenient form obtained by inclosing this equivalent of two transformer units in one case. In this illustration, the high voltage external circuits are shown with a common return wire, and the low voltage external circuits are shown with independent wires; but it is obvious that the use of independent circuits requiring four wires for the two phases, or the use of three wire circuits utilizing one wire for the common return, are free alternatives for use with the primary and secondary circuits.

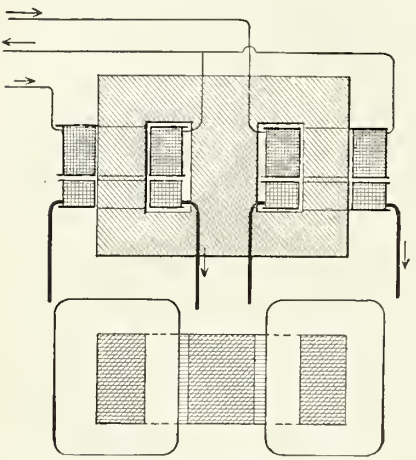


FIG. 291. — Diagram of Circuits and Core of a Quarter-phase Transformer.

A similar combination may be effected in tri-phase transformers, and the magnetic circuits may be coupled in either the wye or the delta arrangement. Figure 292 shows a conventional

diagram of a tri-phase transformer of a form early made by several manufacturers, in which the magnetism in the yokes DD', which join the cores A, B, and C, is $1/\sqrt{3}$ times as great as that in the cores. These yokes make a delta coupling for the magnetic

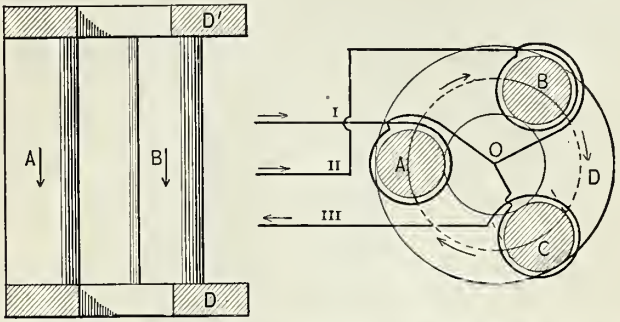


FIG. 292. — Diagram of Circuits and Core of a Tri-phase Transformer with Δ Yokes.

circuits. Figure 293 shows a similar diagram of a tri-phase transformer, in which the yokes are joined so as to make a wye coupling of the magnetic circuits. These constructions allow a considerable economy in use of materials in comparison with

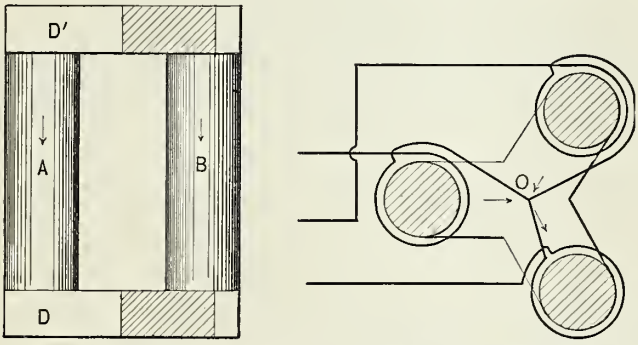


FIG. 293. — Diagram of Circuits and Core of a Tri-phase Transformer with Y Yokes.

separate transformers in the three phases, on account of the fact that the magnetic circuit through the core under each winding is completed through the cores under the other two windings.

In Fig. 294 is shown a diagram of circuits and core for a common type of tri-phase transformer. To obtain an expression for the instantaneous magnetic fluxes in the three cores

W, X, and Y, shown in Fig. 294, assume the magnetic reluctance in the path from A to A' indicated by the U-shaped dotted line through the core W and the path indicated by the corresponding U-shaped dotted line through the core Y to be equal, and of constant value P_w; also let the reluctance of the path directly from A to A' through X be represented by P_x. Consider that sinusoidal alternating exciting currents, 120° apart

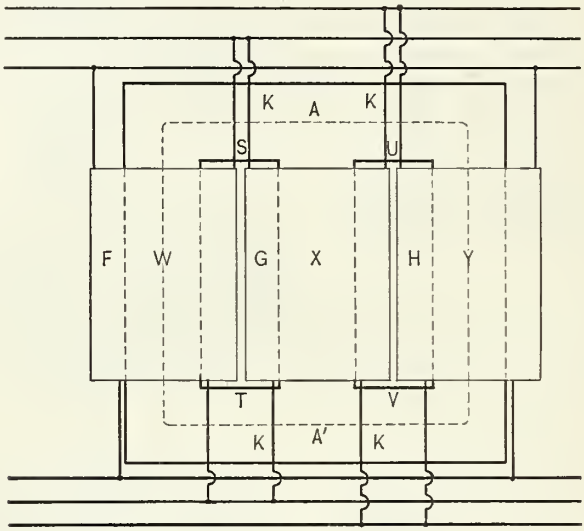


Fig. 294. — Diagram of Circuits and Core of a Tri-phase Transformer having Straight Yokes.

in their phases, flow in the three exciting windings, and let ϕ_w, ϕ_x, and ϕ_y represent values of the magnetic fluxes in the cores W, X, and Y, respectively, at a given instant, and m_w, m_x, and m_y represent the corresponding instantaneous values of the magneto-motive forces exerted in the aforenamed paths through the three cores. Then, using the same conventions in regard to the algebraic signs of the instantaneous values of magneto-motive forces and fluxes in the several parts of the transformer as are set forth in Art. 103 for three-phase electric circuits, the following simultaneous equations represent the conditions:

$$m_w - m_x = \phi_w P_w - \phi_x P_x,$$
$$m_w - m_y = \phi_w P_w - \phi_y P_w,$$
$$m_x - m_y = \phi_x P_x - \phi_y P_w,$$
$$\Phi_w + \phi_x + \phi_y = 0.$$

Solving these equations gives the values of ϕ_w, ϕ_x, and ϕ_y in terms of m_w, m_x, m_y, P_w, and P_x, as follows:

$$\phi_w = \frac{m_w - m_x}{P_w + 2P_x} + \frac{P_x(m_w - m_y)}{P_w(P_w + 2P_x)}. \quad (1)$$

$$\phi_x = \frac{m_w - m_y}{P_w + 2P_x} - \frac{2(m_w - m_x)}{P_w + 2P_x}. \quad (2)$$

$$\phi_y = \frac{m_y - m_x}{P_w + 2P_x} + \frac{P_x(m_y - m_w)}{P_w(P_w + 2P_x)}. \quad (3)$$

Putting $m_w = aI\sin\alpha$,

$m_x = aI\sin(\alpha - 120°)$,

$m_y = aI\sin(\alpha - 240°)$,

a being a constant, these equations become,

$$\phi_w = \frac{aI}{P_w + 2P_x}\{\sin\alpha - \sin(\alpha - 120°) + P_x/P_w[\sin\alpha - \sin(\alpha - 240°)]\} \quad (4)$$

$$\phi_x = \frac{aI}{P_w + 2P_x}\{2\sin(\alpha - 120°) - \sin(\alpha - 240°) - \sin\alpha\}. \quad (5)$$

$$\phi_y = \frac{aI}{P_w + 2P_x}\{\sin(\alpha - 240°) - \sin(\alpha - 120°) + P_x/P_w[\sin(\alpha - 240°) - \sin\alpha]\}. \quad (6)$$

The hysteretic angle of lead being zero under the assumption made, these fluxes come to their respective maxima when $\alpha = 90°$, $\alpha = 210°$, and $\alpha = 330°$, and have the numerical values,

$$\Phi_w = \Phi_y = \frac{1.5\,aI}{P_w + 2P_x}(1 + P_x/P_w) \quad (7)$$

$$\Phi_x = \frac{3\,aI}{P_w + 2P_x}. \quad (8)$$

These values are obtained on the assumption that sinusoidal magneto-motive forces of equal amplitudes and differing successively in phase by 120° are impressed by the coils in the three branches of the magnetic circuit, and also on the assumption that P_x differs from P_w and P_y, but that the last two are equal to each other.

It is obvious from these formulas, that with equal magneto-motive forces, the value of Φ_x is larger than the value of Φ_w and Φ_y, in case P_x is smaller than P_w on account of the shorter

length of the middle magnetic circuit. The maximum fluxes Φ_w, Φ_x, Φ_y are all equal to one another when $P_x = P_w = P_y$; but if P_x is negligible compared with P_w, Φ_x becomes twice as large as Φ_w and Φ_y.

Under the usual conditions of transformer operation, equal voltages, 120° apart in their phases, would be impressed on the three equal primary windings. Under these circumstances, equal fluxes would be set up in the three branches of the magnetic circuit, and equations (7) and (8), which give the relations between the fluxes and exciting currents, show that the exciting currents must differ under these circumstances unless $P_x = P_w = P_y$. Such a difference of currents causes an unbalancing of the supply circuits. Also on account of the difference of form of the magnetic leakage path around the middle limb compared with the leakage paths around the two outer limbs, unbalancing is felt in the secondary voltage.

To partially avoid this unbalancing, the middle leg may be made of smaller cross section than the other two legs, so that P_x is made more nearly equal to P_w and P_y, but this is likely to increase the core losses on account of the increased magnetic density in that part of the core, and it also changes the power factor of the exciting current of that leg compared with the currents exciting the other legs. As the yokes which complete the magnetic circuits at the ends of the limbs obviously carry the same maximum flux as the limbs themselves, if magnetic leakage is neglected, the yokes are usually made of the same cross sections as the outer limbs. The same advantages in respect to reduced weight and bulk and reduced exciting current relate to this form of construction as to the construction with wye and delta magnetic circuits.

The magneto-motive forces of the three primary coils being out of phase with each other, magneto-motive force pulsations between the points S and T, and U and V are created which tend to set up leakage fluxes between these points, which is in addition to the ordinary magnetic leakage between the primary and secondary coils. Pulsations of the third harmonic are most prominent and are in the same phase and the same direction through their respective cores.* Quite large additional third harmonic currents may therefore flow around the mesh made by

* Art. 198.

the primary coils if they are delta connected as shown in Fig. 294. If, however, the coils are connected in wye without a neutral return wire, these currents comprising the third harmonic cannot flow and the resulting leakage magnetism is absent. If the neutral point of the wye is connected to a neutral or fourth wire of the supply system, the currents comprising the third harmonic will flow through the three lead wires and coils in parallel relation and complete their circuit through the neutral wire.

When the three-phase transformer is of the shell type (in which the windings are embedded in the iron), as in Fig. 295, and the primary windings are all connected in the same relative direction, the resultant magneto-motive force acting along the whole length of the common core is

$$\overline{M}_A + \overline{M}_B + \overline{M}_C = 0,$$

where $\overline{M}_A$, $\overline{M}_B$, and $\overline{M}_C$ are the magneto-motive forces in the coils A, B, and C, in case the magnetizing currents are assumed to be sinusoidal, equal in scalar value, and 120° apart in phase. This means that the vector sum of the magneto-motive forces, as in the previous case, equals zero, but in this case any two are in series instead of parallel with reference to the third; also,

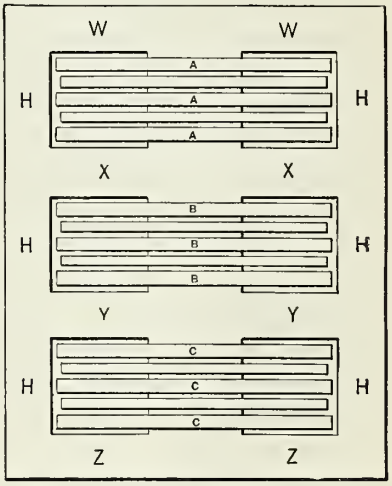


FIG. 295. — Diagram of Tri-phase Shell Type Transformer.

$$\overline{\Phi}_A + \overline{\Phi}_B + \overline{\Phi}_C = 0,$$

where $\overline{\Phi}_A$, $\overline{\Phi}_B$, and $\overline{\Phi}_C$ respectively represent corresponding vector values of the three magnetic fluxes.

The distribution of the magnetism in the core is as follows: The flux in Y is the combination of the fluxes threading coils C and B, and that in X the combination of the fluxes threading coils B and A. As the vector fluxes threading coils C and B are numerically equal, but differ in phase by 120°, the flux in Y

is (neglecting magnetic leakage) equal to vector C-flux plus vector B-flux reversed. That is, the vector addition is made in the same manner as for three-phase electric circuits.* Therefore Y, and for the same reason X, must carry a flux equal to $\sqrt{3}$ times the flux threading either of the coils. The maximum flux in the shell at W, Z, and H has the same value (neglecting magnetic leakage) as the maximum flux threading the coils.

When the middle coil B in Fig. 295 is reversed relatively in respect to A and C, the magneto-motive force formula becomes

$$\overline{M}_r = \overline{M}_A - \overline{M}_B + \overline{M}_C.$$

The vectors on the right-hand side of this formula are now only 60° apart in phase and the scalar value of their resultant $M_r = 2\,M$, where M is the scalar value of the vector magneto-motive force in each coil. But the fluxes in all of the cores must be equal in scalar value and 60° apart in phase in order to produce the proper counter-voltages in the primary coils. Therefore the flux due to $\overline{M}_r$, or

$$\overline{\Phi}_r = \overline{\Phi}_A - \overline{\Phi}_B + \overline{\Phi}_C,$$

where $\overline{\Phi}_A$, $\overline{\Phi}_B$, and $\overline{\Phi}_C$ are the fluxes threading the coils A, B, and C. In order to maintain the proper counter-voltages in the windings and vector sum of fluxes in the main core, flux $\overline{\Phi}_A$ is set up in the magnetic circuit $AWHXA$, $\overline{\Phi}_B$ in $BXHYB$, and $\overline{\Phi}_C$ in $CYHZC$. Fluxes $\overline{\Phi}_A$ and $\overline{\Phi}_B$ are 60° apart in phase in the central core or tongue, so that they are 120° apart in phase in X. The tensor of their sum is then equal to the scalar value of either of the two. The same conditions exist in Y, so that the combined cross section of iron at A, B, C, W, X, Y, Z, and H may be made uniform for the same induction in all parts of the iron. There is thus some saving obtained in material, or reduction in the reluctance of the core, by reversing the middle coil connections. The exciting current is the same in each phase as would be required in three single-phase transformers having equal core reluctances.

When the voltages or currents or both are unbalanced, the conditions still hold that the magnetic flux for each phase must

* Art. 103.

be such that it will produce the proper counter-voltage, and hence the flux under each coil remains proportional to the counter-voltage in that phase; and the three fluxes, however different in value, will combine vectorially according to the same laws as where the circuits are balanced.

139. Some Constructive Features of Constant Voltage Transformers. — The formula

$$E_1 = \frac{\sqrt{2}\,\pi n \Phi f}{10^8}$$

contains the four quantities E_1, n, Φ, and f, which may be varied, but in practice E_1 and f are usually fixed by the requirements of the service, and only n and Φ may be varied. Their relative variation is also limited by the fact that the product $n\Phi$ is fixed when E_1 and f are fixed. In designing a transformer, the relative values of n and Φ depend upon the relative weights which it is considered desirable to assign to the copper and iron in the transformer. When the magnetic density which is safe to use in the core has been chosen, the cross section of the core depends directly upon Φ; while n, and therefore the weight of copper, assuming that the size of wire to be used has been decided upon, is inversely dependent upon the same quantity. The weight of iron depends upon the cross section and length of the magnetic circuit, and must be limited so that the losses occurring in the iron of available quality shall not reduce either the full load efficiency or the all-day efficiency below a reasonable figure.

The number and length of the turns (n) of wire used in each winding and the size of the conductors which are chosen for the purpose of preventing injurious heating, objectionable reduction of full load efficiency, or poor regulation, jointly determine the required weight of copper. The density of current in the windings varies widely, from 400 to over 2000 circular mils cross section per ampere. This density is largely dependent upon the form of the coils. These are seldom made so that the heat liberated by the I^2R loss must travel more than one half an inch before reaching the radiating surface of the coil. It is not unusual to make the density somewhat smaller in the low voltage winding than in the high voltage winding, and the values in the best transform-

ers frequently fall between 1000 and 1500 circular mils per ampere for the high voltage winding and between 1200 and 2000 for the low voltage winding. On the other hand, some designers make the density of current greater in the low voltage windings, while others make the density about the same in both. As the high voltage conductors carry less current and are therefore of smaller cross section than the low voltage conductors, the insulation of the high voltage winding takes up more space in proportion to the space occupied by the conductors ; and this condition becomes emphasized when the voltage exceeds a few thousands of volts, on account of the exaggerated thickness of insulation required for the high voltage. This heavier insulation also makes it more difficult for the heat generated to escape, and it is often necessary to divide the high voltage winding into a number of thin coils separated by ducts for the circulation of air or oil.

The I^2R loss in transformers at full load is ordinarily equal to from $\frac{1}{2}$ per cent to $3\frac{1}{2}$ per cent of the full load capacities. This is divided with approximate equality between the primary and secondary windings. Sometimes this loss is permitted to reach 5 per cent, but in the better transformers it is more often between 1 per cent and 3 per cent. The percentage of loss which is allowable depends primarily upon the transformer efficiency and regulation which are desired, and secondarily upon the duty to be performed, the voltage, and the frequency.

Transformers are of two general classifications, termed the **Core type** and the **Shell type**, these designations depending upon the manner in which the windings and the iron core are disposed with respect to each other. Both forms seem to lend themselves satisfactorily to the design of transformers for ordinary duty, and, in fact, there is no definite line of demarcation between them. In general terms, core type transformers may be defined as those in which each of the principal branches of the magnetic circuit is embraced by windings, as illustrated in Figs. 47 and 296, while shell type transformers are those in which the return branches of the magnetic circuit embrace the windings, as illustrated in Figs. 300 and 301.

Laminations for the cores are made in various shapes. Figure 296 represents the core and coils of a core type transformer,

and Fig. 297 shows a pair of the laminations or stampings of which the core is made up. When the stampings are of this form, they are built up into a core within the finished coils. In every other layer of the core the stampings are reversed end for end so that they break joints. This is for the purpose of reducing the magnetic reluctance of the magnetic circuit to a minimum. It is important that joints in the magnetic circuit shall be as few as possible, and be so made as to be of low reluctance, as otherwise the exciting current may be caused to be of undesirable magnitude. Rolled iron or steel sheets as they come from the rolling mills are usually covered by a tough and closely adherent coating or thin scale of magnetic oxide of iron, which is developed in the process of manufacture of the sheets. This is a poor electrical conductor, and since it sticks to the stamping, it usually affords sufficient insulation between the laminations to prevent eddy currents from becoming excessive. However, to give double assurance, it is usual to dip the stampings into a liquid varnish or enamel and then bake them to solidify the coat before building them into a core.



Fig. 296. — Core Type Transformer, Core and Coils.

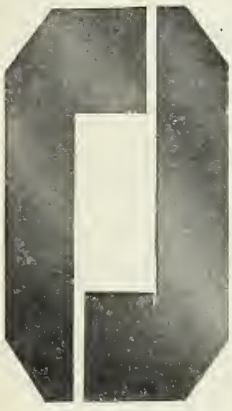


Fig. 297. — Laminations for Transformer shown in Fig. 296.

In Fig. 298 is shown a leg of the core of a large transformer which is composed of rectangular laminations built into a cross section having a cruciform shape. This is for a core type transformer such as is shown in Fig. 47. The complete core comprises two such legs, each covered with windings, and connecting yokes completing the magnetic circuit at their ends. The crossbars con-



Fig. 298. — Cruciform Leg of Transformer Core, built up of Rectangular Laminations.

necting the ends of the legs are also made of rectangular laminations which alternately slip between and butt against those on the leg shown, every other one on the leg being made short to give space for this purpose, or the dovetailing may be done by bunches of laminations instead of by alternate sheets. The type of coil used for the windings of a transformer using the cruciform core is shown in Fig. 299. It is composed of a cotton-covered rectangular copper strip wound edgewise. Where possible to insulate properly proportioned rectangular wire, its use is often desirable, as it occupies less space for a given effective area than round wire. It is also frequently desirable to wind it edgewise, that is, with its greatest width in the plane at right angles to the core, as this construction often makes it practicable to make the complete coil with only one layer of the conductor, thus affording opportunity for efficient cooling, and the construction may be made rigid and strong. Bare strips are sometimes wound in the edgewise coils, with thin strips of vulcanized fiber, varnished paper, or varnished cambric laid on edge between the turns of conductor.



Fig. 299. — Division of a Coil for the Transformer of which Part of the Core is shown in Fig. 298.

Figure 300 shows a shell type transformer with the laminations horizontal. The wire in this construction is wound with its flat faces parallel to the axis of the coil, and the windings are made up of a number of flat sections, each having a thickness equal to the breadth of one conductor strip.

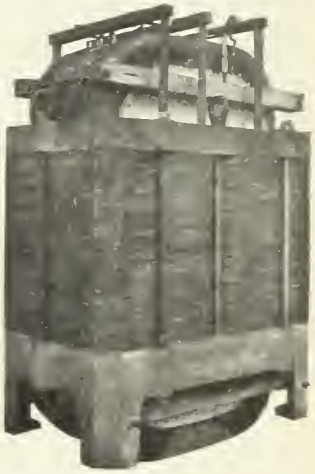


Fig. 300. — Shell Type Transformer with the Laminations Horizontal.

It is usual in the larger transformers, — 50 Kva. capacity and over, — to split up the primary winding into sections or divisions and sandwich between them

the secondary winding, which is also wound in sections. This reduces magnetic leakage and, when free space is left between the coil sections, facilitates the dissipation of heat by providing opportunity for the circulation of air or oil.

Figure 301 shows a shell type transformer in which the laminations and coils are vertical. In this case the coils are made elongated as in Fig. 299, and therefore the sections of the primary and secondary windings are sandwiched within one another, while in Fig. 300 the sections are flat and wide and lie side by side.

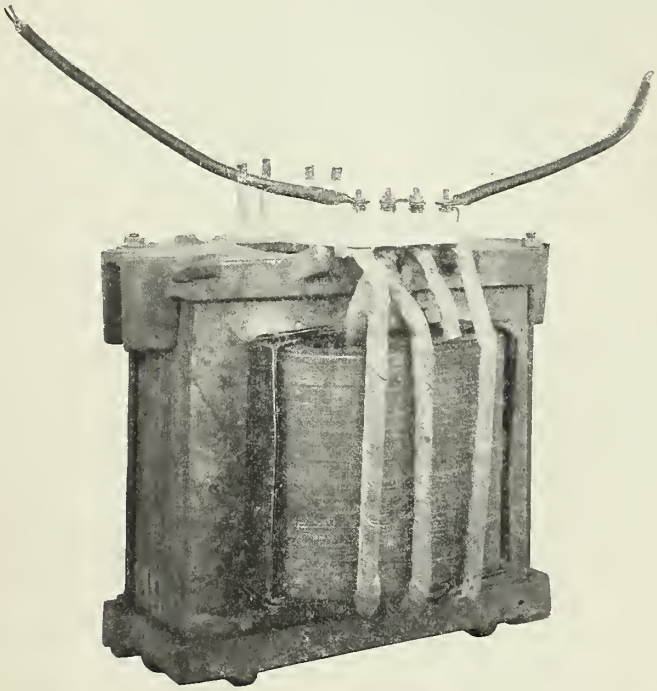


FIG. 301. — Shell Type Transformer with the Coils and Laminations Vertical.

From the illustrations it may be observed that core type transformers usually consist of a simple rectangular magnetic circuit with half of the primary and secondary turns wound upon each of two legs, and the cross sections of the two legs and the two end pieces are equal. Likewise it may be observed that the core of a shell type transformer usually comprises a central branch or leg upon which the windings are located and

which is of double the cross section of the remainder of the magnetic circuit if the latter is closed by two branches as illustrated in Figs. 300 and 301.

Figure 301 *a* shows the construction looking down on the core and windings from the top, of a transformer belonging to a type now much used, in which the windings surround a central branch of the magnetic circuit, and the magnetic circuit is closed by several limbs which join radiating end pieces. This arrangement has proved advantageous for transformers of small and medium capacities, since it affords windings in which the mean length of turn is reasonably short and at the same time affords a magnetic circuit of the requisite cross section while permitting the use of a moderate bulk and weight of iron. This results in satisfactory regulation associated with reasonably good all-day efficiencies. The construction also has some advantages in reducing the total bulk of small transformers without interfering objectionably with the opportunities for getting rid of the heat produced by the operation. It presents a little more difficulty in the mechanical construction of the core, more particularly in respect to obtaining a magnetic circuit of the lowest practicable reluctance without undue expense.

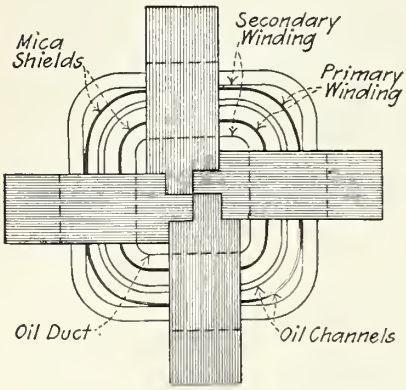


FIG. 301 *a*. — Transformer of Intermediate Type.

Figure 302 shows a tri-phase transformer of the shell type with three cores meeting at the center for carrying the coils of the three phases. It is com-

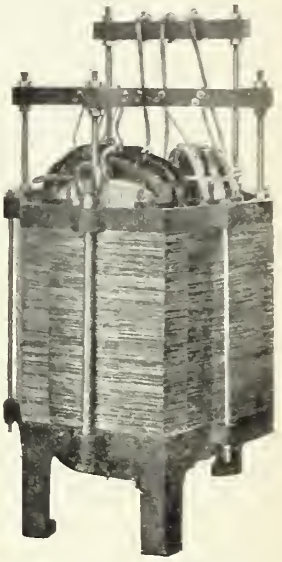


FIG. 302. — Tri-phase Transformer with the Magnetic Circuits connected in Wye Fashion.

mon in this type of transformer to have the coils of the three phases distributed on a single core, as in the diagram of Fig. 295. A core type tri-phase transformer may be arranged as in the diagram of Fig. 294, and the reluctances of the three magnetic paths can be made equal by arranging the limbs at the corners of a triangle, with their magnetic circuits joined in either delta or wye, as indicated in Figs. 292 and 293.

The quarter-phase transformer core may be built conveniently as shown in the diagram of Fig. 291, or it may be built of the shell type, similiar to that shown in Fig. 295.

The various arrangements in which the iron of a transformer may be disposed, which will give approximately the same results in operation, are quite numerous, and the particular one to be selected in any instance is apt to turn upon the question of the cost of manufacture rather than the needs of service. The first and most important element to be dealt with in the construction of a transformer, as in the case of most other electrical machinery, is the disposition of the heat occasioned by the iron and copper losses. Upon the solution of this problem depends the rise of temperature of the transformer per kilovolt-ampere of load when in operation, and its ultimate safe capacity. The windings of transformers are usually embedded more or less in the iron cores, as is illustrated in the preceding figures; and the whole transformer is inclosed in a waterproof iron case, as illustrated in Figs. 47, 303, and 304. The rise of temperature when in operation is due to the heating of the structure by the core losses and the copper losses. If transformers were not encased, but were placed naked in the open air, the entire external surface could be assumed to be effective in dissipating heat by radiation and convection; but on account of the inclosing case, convection of heat to the external atmosphere cannot take place directly, and all the heat must be radiated to the wall of the case or carried thereto through the poor heat conductors which are used to electrically insulate the transformer from its case. The conditions therefore point to the conclusion that, without some special method of cooling, for a given liberation of heat per square centimeter of surface, the working temperature is likely to be higher in transformers than in dynamo field windings. In fact, to prevent excessive working temperatures and permit the highest

safe output per pound of iron and copper, special means of cooling are usually provided for transformers. These are described later.

Transformer coils are usually designed to be lathe wound, and these may be very effectually insulated by a liberal use of mica, varnished cotton cloth, fiber, and wood. It is therefore possible to safely run transformers with the windings at a considerably higher temperature than dynamos. Sixty degrees centigrade (108° F.) may be set as a maximum limit to the safe rise in temperature caused by operation, though so high a temperature is undesirable, both because of depreciation of the insulating materials and the tendency of the core to age unless a non-aging steel alloy is used. A high temperature limit has also a marked disadvantage in causing an undue impairment of the regulation when the transformer is hot, which results from the increased resistance of the windings. Many transformers still in operation exceed the temperature limit named, but most of the best types do not exceed 40° centigrade rise. As the rise in temperature also increases the electrical resistance of the iron core, it decreases the eddy current loss, so that, as suggested by Elihu Thomson, it was early considered advantageous to have the core of a transformer operated at a high temperature while the windings were kept cool. This could not be conveniently arranged in small transformers, but the cooling of the conductors of very large transformers has been experimentally effected by making the conductors tubular and passing a cool liquid through them.

It is practically impossible to fix any averages for the external surface of transformers needed per watt lost in the core and windings, on account of the very varied arrangements of the coils with reference to the core, and the effect of the containing case. In good commercial designs of transformers the superficial area of core and windings varies from 3 to 7 square inches per watt of energy to be dissipated. For transformers of small and medium capacities it is usual to make the design as compact as possible, and no particular trouble from heating is experienced if the losses are not excessive when judged from the criteria of efficiency and regulation, provided the inclosing case is filled with oil for the purpose of improving the facility with which the heat can escape to the outer surface of the case. A highly

refined petroleum oil carefully filtered and dried is used for this purpose, and is commonly called **Transformer oil**, on account of the principal object of its use. Such oil also posesses high insulating qualities, which is an advantage in the transformers. Transformers with which oil is associated for cooling purposes are called **Oil-cooled** transformers. Those without oil are sometimes called **Dry-core** transformers. As transformers increase in size, the weight per unit of capacity decreases, and the superficial area per unit available for dissipating heat decreases at a still more rapid rate, since the bulk and weight are proportional to the cube of the linear dimensions, and the superficial area is proportional to only the square of the linear dimensions; and some device, such as circulating the oil within the case and through ducts in the core and between the divisions of the windings, or blowing air through ducts in the core, must be provided for forcing the more rapid conveyance of heat from the core and windings. Otherwise the heating would be excessive unless the transformer is of unreasonably great and expensive bulk. Dry core transformers, arranged for cooling by a blast of air, are called **Air-cooled** transformers.

A small transformer will evidently cool more readily than a large one of the same type and design. Thus, in a number of transformers of different capacities having the same efficiencies, losses per pound of copper and iron, and voltages, and of similar design, if we let L stand for linear dimensions, A for superficial area, W for weight, K_L for full load losses, and Kva for rated full load output, we have

$$W \propto L^3; \quad A \propto L^2;$$

$$K_L \propto W \propto L^3 \propto A^{\frac{3}{2}}, \text{ approximately};$$

$$Kva \propto L^4 \propto W^{\frac{4}{3}} \propto K_L^{\frac{4}{3}} \propto A^2, \text{ approximately.}$$

It is evident that Kva varies approximately as L^4, since when the dimensions are varied the cross section of the magnetic circuit and hence the magnetic flux is varied as L^2. With fixed voltage, the number of turns in the windings therefore changes proportionately to $\frac{1}{L^2}$ and the space in which the windings go changes proportionately to L^2. Hence with winding space

changed in proportion to L^2 and number of turns in proportion to $\frac{1}{L^2}$, the area of each conductor may be changed in proportion to L^4, and its current-carrying capacity correspondingly changed, provided the necessary insulating space can be reserved and the heat produced by the I^2R losses at full load can be carried off.

It is seen, therefore, that transformers of large sizes should have normally less weight per kilowatt capacity, full load losses lower in proportion, and higher efficiency than those of smaller size, — which deductions are, in fact, borne out in practice. But the dissipation of heat in the larger transformers is more difficult, since $K_L \propto L^3 \propto A^{\frac{3}{2}}$, and therefore the area exposed for cooling in similar transformers of different sizes varies only with the $\frac{2}{3}$ power of the losses, when the losses are taken to be proportional to the weights of the transformers, and it also varies as the square root of the rated transformer capacity given in kilovolt-amperes. Hence it is evident that if a small transformer of a given form heats at full load to the maximum temperature allowed, say 40° centigrade, a transformer of ten times the size but equivalent in form and all other constants of the design will heat to a much higher temperature; for, using primed symbols to represent the larger transformer,

$$\frac{A}{A'} = \frac{(Kva)^{\frac{1}{2}}}{(K'va)^{\frac{1}{2}}} = \frac{(Kva)^{\frac{1}{2}}}{10^{\frac{1}{2}}\,(Kva)^{\frac{1}{2}}} = \frac{1}{3.2},$$

and

$$\frac{K_L}{K'_L} = \frac{(Kva)^{\frac{3}{4}}}{(K'va)^{\frac{3}{4}}} = \frac{(Kva)^{\frac{3}{4}}}{10^{\frac{3}{4}}\,(Kva)^{\frac{3}{4}}} = \frac{1}{5.6},$$

whence

$$\frac{K_L}{A} : \frac{K'_L}{A'} = 1 : 1\tfrac{3}{4}.$$

That is, the full load losses of the larger transformer are about three quarters greater per square inch of the superficial area of the transformer, although the losses are only about half as great per kilovolt-ampere of rated capacity. Had the size been 100 times increased, the rate of liberation of heat would have been over 200 per cent greater per square inch of radiating surface. Therefore, transformers of large size require either the application of special measures for carrying away the heat

generated, or they must be given special designs which would prove bulky and expensive in sizes of 50 Kva. capacity or over.

Several methods for cooling are in use :

For very small transformers — a fraction of a kilowatt capacity — no special arrangements are needed.

For transformers from a kilowatt capacity to those of several hundred kilowatts capacity the most common method is to immerse the transformer in a tank of insulating oil. For the smaller sizes this is not necessary, but it is desirable, as the oil tends to maintain the insulation and close up punctures. The tanks for smaller sizes of transformers, say to about 50 Kva. or less capacity, are frequently made of cast iron with a smooth outer surface, as shown in Fig. 47 ; while those of larger capacity are made of cast iron or riveted boiler plate with a corrugated surface in order to present more radiating surface, as shown in Fig. 303. The action of the oil is to circulate upward through ducts in the warm core and over the surface of the warm coil sections and downward along the cooler interior of the the tank or case. The coils and cores are so arranged that the oil can readily circulate through them ; thus in Fig. 298 the cruciform shaped core leaves triangular channels through which the oil can flow between the core and the cylindrical inner face of the coil. Where necessary, channels or ducts are left between each $1\frac{1}{2}$ or $2\frac{1}{2}$ inches of thickness of the laminations. In transformers of large size space is left for the oil to circulate freely between sections of the coils ; thus in Fig. 302 is shown a transformer intended for oil cooling in which the coils are so separated.



Fig. 303. — Transformer Case having a Corrugated Surface for increasing the Area for Heat Dissipation.

The radiation of the heat from the case is commonly at the rate of one watt (joule per second) to each 4 to 8 square inches

of surface when the case temperature is from 35° to 40° centigrade above the temperature of the surrounding air.

In the case of large transformers, from a couple of hundred to several thousand kilowatts capacity, it is found necessary to artificially cool the oil, as the case will not radiate heat to the surrounding air fast enough to keep the temperature down to a reasonable value. In this case it is common to circulate water in a spiral of pipes located inside of the case above the height of the transformer proper. Such a transformer is said to be **Water-cooled.** The arrangement is illustrated in Fig. 304, which represents the design of a water-cooled transformer, the cooling coils being at *AA*. This figure shows one of the ways for making an oil-proof case of riveted or welded boiler plate, which in this instance does not need to be corrugated since the major part of the cooling is accomplished by the circulating water which flows in the pipes. The figure also shows means which may be used for supporting the cores, coils, and terminals, and for draining the oil from the tank. The latter is accomplished by the pipe containing the valve *B*, which leads to a suitably protected receiving tank. This is most important in order to prevent disastrous results in case the oil in the transformer catches fire. Other constructive details are also illustrated in the figure. The water pipes in this case are covered with cotton tape above the oil to prevent the collection and drip of moisture.

When a water-cooled transformer, such as is illustrated in Fig. 304, is in operation, the oil in contact with the core and windings becomes warm and tends to rise, while the oil cooled by contact with the cooling pipes tends to fall. This sets up a circulation of oil upwards through the ducts and over the surface of the transformer into contact with the cooling pipes, then the cooled oil falls to the bottom of the case through the ample space left around the transformer. In this manner heat is carried away from the transformer and delivered to the cooling pipes. The cooling water may be caused to circulate by a pump and some means utilized to cool it after its exit from the cooling coil. The water may thus be used over and over. For transformers of very large sizes it is sometimes desirable to also give the oil a forced circulation. In this case, oil is drawn from the transformer by means of a pump, passed through a surface condenser or cooler from which the heat is

abstracted by water, and returned to the transformer under pressure, where it is compelled by the design to circulate through the core and windings before again reaching the outlet.

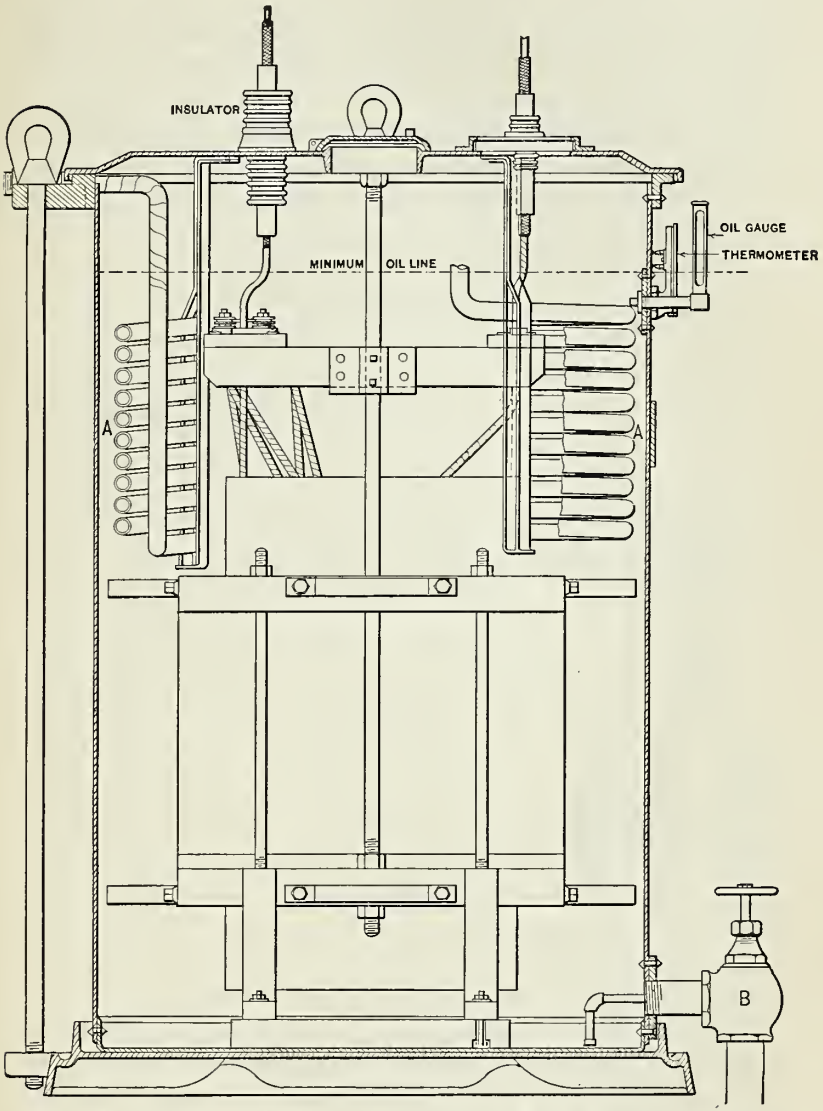


FIG. 304. — Design of a Water-cooled Transformer, showing Constructive Details.

A good method, and one of the oldest, of cooling indoor transformers of low or medium high voltage is by means of a blast

of air. The air blast is produced by a fan, and directed to the transformer by way of an air chest under the transformer case, thence through the core and windings, and thence out of the top cover of the case. The windings and core are designed with ducts to give as free circulation of air as possible. A large number of air-blast transformers can be located over one air chamber and cooled effectively and economically in this way.

Figure 305 shows the rise of temperature of the conductors of an old type air-blast transformer of 200 kilowatts capacity as a function of the period of operation, operated with the blast

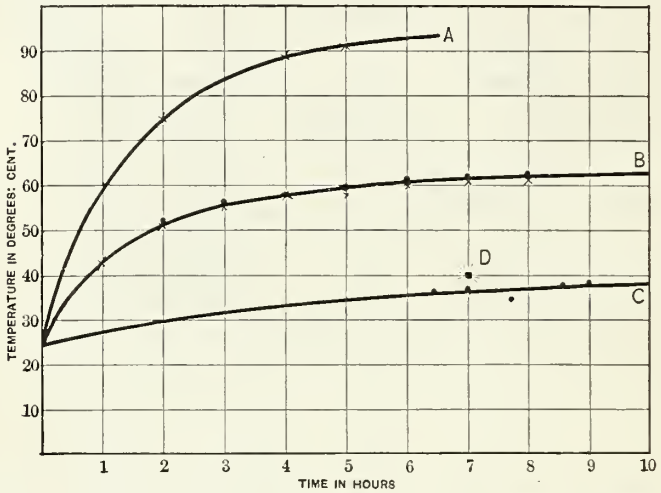


FIG. 305. — Curves showing the Effects of an Air Blast in Cooling Transformers.

and without the blast. Point *D* shows the temperature of the stampings after the transformer has been operated seven hours with blast on; curve *A* shows the temperature of the windings when operated at full load without blast; curve *B* shows the temperature of the windings when operated at full load with blast of 1040 cubic feet per minute; and curve *C* shows the temperature of the air issuing from the transformer.

The insulation in transformers is composed largely of the materials described for generator insulation,* but materials which deteriorate in either hot or cold oil cannot be used in oil-cooled or water-cooled transformers. The usual substances used for the insulation between the cores and coils are mica and

* Art. 36.

varnished cotton cloth or paper, with the addition of blocks of wood or press board where large volumes of insulating materials are demanded. The wire is usually double cotton covered, and varnished cloth is used between the layers. The voltage between layers seldom exceeds 200 volts. In fact, 200 volts per layer is high. Each winding is divided into thin sections, giving a total of not over 3000 or 4000 volts, which figures are also rather high for the best construction. These coil sections are thoroughly impregnated, usually by using a vacuum process, with linseed oil, asphaltum, or compounds derived from coal tar. It is evident that in transformers for high voltages, such as from 60,000 to 100,000 or more volts, the insulation between the cores and coils must have very high dielectric strength so that much mica is employed. The insulation between coil sections is largely accomplished by the cooling oil which surrounds them, as the sections are kept apart by narrow separators of wood or pressboard.

The terminals of a transformer are troublesome to insulate, as they are not in oil, and especially since high disruptive forces are apt to be set up at those points upon opening or closing circuits. Fig. 306 shows at *A* a pair of 66,000 volt leads for a transformer located under roof. The wires themselves are insulated to many times their diameter and are bushed in porcelain where they pass through the iron cover plate of the transformer. At *B* is shown a lead, for a similar voltage, to be used in a transformer intended for use out of doors. The multiple petticoated porcelain insulator is for the purpose of effectively disposing of rain water.

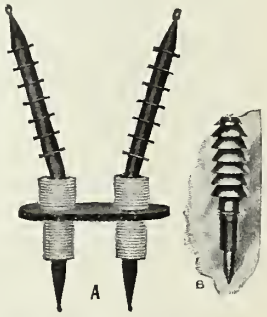


FIG. 306. — Terminals for 66,000 Volt Transformer. *A*, for Indoor Service. *B*, for Outdoor Service.

A desirable form of terminal is now manufactured with alternate layers of insulating material and tin foil, the function of the latter being to distribute the electric stress uniformly through the dielectric.*

The oil in transformers should withstand a breakdown test of about 30,000 volts when the voltage is applied between

* Trans. *A. I. E. E.*, Vol. 28, p. 209.

electrodes having spherical ends of .5 centimeter diameter placed .4 centimeter apart. It should be free from moisture and other foreign substances, such as materials containing sulphur, alkali, or acids; should be a mineral oil manufactured by the fractional distillation of petroleum; and should be very fluid and neither solidify or form wax at 0° centigrade. Its flash point should be at least 170° centigrade.* The removal of moisture which may have got into otherwise satisfactory oil may be accomplished by filtering the oil through common soft blotting paper. It may also be accomplished by heating the oil to a temperature above 100° centigrade and stirring it.

140. Methods of connecting Constant Voltage Transformers and some Features of their Operation. — Single-phase transformers can be connected to single phase circuits either to raise or lower the voltage, as shown by the upper or lower transformers in the diagram of Fig. 307. If the three-wire system is used in the secondary circuit, two transformers in series can be used, but more commonly the neutral wire is tapped to the middle of the secondary winding, as shown by the dotted line in Fig. 307. Lighting transformers generally have their secondary windings divided into two coils with their terminals brought out to terminal blocks so that the three-wire connection can be readily made.

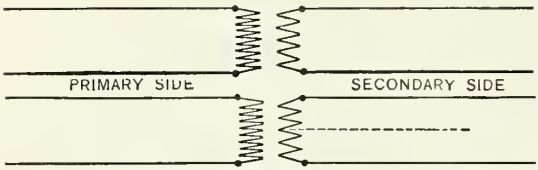


FIG. 307. — Diagram showing Connections of Transformers.

Two or more transformers are sometimes connected with their secondary windings in parallel on the same secondary circuit; in which case it is necessary that they shall have equal ratios of transformation, and that their internal resistances and reactances shall be inversely proportional to their respective rated capacities, or they will not divide the total load proportionally to their rated capacities. Thus, consider two transformers designed for equal primary and no-load secondary voltages and equal rated capacities, but having their total internal impedances, resolved in terms of secondary circuit equivalents, in the ratio

* U. S. Government Specifications.

of the vectors $\bar{Z}_t$ and $A\bar{Z}_t$, where A is a constant, the transformers being connected to the same primary circuit. The currents in the two are respectively proportional to $\frac{1}{\bar{Z}_t}$ and $\frac{1}{A\bar{Z}_t}$. The load on one transformer is therefore less than that on the other transformer in the ratio of $\bar{Z}_t : A\bar{Z}_t = 1 : A$. Moreover, the secondary terminal voltages may not be exactly in the same phase, and an undesirable cross current will flow, so that the arithmetical sum of the currents carried by the two transformers is greater than the current in the load, unless $\frac{X_t}{R_t}$ is equal for the two transformers.

If the ratios of transformation on open circuit are different in the ratio of $E_2 + B$ to E_2, where E_2 is the secondary terminal voltage of one transformer, the transformer having the lower secondary voltage absorbs power from the other until the total load becomes great enough to cause the terminal voltage of the latter to drop to E_2 volts. With further increase of load the first continues to carry the load required to bring its voltage to E_2 and divides further increases with the second transformer in the proportion stated in the preceding paragraph. If the internal impedances are both very low compared to the load impedance, differences in the division of load are little noticeable, but it is always wise to keep these conditions in mind when designing transformers for manufacture or selecting them for use.

When secondary circuits of transformers fed from one primary circuit are to be connected in parallel it is, of course, essential that they be connected in the proper relation so that they will not short-circuit upon each other. It is thus necessary to know or determine the **Polarity** of the windings. The relative polarity of two transformers may be obtained by connecting up one transformer, and then, after attaching the primary terminals of the second to the primary line, connecting one of its secondary terminals directly to a secondary line conductor and the other secondary terminal to the other secondary line conductor through a voltmeter. If the transformers are connected in series instead of parallel relation, the voltmeter will read double the normal secondary voltage. A light fuse may be used instead of the voltmeter, and the blowing of the fuse will

denote wrong connections. Figure 307 *a* shows correct connections for two transformers in parallel. Figure 307 *b* (1 and 2) show incorrect connections for parallel operation, as the transformers in each diagram are short-circuited on each other. Figure 307 *c* shows the connections whereby a three-wire, single-phase, secondary circuit may be obtained by means of two transformers from a two-wire primary circuit when a middle connection to the secondary winding is not available.

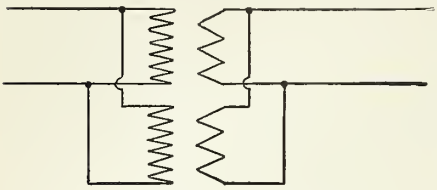


FIG. 307 *a*. — Correct Connections for Two Transformers in Parallel.

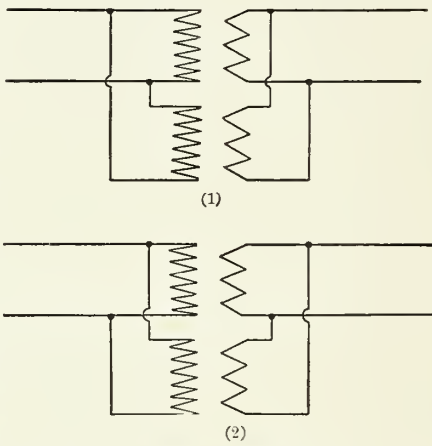


FIG. 307 *b*. — Incorrect Connections for Two Transformers in Parallel.

The transformation of voltage in polyphase circuits may be compassed by using single-phase transformers in groups. A quarter-phase circuit then requires two transformers at each point of transformation, and a tri-phase circuit requires either two or three transformers. The individual transformers must each have a capacity equal to the power required to be transformed by it divided by the power factor of the part which it supplies. As the power factor of an incandescent lamp load is practically unity, and since circuits supplying induction motors are likely to have full load power factors not higher than 0.8, and further since when such motors

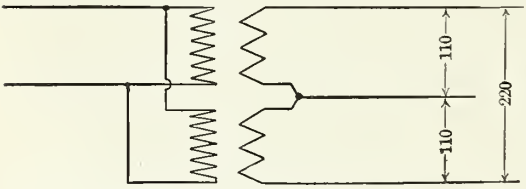


FIG. 307 *c*. — Three-wire Secondary Circuit supplied by Two Transformers.

are only partly loaded the power factor is much decreased, it is evident that transformers which supply currents to motors must be of greater kilovolt-ampere capacity than those which supply equal power to incandescent lamps. This rule applies equally to single-phase and polyphase circuits and it is important to bear in mind when purchasing transformers that their capacities should

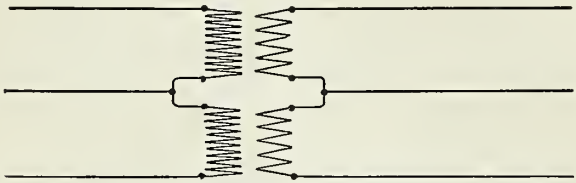


FIG. 308. — Diagram of Transformer Connections for a Three-wire Quarter-phase System.

be suitable for the kilovolt amperes of the load to which they are to be connected. Figure 308 represents the connections for a quarter-phase circuit when a common return wire is used in both primary and secondary circuits. In the case of a four-wire quarter-phase circuit, the transformers may be kept independent, one single-phase transformer being provided for each phase of the circuit. A four-wire circuit may be used for either the primary or secondary circuit and a circuit with common return for the other by associating the four-wire connection with one side of two transformers and the three-wire connection of Fig. 308 with the other side of the transformers. In Fig. 309 are shown the usual connections utilized for groups of three single-phase transformers for three-phase transformation. The figure shows three groups each of which comprises three transformers. In Group 1 the primary windings and the secondary windings of the transformers are both connected in Δ arrangement. In Group 2 the

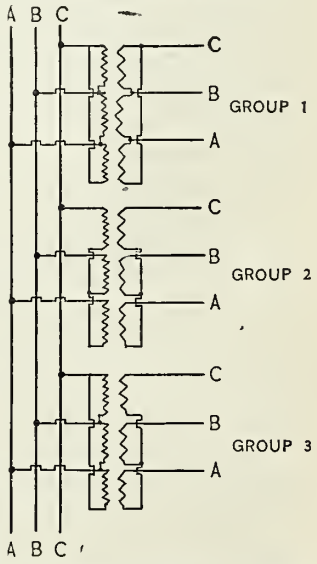


FIG. 309. — Diagram showing Ways of Connecting Three Single-phase Transformers to Tri-phase Lines.

primary windings and the secondary windings of the transformers are both connected in Y arrangement. In Group 3 the primary windings of the transformers are connected in Δ arrangement and the secondary windings are connected in Y arrangement. The third arrangement may be reversed so as to connect the primary windings in Y and the secondary windings in Δ.

When transformers are to be connected in any of these arrangements, the terminals must be joined correctly according to the polarities, or the voltage phases will be unbalanced in the secondary circuit. Thus, when the secondary coils are properly connected and the system is balanced as to the voltages, the three coil-voltages may be laid out graphically as a symmetrical phase diagram, as shown by the vectors represented by *OA*, *OB*, and *OC*, in Fig. 310.

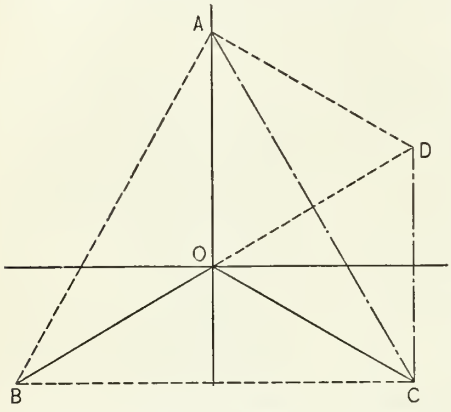


FIG. 310. — Diagram of Y-connected Transformer Voltages in a Tri-phase Circuit to show the Effect of reversing the Connections of One Winding.

By reversing the connections of one of the windings of one of the transformers in a Y arrangement, its secondary voltage is reversed. For instance, reversing the transformer giving the voltage *OB* of Fig. 310 gives the voltage vector *OD* of the diagram, which, using the relations shown in the figure, lies 60° behind *OA* and 60° ahead of *OC* instead of in the proper position 120° ahead of *OA* and behind *OC*. The voltages between the secondary line wires *A* and *B*, *B* and *C*, and *C* and *A* are therefore changed to the values and vector relations shown by the lines *AD*, *DC*, and *CA* in the vector triangle *ADC*, instead of their balanced relations for the proper connections shown by *AB*, *BC*, and *CA*. The diagram shows by inspection that the voltages *AD* and *DC* are numerically equal to the wye voltage *CO* of the circuit, but *CA* is equal to $\sqrt{3}$ times the wye voltage. The angles between *AD*, *DC*, and *CA* are respectively equal to 60°, 150° and 150°.

When the secondary coils are connected in delta, the diagram of Fig. 310 *a* can be applied, but now OA, OB, and OC represent, not only the coil voltages, but also the line voltages. Hence, when the connection of the winding of transformer B is reversed, the voltage between B and C is reversed and the vector diagram of voltages AB, $B'C$ $(= -BC)$ and CA does not close. This leaves an uncompensated voltage $B'B$ to make a circulating current in the mesh, which would burn out the transformers.

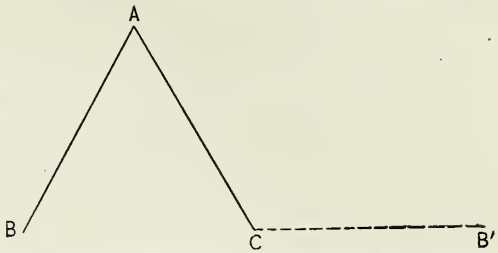


FIG. 310 *a*. — Diagram of Δ-connected Transformer Voltages in a Tri-phase Circuit to show the Effect of reversing the Connection of One Winding.

In a wrongly connected wye system, unbalanced voltages are impressed on the secondary circuit which would ordinarily lead to the flow of unbalanced currents in both primary and secondary circuits, and if maintained would cause inconvenience to users of apparatus of which the load might be composed. In a wrongly connected delta system the transformers are jeopardized by a circulating current caused by the resultant voltage in the mesh. This is generally obviated by use of proper fuses and other circuit breakers, but it is well to guard against such dangers by making polarity tests before cutting the transformers into circuit.

It must be observed that the phases of the voltages between three-phase secondary line wires are opposite to the voltages impressed between the primary line wires when both primary and secondary windings of the transformers are connected both either in delta or in wye arrangement. This is not true, however, when the primary windings are connected in one arrangement and the secondary windings are connected in the other arrangement, as this results in a displacement of the secondary voltage phase 30° from the position of opposition, for the reasons heretofore explained.* The line voltages are shifted forward 30° when going through the transformers from delta to wye and are shifted backward 30° when going through the trans-

* Art. 100.

formers from wye to delta. For this reason groups of three transformers operating in three-phase circuits cannot be connected in parallel on both the primary and secondary circuits unless due consideration is given to the manner of their connections. Transformers connected delta primary and delta secondary and transformers connected wye primary and wye secondary can be paralleled on the secondary side with like groups and with each other, when operated from the same primary circuit; but they cannot be paralleled with transformers connected delta primary and wye secondary nor with transformers connected wye primary and delta secondary. Each of the two last-named arrangements may be paralleled with like groups, but neither may be paralleled with the other.

When the connection of the windings is delta primary and delta secondary or wye primary and wye secondary, it is manifest that the ratio of primary and secondary voltages at no load is equal to the transformation ratio of the individual transformers, but this is not true when the connections of primary and secondary windings differ. When the arrangement is delta primary and wye secondary, the wye voltage of the secondary circuit is equal to the primary impressed voltage divided by the ratio of transformation, and in a balanced circuit the secondary line voltage is $\sqrt{3}$ times as great. When the arrangement is wye primary and delta secondary, the secondary line voltage is equal to the primary wye voltage divided by the ratio of transformation. The primary line voltage is $\sqrt{3}$ times the primary wye voltage, and the secondary line voltage is therefore $1/\sqrt{3}$ as great in a balanced circuit as would be given by dividing the primary line voltage by the ratio of transformation of the transformers.

In Fig. 311 is shown a diagram for connecting two transformers on a tri-phase circuit in what is sometimes called the **Open delta** or **Vee connection.** The two arrangements shown are alike in result, but the transformers are attached to different phases. The secondary voltages of the respective transformers are represented in Fig. 312 by the lines AB and CA, corresponding with the connection shown in the upper part of the figure, while the distance BC represents the resulting secondary line voltage that is maintained jointly by the two transformers between the wires B and C. When three transformers are con-

nected in delta arrangement and one becomes disabled so as to be disconnected from the circuit, a large part of the full load can still be carried by the two transformers that remain in V connection. With three transformers connected in the delta arrangements the total power delivered to the circuit is equal to $\sqrt{3}\, E_2 I_2 \cos \theta_2$, and each transformer delivers one third of this power, or $\frac{E_2 I_2 \cos \theta_2}{\sqrt{3}}$. After the circuit of one of the three transformers has been opened, assuming the load to remain the same, each of the remaining transformers must deliver $\frac{\sqrt{3} E_2 I_2 \cos \theta_2}{2}$ watts to the receiving circuit. Since the current in each line wire remains unchanged as does the voltage between line wires, it is evident that in the transformer coils there will be additional angular displacement corresponding to an angle of $\cos^{-1}\frac{\sqrt{3}}{2}$, that is, $\pm 30°$. The total phase displacement in the two transformers is then respectively $\theta_2 + 30°$ and $\theta_2 - 30°$. From this it will be seen that the current in each transformer increases in a greater proportion than does the power delivered by the transformer. Hence when two transformers are operating in open delta (*i.e.* V connection) on a tri-phase circuit the transformers will not (without being overloaded) deliver their full rated capacity in kilowatts even when supplying power to a non-reactive load. Thus, three transformers each of 200 Kva. rated full load capacity being delta connected to a tri-phase circuit and delivering a total of 600 kilowatts to a balanced receiving circuit having unity power factor, when one of these transformers has its secondary circuit opened by accident each of the two remaining transformers will deliver 300 kilowatts to the receiving circuit,

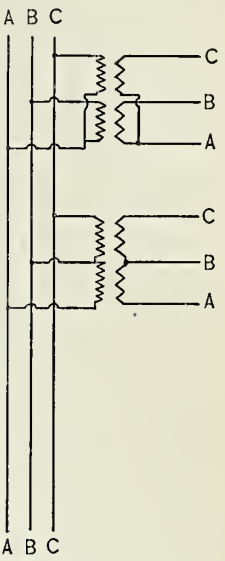


FIG. 311. — Connections for using Two Transformers on a Tri-phase Circuit. V Connection.



FIG. 312. — Diagram showing Voltages of Secondary Phases of a Tri-phase System when Two Transformers are used, and the Effect of making Wrong Connections.

but the current in each transformer will be out of phase with its secondary voltage by an angle of 30°; hence, the kilovolt amperes delivered by each transformer must be equal to $300 \div \frac{\sqrt{3}}{2} = 346$ Kva. and the transformers are 73 per cent overloaded. This is but one of many possible transformer combinations which result in a phase displacement between voltage and current in the transformers; and in every case when such a combination is made the kilowatt output of the transformers is less than the sum of the kilovolt amperes on account of the internal phase displacement of the current.

If the connection of either the primary or the secondary winding of the transformer giving voltage CA in Fig. 312 is reversed, the three secondary line voltages become AB, BC', and $C'A$. The last is equal to $-CA$. This is a workable combination because the third side of the vector diagram BC is not fixed in position or magnitude by its own transformer, as it is in the case of delta connection of three transformers; but it gives unbalanced secondary voltages. The diagram shows by inspection that the voltage BC' is numerically $\sqrt{3}$ times voltages AB and $C'A$, and the angles between the voltages AB, BC', and $C'A$ are 150°, 150°, and 60°. This arrangement is sometimes used to get a 60° phase difference.

A substitute for the V connection may be made with two transformers, in which the windings of one transformer possess $\frac{\sqrt{3}}{2}$ times as many turns as the windings of the other. In this case, one end of each winding of the former is connected to the middle of the corresponding winding of the latter, as illustrated in Fig. 313. This is called the **Tee connection** of transformers. The windings of fewer turns which are possessed by one transformer are sometimes called **Teaser windings**. The potential of A_1, the line end of the primary teaser winding, is fixed by the tri-phase relations of the line voltages, while the potential of the other end is fixed by being midway between the potentials of B_1 and C_1, by reason

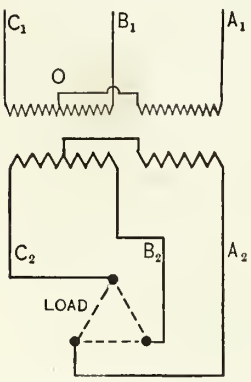


FIG. 313. — Tee Connection of Transformers for Tri-phase Circuit.

of the connection of the teaser at the mid-point O. The vector diagram of voltages is given in Fig. 314, in which BO is the voltage from B_1 to O of Fig. 313, OC is the voltage from O to C_1, and OA is the voltage from O to A_1. The line voltages are represented by AB, BC, and CA. The tensor value of voltage OA is $\frac{\sqrt{3}}{2}$ times that of either line voltage. The voltages of the secondary circuit are proportional to those of the primary circuit, by reason of the correspondence of the primary and secondary windings and, therefore, are also represented by the vectors of Fig. 314. That is, the secondary line voltages must be proportional in magnitude and angular relations to the vectors represented by AB, BC, and CA of the figure.

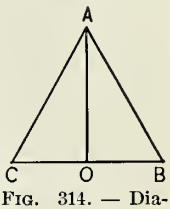


FIG. 314. — Diagram of Voltages produced by Transformers arranged in Tee Connection.

Assuming the system to be balanced, the line currents flowing in the two halves of the coil B_1C_1 or B_2C_2 are numerically equal to each other and are 120° apart in phase. At O they add together vectorially to form the line current which flows through the teaser, which is numerically equal to each of the others and 120° in advance of one and 120° behind the other, in accordance with the law that the vector sum of the currents at a junction must be zero. The kilovolt-amperes capacity of the transformers manifestly must be greater than the kilowatt output even when the load is non-reactive.

The T connection of two transformers for tri-phase circuits possesses the disadvantage, compared with the V connection, that the two transformers cannot be alike unless part of the winding of one is allowed to be idle and its capacity is therefore sacrificed; but the T connection has a countervailing advantage in the fact that the system is not so likely to become unbalanced as when the V connection is used.

The relative transformer capacities required when the delta, wye, vee, and tee connections are used for three-phase to three-phase transformation may be determined by the following relations: For delta and wye, the capacity of each of the three transformers must be equal to one third of the total load in kilovolt amperes. This, for any power factor, is equal to $E_Y I = EI_\Delta = \frac{1}{\sqrt{3}} EI$ (where E_Y, E, I, and I_Δ are the wye voltage,

line voltage, line current, and delta current of the circuit), for in the delta the primary coil current equals $\frac{1}{\sqrt{3}} I$ and in the wye the primary coil voltage equals $\frac{1}{\sqrt{3}} E$. The three required transformers then have a combined kilovolt ampere capacity of $\sqrt{3}\ EI$. If the windings are delta-connected, they must be designed to produce full line voltage and carry a current equal to the line current divided by $\sqrt{3}$, while if the windings are wye-connected they must be designed to produce a voltage equal to line voltage divided by $\sqrt{3}$ and to carry full line current. In the tee connection the coils carry the same currents as the line wires, but the voltage of one transformer having the teaser winding is only $\frac{\sqrt{3}}{2}$ times as large as the line voltage. The kilovolt ampere capacity of the teaser transformer is equal to $\frac{\sqrt{3}}{2} EI$, while that of the main transformer must be equal to EI. The combined capacity of the two transformers required for an output of $\sqrt{3}\ EI$ kilovolt amperes is therefore $\frac{2+\sqrt{3}}{2} EI$. In the vee connection the voltage in the primary windings of each of the two transformers is E and the current I. The currents entering the windings at the outer ends of the vee are equal to the two corresponding line currents, and these vectorially add at the apex at a phase angle of 120°, which gives an equal current for the third wire. Hence the combined capacity of the two transformers should be $2\ EI$.

It should be understood that in the case of the tee connection or the vee connection the relations of the currents and voltages in the transformers so connected are not necessarily the same as the phase relations between line currents and line voltages. In the case of the tee connection the current in the teaser winding has the same angular relation to the voltage of that winding as the line current has to the line voltage, but as previously explained in this Article there is an inherent phase displacement of the currents in the two halves of the main transformer winding irrespective of the power factor of the line. In the case of the vee connection the currents in both transformers are out of phase with the respective transformer

voltages as explained earlier. Because of this condition the total kilovolt ampere capacity of the transformers connected in tee and in vee must be in excess of the total kilovolt amperes supplied by the line or to the load.

When the three transformers are connected in wye in a tri-phase system, a fourth wire may be connected to the neutral point or common junction point, and part or all of the load may be connected from the three independent wires to the neutral wire.* This reduces the load voltage, as shown in Fig. 315, to the secondary coil voltage, while maintaining the voltage between the three-phase wires at $\sqrt{3}$ times that value. When the load is not perfectly balanced, currents will flow in the neutral wire, and it is usual to make the neutral conductor of

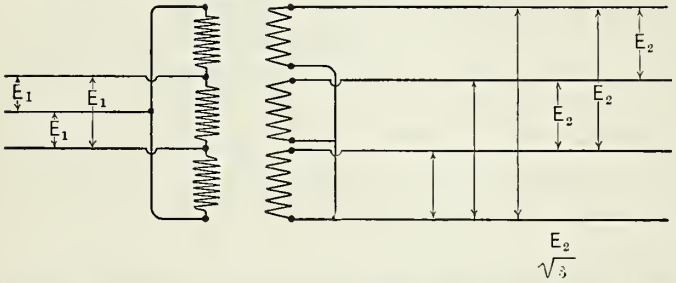


FIG. 315 — Tri-phase Transformer Connections when a Neutral Wire is used.

the same size as the other three; but as the phase voltages are equal to $\sqrt{3}$ times the load voltages, and as the weight of copper is inversely as the square of the voltage, the weight of copper required for the transmission of a given power over a given distance with a fixed loss of power is only $\frac{4}{9}$ as great as when the ordinary delta or wye arrangement is utilized with the same voltage on the load, and these in turn require only $\frac{3}{4}$ as great a weight of copper as a single-phase or a four-wire quarter-phase system with the same voltage between the conductors of each phase. Consequently, the arrangement of Fig. 315 requires only $\frac{1}{3}$ as great a weight of copper as a two-wire single-phase or four-wire two-phase circuit with the same voltage on the load.

The neutral point of a tri-phase system is likely to float from the balanced center when the phases become unbalanced.† That is, the voltage between each of the line wires and the

* Art. 100. † Art. 103. Examples.

neutral wire is dependent upon the character of the load in the several phases.

It is common practice to ground the neutral point of wye connected secondary circuits, both for the purpose of reducing the tendency of unbalancing and of reducing danger which might threaten life or property through having exaggerated voltages arising for any reason between the line conductors and ground. Thus, for instance, if a single phase of a high voltage primary circuit accidentally makes connection into the secondary windings where the neutral is grounded, it may be brought to earth potential, and it cannot rise with reference to the ground potential above the potential of the secondary phase wire with which it comes in contact. The neutral conductors of single-phase three-wire secondary circuits are commonly grounded to obtain similar protection. Variations of voltage between earth and the line wires of an insulated system may also occur from other causes than the illustration just given, but when a system of conductors has a ground connection, whether from a wye neutral or otherwise, the voltages between the ground and the several line wires evidently depend upon the values of the voltages between the wires themselves. Thus, if one line wire of a wye or delta load is connected to ground, the voltages between the ground and the other line wires are equal to the full line voltage. When the wye neutral point is grounded, the voltage between either phase wire and ground is equal to the voltage between the neutral point and the phase wire, which, in a balanced three-phase system, is equal to the line voltage divided by $\sqrt{3}$.

When the neutral points of several generators or transformers are brought into conjunction as by grounding, any third harmonics of the voltages induced in the windings may cause third harmonic cross currents to flow through the ground connections, windings, and line wires, which currents could not exist except for such connections. This condition may be controlled by inserting a small amount of resistance in each conductor leading to ground.

Although grounding at special points may be made designedly with good effect, accidental grounding of otherwise ungrounded systems is generally objectionable, as it may be the cause of serious disturbances. This is particularly true of systems of very high voltage. Even though this relation may not prove serious at the point where accidental grounding occurs, the

effect is carried more or less directly, depending upon the combination of wye and delta connections of the various step-up and step-down transformers, over the whole distribution system and may at one point or another be the cause of accident. It is essential, then, that ground detectors be used on all insulated portions of a distribution system, and that when grounding is desired the ground connection be made with due reference to the voltages and insulation strength over the whole system.

Disastrously high voltages may be caused under certain conditions when the primary circuit of a transformer, having its secondary winding connected into a long high voltage three-phase transmission line of large electrostatic capacity, is open circuited. Thus, suppose a group of three transformers are connected in wye fashion to the primary and secondary conductors of a transmission system, as illustrated in Fig 316. If the primary coil connected into lead A_1 is open-circuited for any reason, and the secondary coil remains connected into its circuit, the voltage between wires B_2 and C_2 may not be seriously affected, but the voltages between A_2 and B_2, and C_2 and A_2, may become dangerously high; for now the coil O_2A_2 acts as a high inductive reactance which is in series with what may happen to be an equivalently high capacity reactance of the line conductor leading from A_2 to the load. This would produce a condition of voltage resonance similar to that assumed in some of the problems of Chapter VI,* and an excessive voltage would appear between A_2 and O_2 and also perhaps elsewhere between the line wire connected with A_2 and the other line wires. Evidently with the conditions, as illustrated, the primary voltages will also be disturbed.

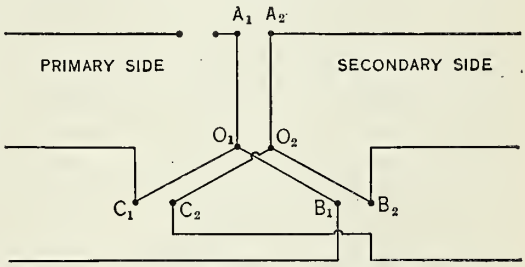


FIG. 316.—Diagram showing Three Transformers Connected in Wye Fashion in a Tri-phase System, with the Primary Circuit of One Transformer Open. Illustration of a Dangerous Condition.

Various other conditions which produce objectionable results may arise when making and breaking connections of trans-

* Art. 81 (Examples *g* to *k*) and Art. 82; also Art. 86.

formers associated with polyphase circuits, so that circumspection must be exercised under such circumstances.

The balance of the load is a most important element in using transformers with their secondary windings connected on single-phase three-wire or on polyphase circuits. It may be readily understood that with the two halves of the secondary winding of one transformer connected to a single-phase three-wire system, any unbalancing of the load will affect the balance of the secondary voltages respectively produced by the two half windings, on account of the fact that greater magnetic leakage will affect the more heavily loaded half winding, unless the two halves are very closely associated on the magnetic circuit. It is therefore important to have each transformer, which is to be used for this purpose, constructed with the halves of its secondary winding so arranged that the magnetic leakage shall be approximately the same in each even if the load becomes unbalanced. When a core type transformer, with the primary and secondary windings each equally divided between the two legs of the core, is to be used to supply current to a three-wire secondary circuit, the half of the windings to be connected between either outside wire and the neutral wire should not be placed on one leg of the core only, but should be composed in equal parts of turns on each of the legs. When transformers are connected in polyphase circuits a similar effect of unbalanced magnetic leakage is produced by an unbalanced load, as may be seen by reviewing the examples given earlier in the book.* The effect of such unbalanced loads is to tend to overheat the transformer coils, generating apparatus, and load appliances on the phases carrying abnormal currents, and to cause unsatisfactory regulation and danger to the insulation over the phases or circuits which have abnormal voltages. Not only do incandescent lamps suffer from the abnormal voltages, but the transformer coils, induction motor fields, and similar apparatus in the circuits having excessive voltages are apt to have their core fluxes go beyond the point of saturation, thus calling for excessive exciting currents which may become, under such circumstances, sufficiently large to cause serious overheating. Many low voltage secondary distributing systems supply three-wire loads from the secondaries of transformers connected to polyphase

* Art. 103.

primaries, and there have been frequent cases where excellent plants of this kind have given most unsatisfactory service because the systems have not been properly balanced. Thus, the three-wire secondaries of the transformers may be unbalanced, and in addition to this defect the sum of the loads of the transformers attached to one phase of the primary supply system may be very different from the loads called for on the other phases. The line and generator IR drops will then vary in the different phase circuits, causing the vector diagram of polyphase voltages to become skewed and the voltages of the several phases at the points of power consumption to become unequal. It is, therefore, essential for good service, when connecting transformers to compound primary or secondary circuits, to first properly balance the load which is to be attached.

The connections of quarter and tri-phase transformers are made circuit by circuit, in the same manner as earlier explained for single-phase transformers.

141. Phase Transformation. — Connections for transforming one polyphase system into another system with a different number of phases may be readily developed from the principles set forth in earlier chapters. Quite a number of commercial devices for this purpose have been proposed. The one most commonly used is a T connection arranged as follows: In Fig. 317, the primary windings of the transformers M and M' are alike and are connected to a quarter-phase circuit.

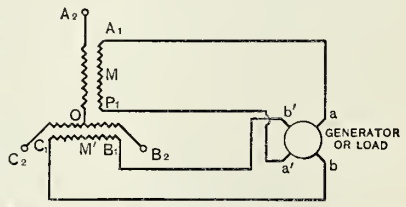


FIG. 317. — Diagram of Arrangement of Two Transformers for converting Quarter-phase into Tri-phase System and Vice Versa. The Tee Connection.

The secondary winding OA_2 of M (sometimes called the **Teaser** winding) is attached to the middle of the secondary winding of M' at the point O. The secondary winding of M has $\frac{\sqrt{3}}{2}$ times the number of turns that there are in the secondary winding of M'. Then, the voltages impressed on the primary windings being numerically equal and 90° apart in phase, the line OB in Fig. 314 represents the voltage between the points O and B_2 in Fig. 317, OC that between O and C_2, and OA that between O and A_2. Voltage OA must be at right angles to OB and OC, as the

phases of the two primary circuits are 90° apart. Thus, the voltages in B_2O and OC_2 being in phase with others and the voltage in OA_2 being in quadrature with them, it is seen that between the points A, B, and C three equal voltages are set up 120° apart, since the number of inducing turns between O and A_2 bears the same relation to the number of inducing turns between B_2 and C_2, that the length of the bisector of an equilateral triangle bears to one of its sides. The apparatus may be used with equal facility for transforming two phases into three phases or vice versa, by impressing two-phase voltages on A_1P_1 and B_1C_1 and thereby producing three-phase voltages between A_2, B_2 and C_2 in the one case, or by impressing three-phase voltages on A_2B_2, B_2C_2, and C_2A_2 and thereby producing quarter-phase voltages respectively in A_1P_1 and B_1C_1.

For some services it is desirable to use a six-phase system, and to do this economically generally requires transformation from quarter-phase or tri-phase transmission lines. Figure 318 shows method whereby this can be done in a simple manner. Here the secondary circuits of the three transformers of a tri-phase primary system are connected at opposite junctions of the six junction points R, M, S, N, T, P (Fig. 318) of a mesh-connected six-phase load. The load makes what is equivalent to the combination of two tri-phase systems. The dotted lines between the load junctions R, M, S, N, T, and P represent the paths of the current in a mesh load and the broken diametral lines represent the paths for a star load. In six-phase systems mesh and star connections call for the same voltages and currents in the individual branches of the load. With the arrangement shown in Fig. 318 the neutral point of the six-phase star is at the same potential as the middle point of the secondary winding of each of the three transformers, and if a neutral return conductor is added to the system, it may be led to those three middle points.

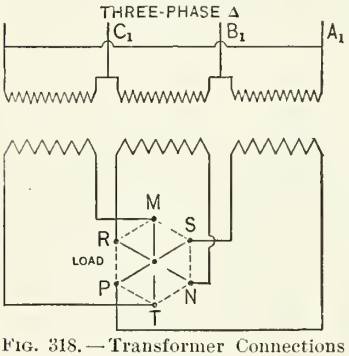


Fig. 318. — Transformer Connections for transforming Three Phases to Six Phases. Opposition Arrangement.

In Fig. 319 each secondary winding is shown divided into

two equal coils. Each set of halves is connected in delta arrangement, one delta being reversed compared to the other in accordance with the connecting diagrams shown in Fig. 319 *a*. The corners of one delta are connected in tri-phase mesh to three points *M*, *N*, and *P* on the load, while the other delta is connected to the points *R*, *S*, and *T*. This arrangement is commonly called the **Double delta** connection.

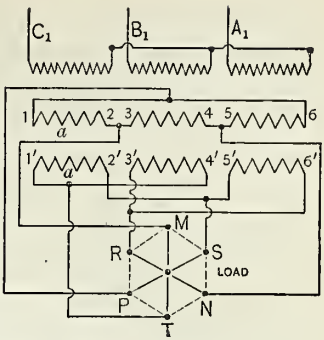


FIG. 319. — Transformation of Three Phases to Six Phases by the Double Delta Arrangement.

Figure 320 shows a similar arrangement, but with the two

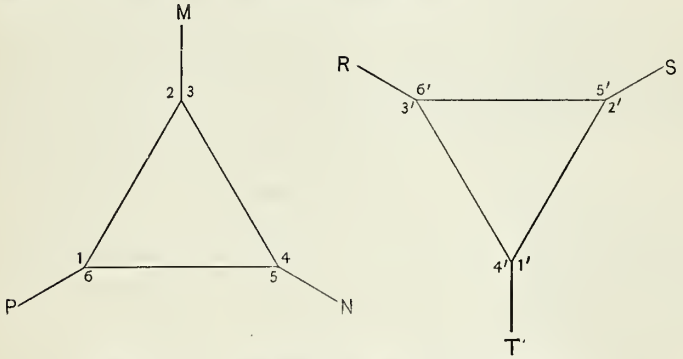


FIG. 319 *a*. — Connection Diagram for Plan of Fig. 319.

sets of half coils connected in wye. The connection diagram is shown in Fig. 320 *a*.

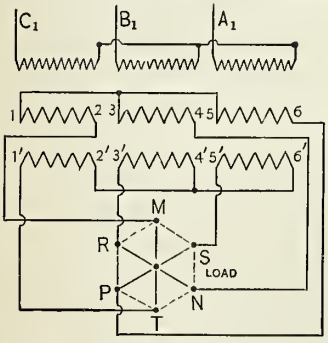


FIG. 320. — Transformation of Three Phases to Six Phases by the Double Wye Arrangement.

The primary windings of the transformers may be in either delta or wye in each of the arrangements illustrated for the transformation from three phases to six phases and the reverse. Which arrangement to select for either the primary windings or secondary windings depends upon the voltages desired and the problems of insulation and current-carrying capacities of the copper in the transformers, second-

ary leads, and load. With a delta system the line current equals $\sqrt{3}$ times the coil current for the balanced three-phase system, with voltages in lines and coils equal; and with a wye

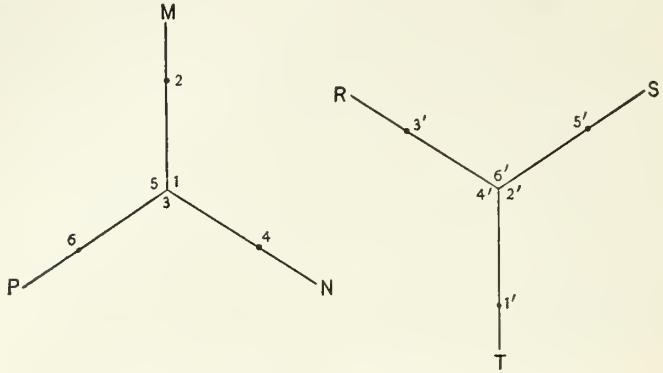


Fig. 320 *a*. — Connection Diagram for Plan of Fig. 320.

connection the line voltages equal $\sqrt{3}$ times the coil voltages, with the currents in lines and coils equal.* For the balanced six-phases, line currents are equal to coil currents, and line voltages are equal to coil voltages in both mesh and star, provided line voltages are measured between electrically adjacent terminals. Figure 320 *b* illustrates the vector relations of the line currents and the coil currents for the six-phase mesh. The line current at corner M is equal to the vector sum of current RM and current $SM\ (= -MS)$. This is the current Mb. The line current at corner S is equal to the vector sum of currents MS and NS, and so on at the other corners.

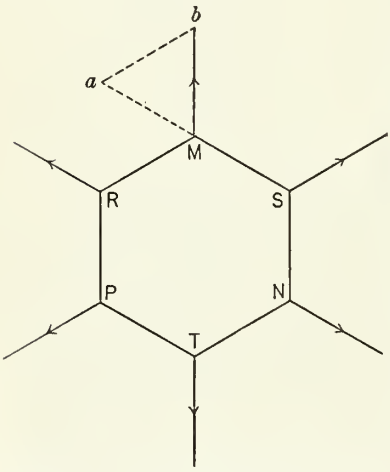


Fig. 320 *b*. — Vector Diagram of Currents in Six-phase Mesh.

Figure 321 shows the tee method of connecting two transformers with divided secondary windings for the purpose of transforming from three to six phases. In this case the primary winding may be connected to a quarter-phase circuit,

* Art. 100.

in which case the primary windings of the two transformers must be alike and the secondary winding of the teaser transformer must contain only .866 $\left(=\frac{\sqrt{3}}{2}\right)$ as many turns as the other secondary winding, or to a three-phase primary circuit, in which case both the primary and secondary winding of the teaser transformer must contain $\sqrt{3}/2$ times as many turns as the corresponding winding of the other transformer, as shown in Fig. 321. Other combinations may be used, but those given are sufficient to indicate in general the manner in which the transformation is accomplished.

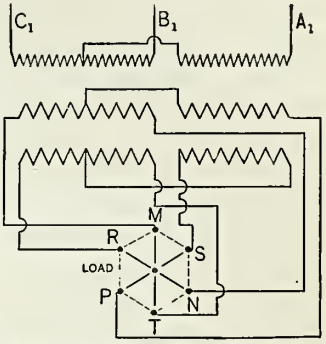
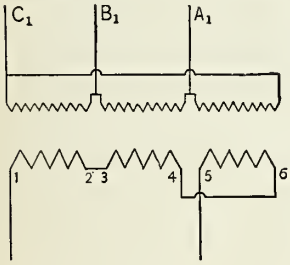


FIG. 321. — Transformation of Two or Three Phases to Six Phases using Two Transformers and the Tee Connection.

Transformation may also be effected from a polyphase circuit to a single-phase circuit for the purpose of distributing the load of a single-phase circuit over the several phases of the polyphase circuit. Each phase of a polyphase circuit is a single-phase circuit, but when it is desired to distribute the load of a single single-phase circuit over the three phases of a three-phase circuit this may be done by connecting three transformers, as illustrated in Fig. 321 *a*, which also exhibits the vector diagram of secondary voltages. It will be observed that the connection of the secondary winding of one of the transformers is reversed in comparison with the normal connection for a delta, and the connecting wire required to close the delta is omitted. The vector diagram shows that the secondary voltage is equal to twice that of one transformer. The current carrying capacity is equal to any one of the transformers, and three transformers are required to transform the full load of two.



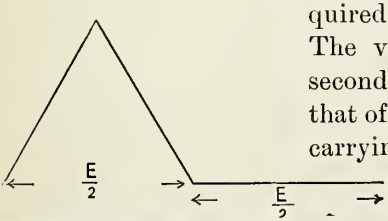


FIG. 321 *a*. — Transformation to distribute Single-phase Load over Three Phases.

Moreover, the results are not to be recommended because the arrangement transmits the pulsating power of the single-phase circuit into the three-phase circuit and unbalances it.

To accomplish the corresponding result with a two-phase primary circuit, the secondary windings of the transformers should be connected as for use with a common return wire, and the single-phase load should be worked off the outside wires.

The transformation of single-phase into polyphase currents by means of stationary transformers may be accomplished by phase-splitting devices, such as by dividing the single-phase circuit into two or more branches and obtaining currents of the desired polyphase angles by inserting properly proportioned resistances and inductive and capacity reactances into the branches. No satisfactory commercial method has been developed, where large units of power are required, which does not include moving parts in the transformer, though phase-splitting devices such as described are much used for single-phase integrating meters and for starting small induction motors.

142. Transformation from Constant Voltage to Constant Current. — The effect of magnetic leakage has been shown in Art. 123. Remembering that the effect is like that produced by self-inductance coils placed in the primary and secondary circuits, the final vector diagram is the same as in the case of a transformer working on an inductive secondary circuit, with an additional correction applied to the angle of lag between the primary voltage and current to account for the direct effect of the leakage on the primary circuit. Such a diagram shows that, as the leakage is increased so that the angle of lag between the mutually induced voltages and the secondary current approaches 90°, the deficiency in the inherent tendency to regulate for constant secondary voltage when constant primary voltage is impressed, becomes so great that the secondary terminal voltage tends to vary inversely with the current ; that is, the voltage ordinates of the transformer regulation curve decrease with increasing rapidity as the current increases. Such a transformer should therefore tend to transform a variable current at constant voltage into a constant current at a variable voltage. This characteristic makes transformers with large magnetic leakage desirable for use for providing constant current for series arc lighting by transformation from a constant voltage

circuit. A transformer constructed for use in this way is called a **Constant current transformer.**

When the lag angle between secondary voltage and current becomes 90°, the transformer can of course do no work, consequently it is impossible to get very exact regulation in thus transforming from constant voltage to constant current, but it is possible to arrange the transformer so that the percentage of leakage reactance can be varied when necessary by partially closing a shunt magnetic circuit by a slab of iron strips, as was first proposed by Elihu Thomson (Fig. 322).

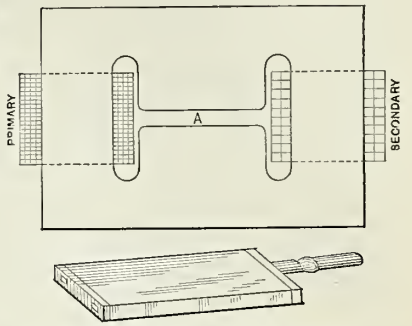


FIG. 322. — Diagram showing a Simple Way in which Magnetic Leakage can be Varied in a Transformer Core.

Figures 323 and 324 show the results of a test of a small transformer, in which the constant current regulation is wholly due to fixed magnetic leakage. In the first figure, one curve shows the efficiency as a function of the current in the secondary circuit, and the other curve shows the secondary terminal volts as a function of the secondary current. The second figure has curves *A* and *B* which show for the same transformers the watts in the primary and secondary circuits as functions of the secondary current. The crosses on the curves in the two figures show the points corresponding to normal load. Close regulation in the transformation into constant secondary current from constant primary

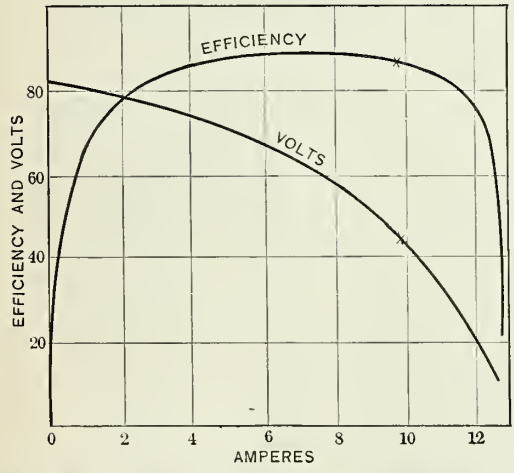


FIG. 323. — Curves of Efficiency and Regulation as functions of the Secondary Current in a Small Transformer having High Magnetic Leakage.

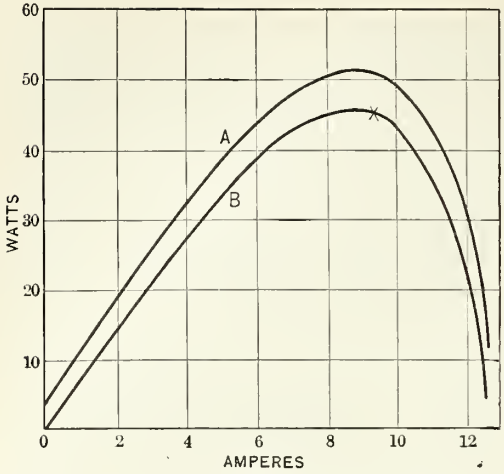


FIG. 324. — Curves of Power Input and Output as functions of the Secondary Current in a Transformer having High Magnetic Leakage. *A*, Primary Kilowatts; *B*, Secondary Kilowatts.

voltage requires the use of accessory means, such as means to vary the magnetic leakage.

In well-built transformers designed for constant secondary voltage, magnetic leakage is not likely to be of much magnitude, and in fact it can only be brought to a large value by making the space occupied by the primary and secondary coils very large compared with the cross section of the iron core, by using iron of a low permeability, or by specially arranging leakage paths. In Fig. 325 is shown a sketch of a modern constant current transformer in which it may be observed that the magnetic circuit is long and the space within which the coils are wound is large. The letters *C*, *C* indicate the laminated iron core, the central limb of which is embraced by the primary and secondary coils *A* and *B*. The coil *A* is movable and its position is determined by the balance between the weight of the counter-weighted coil and the magnetic repulsion between the coils. Thus, if the repulsion just balances the weight of the movable coil and mechanism when certain currents flow in the two coils, any change in the currents will cause the movable coil to

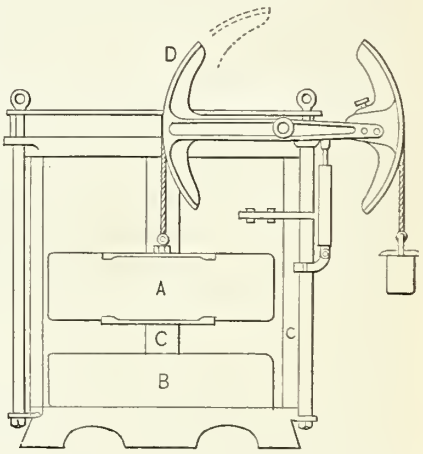


FIG. 325. — Constant Current Transformer.

move up or down. Instead of only one, both coils may be made to move.

The transformer being connected to a secondary circuit comprising translating devices, such as arc lamps in series, if one of the units of load is short-circuited for the purpose of removing it from operation, the current in the circuit will rise on account of the reduced impedance. Thereupon, the repulsion between the two coils is increased and the movable coil retreats farther from the stationary coil, thereby causing an increase of the magnetic leakage and a reduction of the secondary terminal voltage. The counter-weight being suspended on an arc of slightly decreasing radius, the coil promptly finds a position of balance, with the secondary current at substantially its normal value. If, on the other hand, additional load is added to the circuit, the current will at once correspondingly decrease, the repulsion between coils also decreases, the movable coil will move toward the fixed coil, the magnetic leakage will be thereby decreased and the secondary terminal voltage increased, and the coil will find a new position of balance with the secondary current again at substantially its normal value.

The full load secondary voltage evidently occurs when the coils are nearly as close together as the construction permits, and the load impedance is at the highest value through which the transformer is designed to send the normal secondary current; while at no load, which is the condition occurring when the secondary circuit is short-circuited, the coils are at the extreme ends of the central magnetic core. When a heavy load is suddenly reduced, preliminary hand regulation may be necessary with such a transformer to prevent unsafe currents or voltages occurring while the movable coil is moving to its proper position, but as a rule the devices are considered automatic. The secondary terminal voltage when the secondary circuit is open, that is, when the load impedance is infinite, should approach a value equal to the primary voltage divided by the ratio of transformation, as in such a case the coils will be at their nearest approach to each other, since the secondary current is zero and no repulsion exists between the coils.

The circle diagram for such a transformer when the load is considered to have a power factor of unity and the ratio of transformation is unity is shown in Fig. 326. For con-

venience in this figure the negative resistance loci are used for representing secondary quantities. In the case under consideration the reactance and resistance of the equivalent circuit both vary and in such proportion that the secondary current is maintained constant. The secondary current locus is therefore represented by the semicircle A, I_2'', I_2', I_2, B.* The construction is otherwise of the same principles as shown in Figs. 277 to 281 inclusive. At full load the leakage reactance voltage drop, represented by the line Q_1E_1, is at the relatively low value which occurs when the transformer coils are close together. At no load, that is, the secondary short-circuited, the leakage reactance voltage drop and the resistance drop in the two coils must combine to equal the impressed voltage OE_1. The leakage voltage then has the value of E_1Q_1''. The resistance drop remains numerically equal to K_1Q_1, since the secondary current is maintained constant and the ampere-turns of the primary winding therefore must stay constant, except as the variation of the magnetic leakage may affect the losses and thus affect the exciting component of the current. That is, reducing the series load by short-circuiting the translating devices does not decrease the primary current, but it causes the primary angle of lag to increase and therefore reduces the power absorbed from the primary supply circuit. The primary current is found as in the constructions in the earlier figures referred to, and is shown for one position of Q by the line S_1I_1, where S_1O is the exciting component.

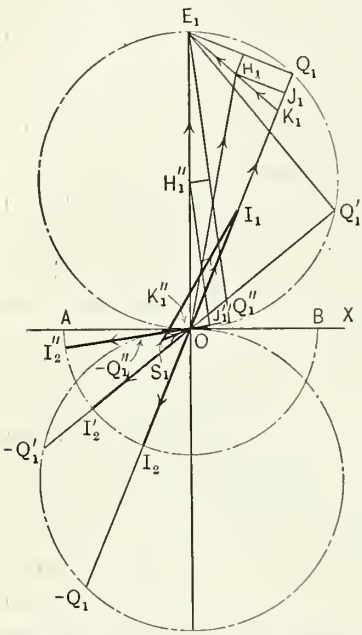


FIG. 326. — Diagram showing Circular Loci for Constant Current Transformer.

If the load impedance is increased beyond the point required to bring the two coils of the transformer to their position of

* Compare with Art. 70.

nearest approach, which represents the condition of maximum load, the primary and secondary currents will decrease along a locus determined by the effects of a varying resistance and constant reactance in a circuit.*

The equivalent impedance combination required to represent the reactions of this type of transformer can be built up as in the case of constant voltage transformers for any value of the load, but the equivalent coil reactance is not fixed but varies with the position of the movable coil, and the parallel impedance supplying the exciting current should be connected at the position indicated by *NS* in Fig. 282, since the mutually induced voltage varies widely.

Figure 329 shows a constant current transformer at *A*, which has a double secondary circuit arranged to operate two series of arc lamps. A plug switch is provided in each secondary external circuit, as shown at *B*, to be used for short-circuiting either one at will. Other plug switches are provided at *C* and *D* to be used for disconnecting the external secondary circuits from the transformer and for disconnecting the transformer from the primary supply circuit. An amperemeter is arranged so that it may be introduced into either lighting circuit by means of a switchboard plug, and a lightning arrester is connected to each circuit.

143. Series Transformers. — When transformers are connected in series with a line carrying a load as shown in Fig. 327 they are usually called **Series** or **Current transformers**. Such transformers in commercial service are usually of very small capacity, such as for loads of from ten to fifty watts in the secondary circuit, and are provided for service with electrical measuring instruments, circuit breaker relays, and the like. The current in the secondary coil divided by the ratio of transformation takes a value equal to the vector difference between the total primary current and the exciting current component. The exciting current varies with the load for the reasons next explained ; but in a well-designed machine it is relatively small for all conditions of load from a short circuit of the secondary terminals up to an impedance in the secondary circuit which is considerably greater than that of normal full load. The voltage across the primary terminals equals the whole transformer impedance, including

* Art. 70.

that of the secondary load, reduced to primary equivalents, multiplied by the primary current; consequently, when the primary current is constant, if the impedance of the load increases, the secondary current tends to fall, but the exciting component increases so as to maintain the closed vector relation between secondary ampere-turns, exciting ampere-turns, and total primary ampere-turns. If the impedance of the load decreases, the secondary current tends to increase and the exciting component decreases. The tendency, therefore, is for the magnetic flux, and therefore the secondary voltage, to vary so as to maintain the secondary current in approximately fixed relation to the primary current.

The secondary mutually induced voltage equals the primary

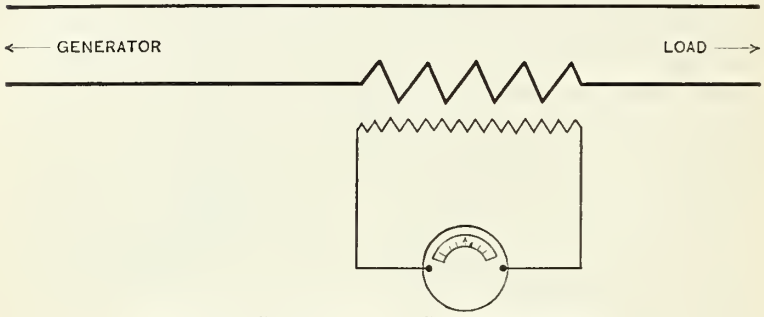


FIG. 327. — Connections for a Current Transformer.

mutually induced voltage divided by the ratio of transformation. For the conditions of constant impedance in the secondary circuit, the secondary current is therefore substantially proportional to the primary current and in nearly opposite phase, so long as the maximum magnetic density in the transformer core is sufficiently well below the point of saturation so that it may be considered to be substantially proportional to the exciting current. When the transformer secondary load is the current coil of a wattmeter or an amperemeter, as is usually the case, this condition is approximately realized.

If the load impedance is increased so that the saturation of the magnetic core is high, a large portion of the primary current may be used for exciting purposes, leaving a comparatively small portion equal and opposite to the equivalent secondary current. It is thus seen that the ratio of the secondary current to the primary current may vary considerably in case the impedance of the secondary circuit varies, especially if the latter is rather large,

so that when a series transformer is used to supply current to the coils of a measuring instrument, it should be used only with instruments with which it was designed to be associated, or the instruments should be calibrated in association with the transformer. When more than one instrument coil (such as the current coil of a wattmeter and the coil of an amperemeter) is to be associated with a particular series transformer, the instrument coils should be connected in series with each other in the transformer secondary circuit. The transformer should be designed so that the magnetic density in the core is sufficiently far below the point of saturation so that the exciting current is small and approximately proportional to the impressed voltage and hence to the secondary current, over the entire range of the readings of the instruments from zero to their maximum readings.

When the secondary circuit of such a transformer is opened, the total current flowing through the mains into which the primary coil is inserted, acts as an exciting current. The form of the wave of this current is largely fixed by the constants of the load attached to the mains, a small current transformer having little effect upon it, so that the mutually induced voltage and the secondary current may be very irregular in form if the magnetization of the core is carried beyond the point of saturation. This is the same condition as is shown in Fig. 264, which gives the curves of current, magnetism, and induced voltage in an induction coil through which a current of fixed form flows.* The secondary voltage when the secondary circuit is open depends upon the magnetic reluctance of the core, the current in the main wires, and the ratio of the windings. As the magnetic density in current transformers designed to be used for measuring instruments is made very low through the range of currents to be measured, it is evident that the secondary circuit voltage may rise to a high and dangerous value when a large current flows in the mains, if the secondary circuit is open, and in any event the magnetic flux is likely to rise to an excessive density and cause injurious heating from the hysteresis and eddy current losses. The secondary coils of such transformers can be short-circuited at any time, however, without danger, as they will not then take a greater current

* Art. 116.

than that in the primary circuit divided by the ratio of transformation. It is a safe rule never to open the secondary circuit of an instrument transformer when current is flowing, and it should be short-circuited when connected to a supply circuit but not provided with a load.

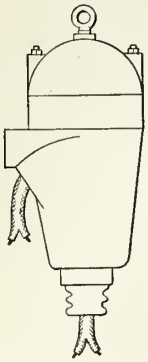


FIG. 328. — Current Transformer in its Case.

Figure 328 represents a commercial current transformer complete in a neat iron case. Figure 329 shows a current transformer at *E* and a potential transformer at *F* connected to a registering watt-hour meter.

When a wattmeter is used in association with a current transformer and a potential transformer, a good deal of caution should be observed in respect to its readings unless the instrument has been calibrated in association with the same transformers and with due regard to the power factor of the load which it is to be used to measure. The accuracy of the wattmeter readings is dependent upon the preservation of the same angular relation between the current in its current coil and the voltage impressed on its voltage coil as exists between the main circuit current and voltage; and also upon the transformation of both current and voltage occurring in fixed ratios. The potential transformer, being a small constant voltage transformer, its secondary terminal voltage differs in phase very nearly 180° from the phase of the primary impressed voltage, but there is a very small deviation from an exact 180° relation which is caused by the magnetic leakage and the primary *IR* drop. In the current transformer, the secondary current differs substantially 180° from the phase of the primary current except for the effect of the exciting current.

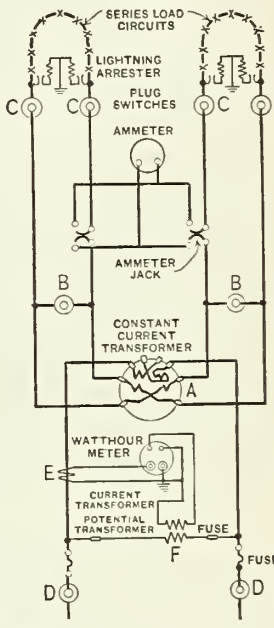


FIG. 329. — Constant Current Transformer with Double Secondary Circuit connected to Series Load. Also Instrument Transformers.

Each of these causes of inaccuracy in apparent lag angle is quite small in good commercial transformers and has relatively little effect on readings of the instrument when the load has a high power factor, but even such small deviations may introduce a large error into the readings when the power factor of the load is small.

144. Impedance Coils, Compensators, etc. — The design of **Reactance coils, Impedance coils,** or **Choking coils,** as they are variously called, is carried out in very much the same manner as the design of a transformer. An impedance coil consists of a magnetic circuit with a winding of small resistance but large inductance. It may be used in lieu of a rheostat to modify the current in an alternating-current circuit and with less loss of power than would be caused by the rheostat. The magnetic circuit and winding are proportioned in exactly the same manner as the magnetic circuit and the primary winding of a transformer, using the formula $E_1 = \frac{\sqrt{2}\,\pi n \Phi f}{10^8}$. Coils of this type are used for a variety of purposes where it is desired to throttle the flow of current without the attendant loss of power which always follows the use of resistances.

Thus where an arc lamp is used on a constant voltage alternating-current circuit, a reactance coil is ordinarily used to reduce the voltage from the voltage on the wires to that required at the lamp and to aid in maintaining the stability of the flow of current. Such an arrangement is illustrated in Fig. 330, where the circuit connections are indicated at the left hand, and the vector diagram of voltages measured across the line, between the arc terminals, and between the coil terminals, are indicated at the right-hand. The voltages upon the distributing circuits in a theater, or upon the feeders of any plant furnishing alternating currents for incandescent lighting, may be regulated by impedance coils. Figure 331 shows one of the original Thomson impedance coils, in which the reactive

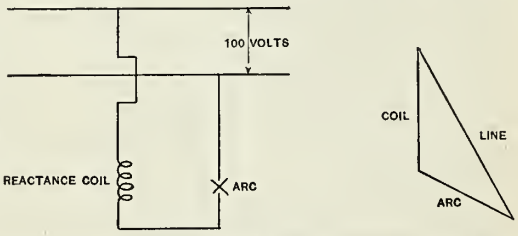


FIG. 330. — Reactance Coil in Series with an Arc Lamp.

effect is varied by moving a heavy copper shield A so as to more or less inclose the winding B, instead of varying the number of turns of the winding included in the circuit. This shield acts like the short-circuited secondary of a transformer, and therefore reduces the apparent impedance of the windings as it approaches them. Figure 332 is a regulator for a constant-current circuit such as may be conveniently used in series lighting circuits, in which A is the impedance coil winding and B is an iron core for the magnetic circuit. By means of the regulating mechanism C, the core B is automatically thrust varying distances into A, thus holding the load current constant very much as is done in a constant current regulating transformer.*

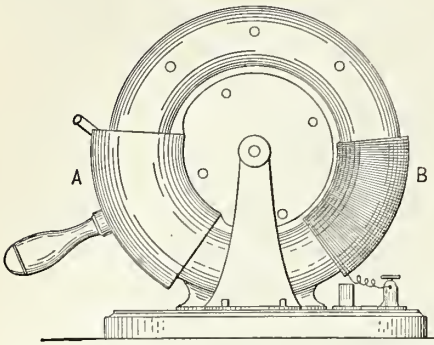


FIG. 331. — An Early Type of Voltage Regulator in which the Reactance of the Coil is varied by Means of a Movable Copper Shield.

It is evident that the reactance of an impedance coil is more economical for use in reducing voltages than a simple resistance, since pure reactance in a circuit neutralizes part of the impressed voltage without absorbing power, though it has the disadvantage of lowering the power factor.

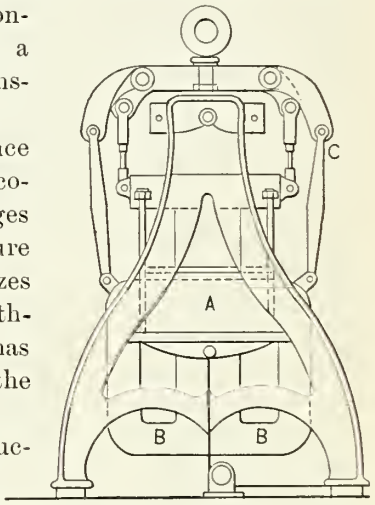


FIG. 332. — Regulator for a Constant Current Circuit.

There is another type of inductive apparatus which is much used and which goes under the name of **Autotransformer** or **Compensator**. This consists of a single winding on a proper magnetic circuit, to which single winding both the primary and secondary circuits are connected. Figure 333 shows the connec-

* Art. 142.

tions of a 220-volt compensator which feeds two 110-volt secondary circuits. In this case the function of the compensator is to equalize the voltage between the two secondary circuits regardless of their relative loads. This purpose is fulfilled fairly well, although when the two loads are unbalanced the voltages delivered by the two halves of the compensator are different, due to the differences of drop caused by the local impedances of the two halves with different currents flowing through them. When the load is balanced, no current flows through the neutral wire which is connected to the compensator at *O*, and the compensator winding carries only sufficient current to excite the core so that a counter-voltage is produced which is equal and opposite to the vector difference of the impressed voltage and the *IR* drop caused by the exciting current. Upon unbalancing the load more power is absorbed on one side of the neutral wire than on the other, and a local current flows through the neutral wire and part of the compensator winding. In this case the halves of the compensator act as a transformer. Considering the case when one side, *A*, of the three-wire circuit is fully loaded, and the other side, *B*, is without load, — then, neglecting the compensator core losses and exciting current, the current in wire *G* is zero and the current in wires *E* and *F* is twice as great as the current in wires *C* and *D*, since the power delivered all comes from the primary circuit but the delivery voltage is only one half as large as the primary voltage. The compensator must operate as a transformer to transform one half of the power required for the load from the *B* side of the circuit to the *A* side of the circuit. The current in the neutral wire therefore divides at *O* into two halves, which flow respectively in opposite directions through the winding. A similar condition exists for any degree of unbalance, so that one half the current in the neutral wire flows in opposite directions through the two halves of the compensator winding when it

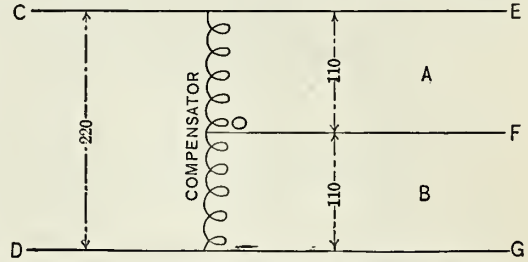


Fig. 333. — Compensator used for Maintaining the Voltages of a Three-wire Circuit.

is used to halve the primary voltage as in this instance, and the wire of the winding need be made only large enough to carry without overheating one half the current in the neutral wire at the time of the maximum degree of unbalance that can occur in the circuit concerned. A compensator therefore requires less copper under these conditions than is required for a regular transformer.

Figure 334 shows the connections of a 330-volt compensator which supplies a 1100-volt secondary circuit or *vice versa*. In this arrangement suppose that *a*, *b* are the terminals of the primary circuit, and *b*, *c* of the secondary circuit. Then the exciting current is an element of the primary current, and passes from *a* to *b*. The induced voltages between *a* and *b*, and *b* and *c* are proportional to the number of turns between the terminals, as in a transformer. The secondary current is the combination of two currents, flowing in *ac* and *bc*, which produce equal and opposite magneto-motive forces. Neglecting the core losses and exciting current, the current in *ac* is equal to the current in the secondary circuit divided by the ratio of the turns in *ab* to the turns in *cb*, that is, divided by the ratio of transformation, since the power delivered to the secondary circuit all comes from the primary circuit. The current in the windings between terminals *cb* must therefore be less than the current in the secondary circuit in the ratio of

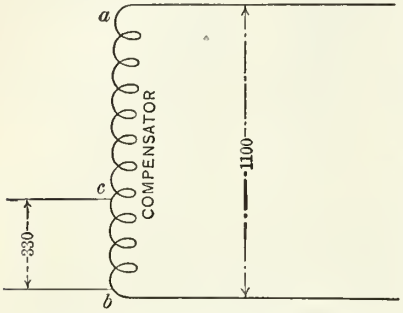


FIG. 334. — Compensator arranged to Transform Voltage from 1100 to 330 Volts or *Vice Versa*.

$$\frac{\text{turns } ab - \text{turns } cb}{\text{turns } cb} \div \frac{\text{turns } ab}{\text{turns } cb}.$$

The current in the winding between *c* and *b* is therefore equal to the difference between the secondary current and the primary current. Thus, it is evident that the autotransformer acts very much like a transformer; and modified transformer diagrams and arrangements of substituted impedance can be as readily worked out for them as for transformers. The sizes of wire in *ac* and

bc must be made with reference to the current in each part at full load, and the two parts should be sandwiched together or otherwise arranged so as to keep down magnetic leakage to the limit required. The magnetic circuit is made of the same materials and is subject to the same conditions as are the cores of transformers, but for equal current densities in the windings the weight of copper is less in the autotransformer by an amount proportional to twice the weight of the primary winding divided by the ratio of transformation.

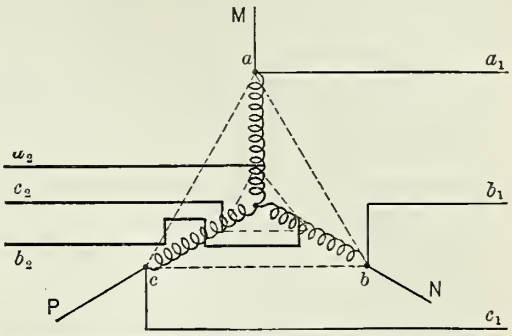


FIG. 335. — Connection of Compensators to a Tri-phase Circuit in Wye, — Step-down or Step-up.

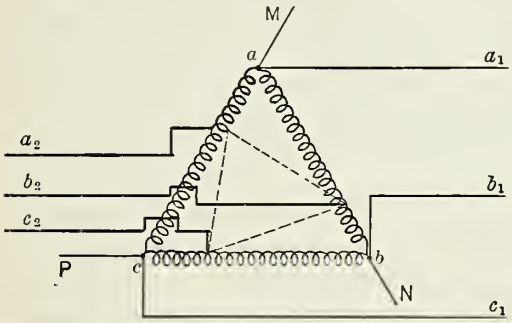


FIG. 336. — Connection of Compensators to a Tri-phase Circuit in Mesh — Step-down or Step-up.

Autotransformers may be connected in wye to a tri-phase circuit as shown in Fig. 335, when the secondary parts are grouped around the neutral point; or in delta, as shown in Fig. 336, though in this case, as can be seen from the figure, the lowest secondary voltage possible is one half of the primary voltage.

Figure 337 shows autotransformers connected into a tri-phase circuit for starting a motor at lower voltage than that of the

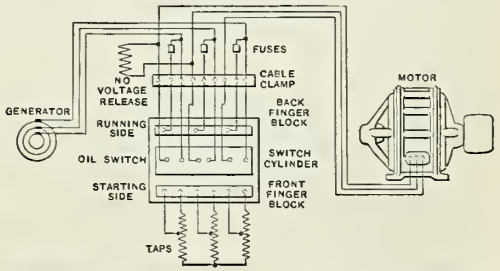


FIG. 337. — Compensators connected for starting a Motor at Reduced Voltage.

line. Here any one of three sets of secondary taps may be used. The switching devices are so arranged that the autotransformer is only in series during the starting period. Polyphase autotransformers can be built with the cores like those of polyphase transformers. Regulation or starting taps can be taken from the polyphase secondaries of banks of ordinary transformers used in supplying load to motors or other apparatus. The connections can in such cases be made similar in principle to those shown in Fig. 337 or sometimes as those in Figs. 335 and 336. Where special transformers are used for supplying a motor this arrangement is desirable, as the transformers thus perform the dual functions of furnishing the proper operating voltage to the machine and acting as a regulator or starter.

One or both of the secondary terminals of an autotransformer may be arranged so that the position of connection to the winding may be varied, and by that means the secondary voltage may be caused to vary through any desired range, while the primary current changes only so far as is required by any change in the power absorbed by the secondary circuit. The last step may bring the secondary circuit terminals into connection with the primary circuit terminals, thus putting full voltage on the secondary circuit, as, for instance, in the case of autotransformers in three-phase circuits, the final step puts the secondary terminals in connection with the primary terminals, as shown by *a*, *b*, *c* of Figs. 335 and 336. Autotransformers are commonly used for voltage regulators in intermittent service, as, for instance, in place of rheostats for starting alternating current motors. They are evidently lighter in weight per kilowatt of energy transmitted, and are therefore desirable for use when neither the primary nor secondary voltages are of such a high value that it is unwise to have the primary and secondary circuits electrically connected together.

Figure 338 shows a small autotransformer intended for voltage regulation, which may be used either to reduce the voltage in the circuit (in which case it is called a **Dimmer**) or to raise it (in which case it is called a **Booster**) depending upon how its internal connections are made. The regulation is effected by pushing

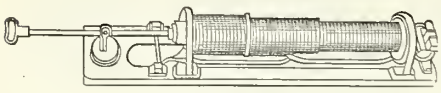


FIG. 338. — Autotransformer arranged telescopically so that it can be used as a Regulator.

a part of the winding in or out of the solenoid formed by the remainder. Numerous devices of this kind, which depend upon moving the primary and secondary coils with reference to each other, or the core with reference to both, are manufactured.

145. Calculation of Magnetic Leakage. — By properly placing the windings with respect to each other and to the magnetic circuit, the magnetic leakage of transformers may be reduced to reasonably small values. Thus in Fig. 339 the primary and secondary windings P and S are so placed that a short-circuiting of magnetic flux along the path indicated by the dotted lines is to be expected; but when the windings are arranged as shown in Fig. 340, the leakage is not as great, because of the greater reluctance in the magnetic circuits of the leakage paths caused by their greater length and lesser cross section; while if each of the windings is divided and the primary and secondary parts sandwiched together, the leakage may be made very small.

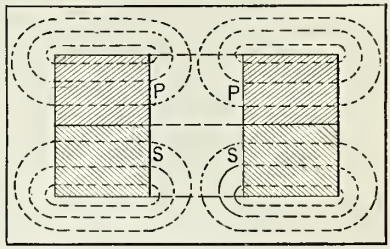


FIG. 339. — Transformer Windings arranged in Shell Transformer of Given Dimensions so that Leakage is large.

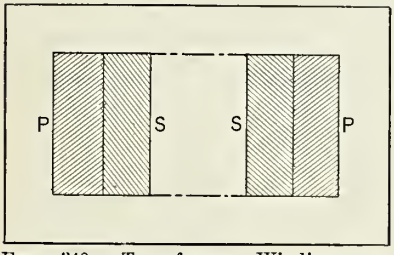


FIG. 340. — Transformer Windings arranged in the Shell Transformer shown in Fig. 339, but so that Leakage is reduced.

The magnetic leakage may be calculated with approximate accuracy by the method indicated below.

Since the currents in the primary and secondary coils of the transformer are in practical opposition of phase, their magnetizing effects are opposite. This tends to cause lines of force to short-circuit through the coils, as shown in Fig. 339, the tendency being greatest at the plane where the coils touch each other, since the magneto-motive force is there the greatest, and falling off to zero at the outer edges of the coils, so that the magnetic leakage will differ for each layer of wire in the coils. The effect of leakage must therefore be calculated for each layer, and the total effect may then be summed up.

In Fig. 341 the ordinates of the line $A'BA''$ are proportional to the total ampere-turns acting at any point to cause leakage lines to pass through the coils P and S. These ordinates are equal to the number of turns in a coil between the foot of the ordinate and the outer edge of the coil, multiplied by the current flowing in the turns. The ordinate at the outer edge of each of the windings is evidently zero, and at the plane between the windings reaches a maximum equal to practically $n_2 I_2$. The number of the leakage lines of force inclosed by any layer is proportional to the corresponding ordinate of the lines $C'DC''$, when these ordinates are respectively equal to

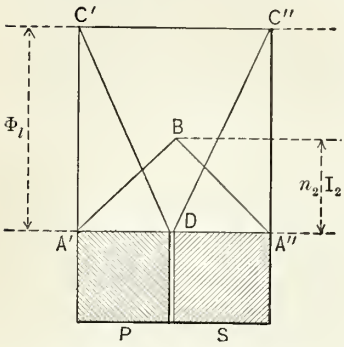


FIG. 341. — Diagram for showing the Relation of Transformer Leakage Magneto-motive force to Magnetic Leakage.

$$x = \frac{1.25\sqrt{2}\ ya}{l},$$

where x is the desired ordinate, y is the mean ordinate of the line $A'BA''$ taken from the neutral plane at D to the point under consideration, $a = m\,n$ is the area of the coil between the neutral plane and the point under consideration, n being the length of the coil perpendicular to the plane of Fig. 342, and l is the average length of the leakage lines of force through the coils (Fig. 342). The maximum value of x falls at the outer edges of the coils and is

$$x_m = \frac{1.25\sqrt{2}\ n_2 I_2 A_1}{2l},$$

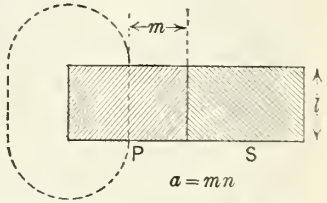


FIG. 342. — Diagram showing the Dimensions of Leakage Paths in a Transformer.

where A_1 is the total iron surface presented to a coil from which leakage lines emerge; and the average number of leakage lines inclosed by the different layers at the instant of maximum leakage is

$$\frac{x_m}{2} = \frac{1.25\ n_2 I_2 A_1}{2\sqrt{2}\,l} = \frac{.45\ n_2 I_2 A_1}{l}.$$

The inductive effect of this leakage on the secondary winding is equal to

$$\frac{\sqrt{2}\,\pi n_2 \frac{x_m f}{2}}{10^8};$$

and an equivalent effect is produced on the secondary voltage on account of the leakage of lines of force through the primary coil. If A is taken to represent the total area of iron presented to both windings from which leakage lines emerge, the formulas become

$$\Phi_l = \frac{1.25\, n_2 I_2 A}{\sqrt{2}\, l} \text{ and } E_l = \frac{\sqrt{2}\,\pi n_2 \Phi_l f}{10^8} = \frac{4\, n_2^2 I_2 A f}{10^8 l},$$

where E_l is the combined leakage voltage of the primary and secondary coils, the former reduced to secondary equivalents. The inductive effect due to magnetic leakage has already been shown to be in quadrature with the mutually induced voltages of the transformer.* The effect of the leakage voltage is determined by the formulas or diagrams already given.† The formulas given above are for the parts of the coils under the iron only, and the iron portion of the leakage paths was assumed to be of negligible reluctance. Leakage around the ends of the coils has some effect, and the leakage paths in the iron have a low reluctance which cannot always be considered negligible. As a result the formulas derived do not give the total magnetic leakage exactly and though approximately correct they should be used with discretion.

146. Current Rushes and Surges. — It was shown earlier that the exponential term in the complete equation for an alternating current in an inductive circuit is ordinarily negligible, but under certain conditions its effect for a few periods after the current is started in a circuit may be considerable. This question was investigated by Fleming ‡ and others § with especial reference to the action of transformers when first switched on to an alternating current circuit. If a transformer is switched on to a

* Art. 124.

† Arts. 123, 124.

‡ *Jour. Inst. Elect. Eng.*, Vol. 21, p. 677.

§ Hay, On Impulsive Current Rushes in Inductive Circuits, *London Electrician*, Vol. 33, pp. 229, 277, and 305.

circuit, the current does not instantly assume the final form of the wave, but comes gradually to its final form through a short interval of time. The length of the interval and the magnitude of the early current depend upon the instantaneous reactance of the circuit, the frequency, and the point in the voltage wave at which the connection is made. The instantaneous reactance of the circuit depends upon the magnitude of the residual magnetism in the core at the instant of switching in the current, and its direction compared with the instantaneous impressed voltage at the instant of switching in. If the instant of switching on to the circuit is that at which the impressed voltage is passing through zero, the current in the transformer is less during the early interval than its final value; while if, at the instant of switching on, the impressed voltage is passing through its maximum value, there may be quite an excess of current flow through the circuit for a short time, on account of the relations which exist between the instantaneous impressed and counter electric voltages during the first half period. If the residual magnetism has a direction in the core corresponding to the magneto-motive force of the current at the instant of switching in, the apparent reactance may be very small, and the instantaneous current rush be correspondingly large; but if the residual magnetism is in opposition to the magneto-motive force of the current at the instant of switching in, the quick reversal of this magnetism may give an apparent large reactance for the instant and the current may rise gradually. The abnormal state of the current can only exist for a very short time unless the reactance of the circuit approaches a condition of resonance.

When the transformer circuits contain resistance, self-inductance, and capacity, the disturbances upon opening or closing the switching devices controlling the circuits may be excessive and may cause the flow of very large momentary currents or the generation of abnormal voltages.* Troublesome effects of this nature are most apt to occur on high voltage transmission lines and where large units of power are used. The exceptional care with which transformer terminals are insulated is in part necessary on account of the unusual conditions that exist upon opening or closing circuit switches or upon sudden shifts of load.

* Art. 59.

147. Methods of Testing Transformers. — The commercial output rating of a transformer should be the number of kilovolt amperes it will supply to a load at unity power factor, without heating beyond a specified rise of temperature, when the rated full load voltages are used in the primary and secondary circuits, and when the voltage and current waves are approximately sinusoidal. Commercial efficiency and regulation have already been defined.* The most important tests of transformers are those for determining the temperature rise at full load, and at any overload for which the transformer is designed ; the regulation between no load and full load, or at other proportions of load, as desired ; the core losses and the copper losses, from which to compute the efficiency ; and the dielectric strength of the insulation between the primary and secondary coils and the coils and core. The efficiency and regulation are often desired also for loads with power factors other than unity. In addition to tests to determine those quantities it is often desirable to obtain the values of the exciting current under various conditions.

It is evident that having obtained the values of the coil resistances and leakage reactances, the iron losses, and the value and power factor of the exciting current, the more important quantities related to the transformer operation except the heating can be closely determined for various loads and power factors by the use of the transformer circle diagrams and formulas.

All tests should be made with load currents and voltages of rated value and frequency, and approximating to sinusoidal form, that is, having ordinates not varying, at any instant, more than 10 per cent from that of the equivalent sinusoid ; with a room temperature at 25° centigrade or with the quantities affected properly corrected ; with the barometer at 760 mm., or with proper corrections for other conditions ; with the rise of temperature of the transformer at the normal value for the conditions of the test ; and with the apparatus operating under the normal conditions for which it has been designed and rated.

In this country the Standardization Rules of the American Institute of Electrical Engineers are generally accepted as authoritative.† In these rules are given detailed statements of the

* Arts. 123, 137.

† *Trans. Amer. Inst. Elect. Eng.*, 1907, pp. 1797–1825, and *Year Book, ditto*, 1909.

conditions under which tests of the electrical apparatus, including transformers, should be made; and they should be consulted before making tests of the kind indicated above. The following few paragraphs give in outline some of the simpler methods for testing transformers.

Wattmeter Method. — A wattmeter may be placed in the primary circuit of a loaded transformer and another in the secondary circuit. Then the ratio of their readings is the efficiency of the transformer. Or, a wattmeter may be used in the primary circuit, and an amperemeter and voltmeter can be used to determine the output if the transformer is worked on a purely non-reactive load, though it is always wisest to use a wattmeter on alternating-current circuits for measuring power. The wattmeter was early used by Fleming in an extended series of transformer tests, and found to be satisfactory. It has now become the standard method for measuring electrical power. Before taking the readings the transformers should be run at full load until the temperature has become constant in all parts, which requires from five to fifteen hours, depending upon the size. Also, such other conditions as are specified earlier in the article and in the rules referred to should be carefully observed. This method requires a supply of power, and a secondary load for absorbing it, equal to not less than the full load capacity of the apparatus, or larger if the efficiencies at overloads are required. When very large transformers are under test this may be impossible or undesirable, in which case one of the methods given later may be adopted. In transformer manufacturing establishments it is often possible to feed back the secondary test load into the load circuits of the works. It is common in testing to obtain the efficiency at $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $1\frac{1}{4}$, and sometimes $1\frac{1}{2}$ the rated output. For great accuracy these efficiencies should each be obtained after the temperature has become constant. It is also not unusual to make the efficiency tests for several power factors if the transformer is to be used on loads of various power factors.

Figure 343 shows a tri-phase transformer connected up with tri-phase wattmeters in the primary and secondary circuits as required for making an efficiency test, though ordinarily two single-phase wattmeters would be used in place of each tri-phase instrument. On the low voltage load side the wattmeter current coils are attached to the secondaries of current

transformers (*C.T.*), while on the high voltage primary circuit the instrument voltage coils are attached to the secondaries of potential transformers (*P.T.*) in addition to the current coils being connected to current transformers. In making the test, two voltmeters and amperemeters would ordinarily be also connected into two of the circuits of the primary and secondary leads.

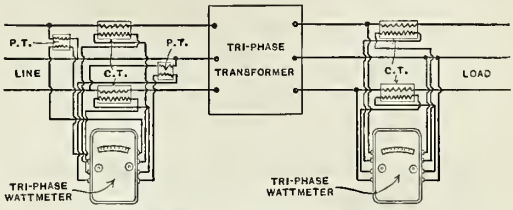


FIG. 343. — Tri-phase Wattmeters connected to a Tri-phase Transformer for Testing. Connections are also shown for Current and Potential Instrument Transformers.

The method here described has the disadvantage of depending for its result on the ratio of two instrument readings that are nearly alike, and error is therefore introduced into the efficiency measurement nearly in proportion to the errors inherent in the instrument readings. Additional errors are introduced when current and potential transformers are used. This method, therefore, should only be used for purposes of checking results of other methods or under circumstances in which other and more accurate methods are not available.

Stray Power Methods for obtaining Efficiency. — A very convenient method of measuring the efficiency of transformers is to determine the various losses directly, and thence the efficiency by calculation. The iron losses may be determined by measuring with a wattmeter the power absorbed by the transformer when the secondary circuit is open, and may be considered to be constant for all loads with sufficient accuracy for commercial purposes, since the core magnetism changes but little with changes of load in constant voltage transformers. The copper losses for any load are readily calculated when the secondary and exciting currents and the primary and secondary resistances are known. The exciting current may be measured at the same time that the iron losses are determined, by the insertion of an amperemeter into the primary circuit with the wattmeter; and the resistances may be measured by a bridge or voltmeter and amperemeter method

using direct currents. For a given load, the secondary current is a fixed quantity. The efficiency is then, practically,

$$\frac{P_2}{P_2 + P_c + I_2^2 R_2 + \left(\frac{I_2}{s} + I_\mu\right)^2 R_1},$$

where P_c represents the measured iron losses and s the ratio of transformation. Due care must be used that the temperatures are right, or else proper corrections must be made to the coil resistances. The iron loss will vary only slightly with the temperature and need not ordinarily be corrected.

A still more convenient method, which may be readily used in central stations for testing transformers, is to measure the iron losses by a wattmeter, as explained above. The copper losses may then be measured by short-circuiting the secondary winding through an amperemeter, and adjusting the primary voltage until the full load current, or any desired fraction thereof, passes through the amperemeter. The reading of a wattmeter in the primary circuit is now nearly equal to the copper losses for the temperature of the windings at the time, since the voltage and maximum magnetic density must be very small, and the iron losses are therefore almost or entirely negligible. The exact copper losses may be determined by measuring and correcting for the small iron loss. The tests for the iron losses may often be conveniently made by using the low resistance coil of the transformer as the primary coil (Fig. 344).

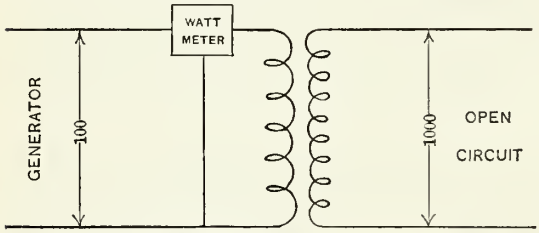


FIG. 344. — Transformer arranged for measuring Iron Losses. Low Voltage Circuit used as Primary.

This method of testing may be used with satisfaction where numerous transformers must be tested, since the losses and efficiency may be determined expeditiously and with the expenditure of little power. When combined with a run of several hours with full load current, the secondary circuit

being made up of impedance coils, the method proves economical for shop tests.

Ayrton and Sumpner early devised a method which is frequently used in which two transformers of the same size and make are opposed to each other, and which is often called the opposition method. The method of connecting is shown in Fig. 345, in which A and B are the transformers to be tested, with their primary windings connected in relatively opposite directions to the leads and their secondary windings in series with each other. The voltages of the secondary windings are thus in opposition. A transformer T is inserted with its secondary winding in circuit with the secondaries of A and B. By varying the resistance R, the voltage of T may be regulated so that any desired currents will pass through the primary and secondary windings of A and B. Then the output of T measured by the wattmeter W_2 will give approximately the copper losses of A and B plus the loss in leads and instruments. The power supplied to the primary circuits of A and B by the alternator, measured by the wattmeter W_1, gives approximately the iron losses of the two transformers. From the data thus derived the efficiencies may be computed on the assumption that the losses are equally divided between the transformers A and B. Various modifications of this method may be made, such as putting an autotransformer in place of T, or the two primary circuits may be fed from two branches connected to the mains, but with the voltage of one regulated above normal and the other below normal, so that the difference will be sufficient to send

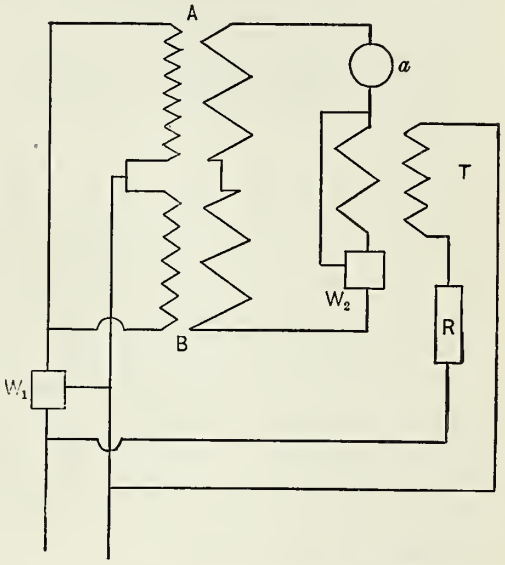


FIG. 345. — Transformers connected up for Testing by the Feeding Back or Opposition Method.

full load current through the short-circuited joint secondary circuit. Wattmeters in the primary branches will then, together, measure the entire losses of the two transformers. An amperemeter should be inserted in the primary or secondary circuit by which to regulate the current.

Either the high or low voltage coil may be used as the primary coil in making transformer tests, as is convenient. The last method described is largely used in manufacturing establishments or when two transformers of similar make are to be tested.

It must be borne in mind that the maximum magnetic density in the core which corresponds to any particular impressed voltage as read by voltmeter is smaller when the voltage wave is peaked and larger when the voltage wave is flat-topped, than when the wave is sinusoidal,* and the core loss is therefore less for a transformer run on a circuit with a peaked voltage wave than when the voltage wave is flat-topped, other things being equal, and correction must be made for that.

Regulation Tests. — The regulation of transformers which are used in incandescent lighting is a matter of much moment, and regulation tests are of almost equal importance to the tests of losses and temperatures. An ordinary method of making regulation tests is to place a voltmeter across the primary circuit and another across the secondary circuit of the transformer to be tested, care previously having been taken that the transformer is at the specified temperature and other conditions called for in the testing rules heretofore referred to are observed. At no load, the reduced equivalent readings of the instruments should be equal, and the numerical difference between the reduced readings at any other load gives the drop of voltage corresponding to that load. The reduced readings are gained by dividing the reading of the voltmeter in the high voltage circuit by the ratio of transformation of the transformer. This method lies under the serious disadvantage that the result to be measured is a small quantity equal to the difference of the much larger observed quantities, and the ordinary errors inherent in instrumental readings therefore are likely to introduce large errors in the value found for the voltage drop. The drop of voltage, measured in this way, includes the IR drop in the windings and the drop due to magnetic leakage, both of which increase with the load. The

* Arts. 115, 134.

magnetic leakage drop may be determined by subtracting from the total drop, the value of the IR drop which is calculated from measured resistances and currents. The scalar value of the induced leakage voltage is obtained from the expression

$$E_L = -\sqrt{E_d^2 - E_R^2}$$

when E_d is the total drop measured and E_R is the total drop in the secondary voltage due to the resistance of the two coils. Having obtained E_L and E_R, and the exciting current, for a given voltage and load, the circular transformer diagram loci may be drawn as earlier explained.* The transformer reactance equals

$$X_L = \frac{E_L}{I_2}.$$

A much more accurate regulation test may be made by using two transformers of equal transformation ratios and one voltmeter. The primary circuits are separately connected to the supply mains, and the secondary circuits are connected together on one side, terminals of like polarity being tied together. A voltmeter of high resistance or an electrostatic voltmeter is connected between the other legs of the secondary circuits. The reading of this voltmeter with a load on one transformer, when the other is unloaded, gives the total drop of voltage caused by loading the former.

Regulation ratings are usually made with non-reactive loads, though the regulation for loads of lower power factor is often very important. The regulation of a transformer is changed for the worse by introducing inductance into the secondary circuit, and for the better by introducing electrostatic capacity into the secondary circuit, as has already been proved.† Regulation tests on a reactive load are made in the same manner as explained above. The power factor of the load may be obtained by dividing the wattmeter reading of the load by the product of the secondary amperes and volts.

The efficiency and regulation of polyphase transformers can be obtained by the same methods as have been explained for single-phase transformer tests, using the methods given earlier for measuring polyphase power and power factor.‡

Heating.—The rated capacity of a transformer is determined

* Arts. 130, 132. † Art. 132. ‡ Art. 104.

mainly by the amount it rises in temperature due to the heat caused by internal losses of power. The rise of temperature is affected by the temperature of the surrounding air, the barometric conditions, and extraneous movements of air over the apparatus. Therefore these conditions during a test should be made to accord with the accepted standards.* The temperature of the conductors can be accurately obtained by measuring their resistance while at the temperature of the room in which the test is being conducted and again after their temperature has become constant under load. The change of temperature can then be determined by substituting in the formula $t_2 - t_1 = \frac{R_2 - R_1}{a\ R_0}$, when a is the temperature coefficient of copper with respect to zero degrees centigrade and may be ordinarily taken as .0042, R_0 is the resistance of the windings at zero degrees centigrade and is obtained from the formula $R_0 = R_1/(1 + at_1)$, R_1 is the resistance measured at the initial temperature t_1, t_1 is the temperature of the transformer when the first resistance measurement was made, R_2 is the resistance measured at the final temperature, and $t_2 - t_1$ is the rise of temperature to be determined.

The temperature of core, case, and insulation can be measured by thermometers, as can also that of the windings as a check upon the resistance temperature measurement. The bulbs of the thermometers must be carefully protected from improper radiation. In transformers with water circulation, air-blast, or forced oil circulation the temperatures at inlet and outlet, and the volumes of the water, air, or oil respectively introduced per unit of time, should also be measured. From these data and the specific heats of the cooling agent the amount of heat disposed of by the special cooling device may readily be obtained. In selecting a transformer, the amount of heat to be removed by special cooling devices should be specified. No part of the transformer should exceed 50° centigrade rise in temperature for the full load it is to carry, and from 5° to 10° lower is preferable, referred to an initial temperature of 25° centigrade.

Dielectric Strength of Transformer Insulation.— The dielectric strength of the insulation of transformers is tested by applying sufficiently high voltages between the primary coil and the core or the secondary coil and the core connected together, to insure

* *Trans. Amer. Inst. of Elect. Eng.*, 1907, pp. 1815-1818.

an insulation factor of safety which will be a sufficient protection against accident.* For transformers in which the high voltage coil is for 550 to 5000 volts the accepted testing voltage for this purpose is 10,000 volts when the secondary coil connects directly with power consumption circuits. The general rule for other transformers is that the test voltage shall be twice that of the high voltage coil. The test voltage should be applied for the period of one minute. When the test voltage exceeds 10,000 volts, great care must be exercised that the voltage be brought gradually to its full value, and while being maintained no large variations occur in it, as otherwise destructive oscillations may be set up.

The testing voltage may be measured by a voltmeter in connection with a potential transformer if necessary or by means of a standard spark gap. Where high test voltages are used, the connections must be made with great care to prevent a spark occurring due to a break, which is especially apt, on account of its own oscillatory character, to cause trouble. For this same reason the spark gap, when used, should be in series with a large resistance. The testing voltage should be sinusoidal, under which conditions the stress imposed on the dielectric is 1.41 times the testing voltage observed by the voltmeter. The ordinary commercial frequencies are all equally effective for testing purposes.

Special testing transformers are often employed for testing the dielectric strength of machines. These are designed with relatively small electrostatic capacity and large impedance which tends to reduce danger due to a breakdown of the dielectric from the causes named above.

Determination of Wave Shape.—Various methods of checking the shape of voltage and current waves used in testing have been used. One of the oldest of these is by means of a **Contact maker** revolved synchronously with the generating apparatus and so arranged that instantaneous values of currents and voltages may be obtained at any series of angles of advance during a cycle; but undoubtedly the most convenient and satisfactory apparatus to use for the purpose is the **Oscillograph.** Briefly, this consists of a looped ribbon wire, mounted so as to have a relatively very high natural frequency of mechanical vibrations,

* *Trans. Amer. Inst. Elect. Eng.*, 1907, pp. 1811–1815.

which is placed within a strong constant magnetic field. Through this loop is passed a current proportional at each instant to the current or voltage wave that it is desired to trace. The loop is then influenced by a torque proportional at each instant to the value of the current within it, acting on one half of it in one direction and on the other half in the opposite direction. By a minute mirror attached at one edge to one strand of the loop and the other edge to the other strand of the loop, it is possible to cast a beam of light upon a film or screen which repeats very accurately a tracing of the wave required. The subject of curve tracing is more fully discussed later.*

Methods used in Some Historically Important Tests. — The contact maker was used, in the early days before reliable wattmeters had been invented, for tracing the voltage and current curves of a transformer; and from the data thus obtained efficiency and other characteristics were calculated. Among the historic investigations of this kind were those by Ryan and Merritt,† John Hopkinson,‡ Mordey,§ and others. A large calorimeter was sometimes used in which the transformer was immersed for the purpose of obtaining its losses. ‖ The power in the transformer circuits was sometimes measured by the split dynamometer or combinations of the three voltmeters and three amperemeters method.

That the value of the iron losses is largely independent of the load carried by a transformer was first conclusively proved experimentally by Ewing. The same thing was also proved by Fleming's experiments. Figure 346 is plotted from one of Fleming's tests made on a transformer of 4000 watts capacity. The ordinates of line AB represent the differences of the power in the primary and secondary circuits as measured by wattmeters. The calculated copper losses are represented by the ordinates of the line CD. The difference of the ordinates of the lines AB and CD at any point is the iron loss for the par-

* Art. 157.

† *Trans. Amer. Inst. E. E.*, Vol. 7, p. 1.

‡ Hopkinson's *Dynamo Machinery and Allied Subjects*, p. 187; *Elect. World*, Vol. 20, p. 40.

§ *Jour. Inst. E. E.*, Vol. 18, p. 608.

‖ Duncan, *Electrical World*, Vol. 9, p. 188; Roiti, *La Lumière Electrique*, Vol. 35, p. 528.

ticular load. The lines AB and CD are approximately parallel, which shows that the iron losses were practically constant, regardless of the load. Therefore the conclusion was drawn that the stray power methods of testing transformers give efficiencies which are entirely reliable.

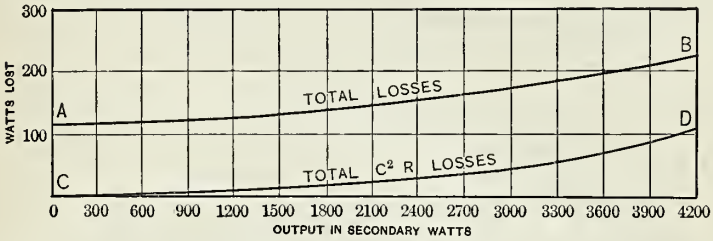


FIG. 346. — Curves showing Results of a Classic Experiment for determining Iron Losses with Varying Load.

Ewing's very neat plan for proving this point was designed to get at the matter directly. Two small transformers were made up exactly alike, the cores of which consisted of insulated iron wire wound into the form of a ring. Over this were uniformly wound two layers of wire making a primary coil, and another two layers making a secondary coil. In operating, the primary and secondary coils were respectively connected in series, but the two halves of each coil in one transformer were so connected as to be in magnetic opposition (Fig. 347). The core of one transformer was therefore magnetized and that of the other was not. While the I^2R losses at any load were equal in the two, the transformer with magnetized core heated more when put in operation than the other transformer, but the temperature of the second was brought to equality with that of the first by passing a direct current through the insulated wire of the core. The power expended in heating the second core by the direct current was thus equal to that expended in the first core due to iron losses. The equality of temperature was determined by means of thermo-electric couples embedded in the cores, which were connected in series with each other through a galvanometer (Fig. 347). In this experiment it was found that, after a balance of temperature was once obtained, it was unaltered by any changes in the loads of the transformers, thus showing that the core losses

in the magnetized transformer were independent of the load.*

These tests fully confirm the *a priori* deduction that the core loss must be substantially independent of load in an ordinary constant voltage transformer, which follows from the relations of counter-voltage to impressed voltage. Neither the conclusions nor the experiments apply to transformers in which

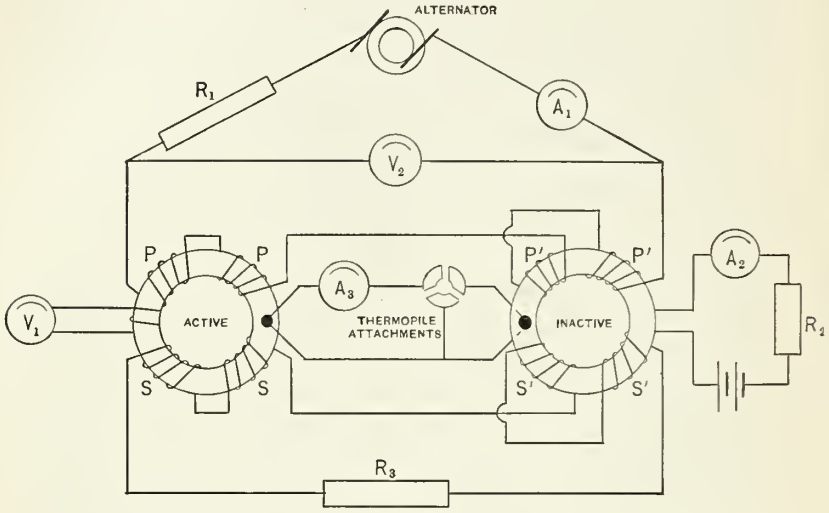


Fig. 347.—Ewing's Apparatus for determining Iron Losses.

IR drop or leakage reactance drop absorb a considerable part of the impressed voltage, however; because in them the mutually induced voltage and therefore the maximum magnetic density in the core must vary with the load if the primary impressed voltage is maintained constant, and the core losses vary when the maximum magnetic density is varied provided the cycles per second remain the same.

* Ewing and Klaassen, Magnetic Qualities of Iron, *London Electrician*, Vol. 32, p. 731.

CHAPTER XI

SYNCHRONOUS MACHINES — ALTERNATORS, MOTORS, ROTARY CONVERTERS, FREQUENCY CHANGERS

148. Losses in an Alternator. The principal internal losses of alternators are caused by: (1) I^2R **loss** in the conductors on armature and field magnet: (2) **Eddy currents** in armature cores and field magnet: (3) **Eddy currents** in armature conductors: (4) **Hysteresis** in armature cores: (5) **Friction** of bearing and brushes, and air friction, often called **Windage.**

In well-designed direct-current dynamos, the pole pieces usually cover not less than two thirds of the armature surface. In alternators, the poles usually cover about one half of the armature surface, or a little more.* This would make it appear, upon a superficial examination, that the field ampere-turns, and therefore the field losses, must be much greater in the alternator. However, since alternator armatures are made proportionally larger in diameter (usually exterior to a rotating field magnet) in order to give space for the windings and to avoid excessive magnetic leakage, the proportional excitation really required need not be much increased when the magnetic circuit is well designed. In the same way, while not much more than one half of the armature surface is covered with wire in a single-phase machine, the surface for winding is made much larger by increasing the diameter, while the number of revolutions per minute of the rotating part is not much reduced. Consequently, the voltage produced in a given length of conductor is commensurable in the two classes of machines running at equal speed. When windings for two or more phases are wound upon the armature, the entire surface is covered, and as a result, taking into account the advantages in design named above, polyphase machines can be made to give a larger output per pound of material, without excessive heating, than is usual for direct current machines.

* Art. 21.

The fact that the copper is usually divided among more cores increases the length of wire on alternator field magnets for a given magnetizing power, as compared with the field magnets of direct-current machines. On the other hand, the losses may be brought by careful designing to approach the average values used in direct-current machines. For large machines running from 200 to several thousand kilowatts capacity, the combined armature and field copper losses in good practice gradually reduce to a combined value of between one and two per cent for the very large sizes. A certain line of 5000 kilowatt machines built by a large manufacturer in this country has a combined I^2R loss of approximately one per cent, while a similar line of 500 kilowatt machines has close to two per cent I^2R loss. The loss is about equally divided between the field and armature windings.

Eddy currents in alternator cores are apt to cause greater loss than in ordinary direct-current machines unless greater precautions are introduced for preventing them. Thus, in the armatures of direct current machines of considerable capacity, the magnetic cycles in the core seldom reach twenty-five complete periods per second. On the other hand, the commercial frequencies for alternating currents are now 25 and 60 periods per second. The number of magnetic cycles per second in alternator armatures is evidently equal to the frequency of the induced voltage and current. Since the heating in the core discs which is caused by eddy currents is proportional to the square of the induced voltage and therefore to the square of the number of magnetic cycles per second, it becomes particularly important that the discs composing alternator armatures be thin and well insulated from each other. Therefore, the iron oxide composing the mill scale on the surface of the laminations is not always to be depended upon for insulation, and the stampings may be covered with an insulating enamel. However, when some of the newer irons giving low losses are used, the oxide proves sufficient. All burrs caused by punching the discs or truing the surface of the core should be carefully avoided or removed.

Eddy currents in the pole pieces are felt quite severely in some types of alternators, but the loss caused by them can always be brought within reasonable limits, in well-designed

machines, by making the pole faces of laminated iron, which is cast or dovetailed into the rotating spider frame. Tufting of the magnetic flux in the polar space due to the effect of the armature teeth is apt to be one of the causes of eddy currents in the pole pieces. This trouble can be reduced by careful design of the teeth.

The loss due to eddy currents in armature conductors of a fixed size has a tendency to be greater for alternating than for direct-current dynamos, on account of the more frequent and sudden changes in the strength of the field through which the conductors pass. However, since alternators are, in general, built for considerably higher voltages than direct-current machines designed for a similar duty, the conductors on the alternator armatures are of proportionally smaller cross section. This reduces the relative eddy currents to such an extent that they are not particularly noticeable except in very large or special machines. The common practice of winding armature coils with copper ribbons or bars set on edge (the broad side parallel with the lines of force) also tends to decrease eddy currents. In very large machines built to generate voltages not exceeding 2000 volts, armature conductors of a large cross section become essential. In such cases they can be made up of insulated strips connected in parallel. The uniform practice, however, of building alternator armatures with embedded conductors avoids all great difficulty from eddy currents of this character, since the relatively high frequency of the magnetic cycles in the core makes it undesirable to saturate the core teeth and but little stray magnetism from the field magnets gets into the slots.

The effect of hysteresis in iron core armatures is proportional to the number of magnetic cycles per second, and is therefore usually greater in alternator armatures, for a given magnetic density, than in those of direct-current machines. There are only two ways of decreasing the hysteresis loss per cycle and per unit volume: (1) by reducing the magnetic density; (2) by improving the quality of the iron used in the core. Reducing the magnetic density serves to decrease the eddy current loss also, and is therefore doubly advantageous. With the best quality of ordinary soft steels, the magnetic density used in the cores of alternator armatures and in the field cores varies commonly from 10,000 to 12,000 lines of force per square centi-

meter, and tends toward the former figure for machines having frequencies of 60 cycles per second. In the teeth the density runs up to from 15,000 to sometimes even 20,000 lines per square centimeter. For machines of 25 cycles frequency the magnetic densities tend to mark the higher values given. When the new silicon and other alloy steels, which have very low hysteresis constants, are used, the magnetic densities for 60-cycle machines also give values approaching 10,000 lines of force per square centimeter or slightly more in all parts of the magnetic circuit. In the case of these steels the low point of magnetic saturation largely determines the density it is possible to use. Halving the magnetic density tends to quarter the eddy current loss for an equal number of cycles per second. In the same way, the hysteresis loss varies nearly in the proportion of $B^{1.6}$, or in other words, halving the magnetic density decreases the hysteresis loss per unit volume to one third. On the other hand, the cross section of iron must be increased to decrease the magnetic density per square centimeter. This makes the careful selection and handling of the iron designed to enter alternator cores of special importance. The total iron loss varies, in commercial 60-cycle alternators of good design, from 1.5 to 2 per cent of the rated output capacity, and this is increased about .25 per cent for alternators having a frequency of 25 cycles per second. High voltage alternators are apt to have slightly higher iron losses than those for low voltage on account of the greater depth of teeth required on account of the additional conductor insulation.

149. Armature Ventilation, Density of Current in Conductors of Alternators, etc. — Since the hysteresis and eddy current losses are apt to be greater in alternator armatures, it is usually necessary that more opportunity be given for cooling than is the case in direct-current machines. The real effect of the velocity of rotation upon cooling is dependent largely upon the design of the rotating field magnet, or, if the field magnet is stationary, upon the fanning arrangements made in the construction of the armature; but for a given type of construction, the cooling effect is roughly proportional to the velocity, as was early indicated by the experiments of Rechniewski.* Consequently high

* *Bulletin de la Société Internationale des Electriciens*, 1892; *Electrical World*, Vol. 19, p. 336.

surface velocity is desirable for the purpose of cooling. Internal ventilation of rotating armatures is also rendered more effective by reason of the high surface velocity. For the purpose of internal ventilation a revolving armature is virtually converted into a centrifugal blower, which sucks in air at its center, along the shaft, and ejects it from the surface. For this purpose the openings in the spider which supports the core communicate with the surface through radial ducts, as seen in Figs. 78 and 92. The rotation of the armature then causes a continuous circulation of air through the ducts, which is proportional in volume to the surface velocity of the armature. Sometimes special wings or projections are placed on a rotating armature which greatly assist in ventilating the field magnet and also the armature. These remarks apply with particular force to rotary converters, which always have rotating armatures. When the field magnet revolves, which is the case in all large size modern alternators, the poles cause a vigorous circulation of air, which serves in lieu of the blower action of a revolving armature, except in the case of steam turbine driven field magnets, which are generally built without projecting poles so as to give a cylindrical surface, and such field magnets are provided with blower ducts similar in character to those of revolving armatures. The stationary armature has radial ventilation ducts between the laminations, from $\frac{1}{4}$ to $\frac{1}{2}$ inch in width and spaced from $1\frac{1}{2}$ to $2\frac{1}{2}$ inches apart. When the field poles are properly arranged, large volumes of air are forced through the ducts. Figures 80, 81, and 82 show the ventilating ducts of stationary armatures quite clearly, while Fig. 113 shows the adaptability of the rotating crown of poles on a rotating field magnet to act as a fan. The blower ducts of a steam turbine driven field magnet ordinarily deliver air into corresponding ducts of the stationary armature of the machine. The peripheral velocity of rotating field magnets varies from 3000 to 20,000 feet per minute, the latter figure being marked in certain high speed alternators of large capacity, while the velocities near the lower limit are suited to low speed small machines.

With all precautions to avoid excessive heating in alternator cores, careful design is demanded to prevent them from heating to a higher temperature than is desirable, especially when they are of the modern high speed types. It then becomes a matter of some

concern to determine the possible amount of energy which may be expended in the armature conductors without placing an excessive additional burden upon the cooling surface. There is no valid reason for admitting a higher temperature limit in alternator armatures than is allowed in direct-current machines, as was formerly done, and consequently, the radiating surface per watt of the power to be dissipated should be the same, where the ventilation and construction are similar. On account of the large amount of heat caused by core losses which must be radiated, it is not safe to allow an armature I^2R loss of one watt for each square inch of cooling surface. In common practice, for each watt of I^2R loss an average of from $1\frac{1}{2}$ to 2 square inches of outside winding surface is allowed. This constant is sometimes made much larger, but seldom smaller. This makes the total watts to be radiated, about 2 to 6 watts per square inch of the surface.

It will be appreciated that the velocity of an alternater not only determines the amount of blower action possible in a machine of given type, but also determines the form of the masses of material from which the heat must be extracted. Thus, in a low speed alternator arranged for direct connection to a reciprocating engine, the diameter of the armature must be made large relatively to its length in the direction of the shaft. While in a high speed machine, arranged for connection to a steam turbine, the diameter of the armature is relatively less and its length greater. This difference is caused by the different numbers of poles required and the permissible circumferential velocity. Further, the high speed machines, up to certain limits of speed, as shown by Fig. 348*, have a better weight efficiency and thus have a less volume of material per watt loss.

The density of current in conductors wound upon iron core armatures in good practice should usually not much exceed one ampere for from 300 to 1000 or more circular mils cross section. The density of current that is allowable in either armature or field windings varies quite widely with the dimensions of the coils and the efficiency of their ventilation.

Since the field cores of alternators are usually quite thin, the windings are often of a depth equal to as much as one half

* Behrend, *Electrical Review*, Vol. 45, p. 375 *et seq.*

the thickness of the cores. At the same time this depth is generally no greater than that on many direct current dynamo field magnets. The same radiating constant can therefore be safely used, that is, from .35 to .40 watt per square inch of the outer surface of the winding when the armature rotates as in rotary converters, and 1.5 to 3 watts per square inch when the field magnet rotates — the latter value depending upon the speed and

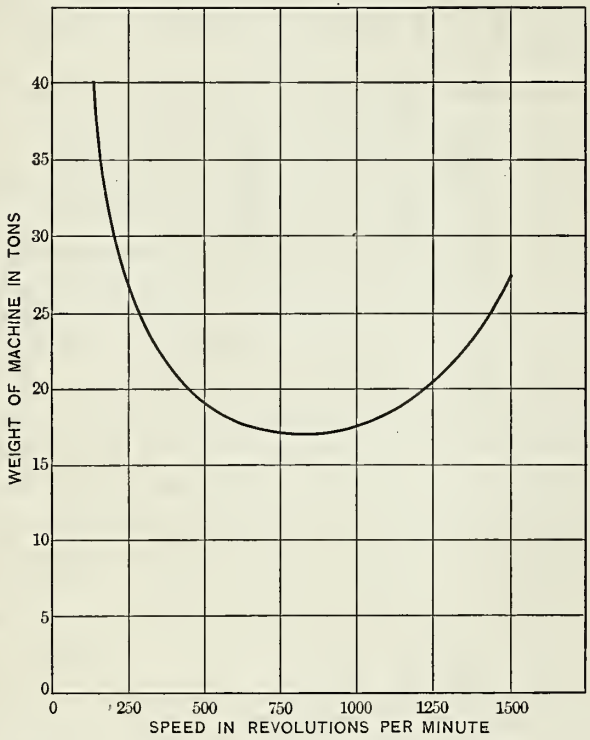


FIG. 348. — Relation of Speed to Weight in Alternators, for a line of 1000 Kilowatt, 3-Phase, 25-Cycle Alternators.

design of the field magnet. The ratio of winding depth to thickness of core is widely variable, and the radiation constant can wisely be based upon the external surface of the windings. The field cores should therefore be made of such a length that the area of the external surface of the windings on rotating field magnets, in square inches, may be from $\frac{1}{3}$ to $\frac{2}{3}$ times the I^2R loss in watts. The form of the field magnet, and therefore the effect of the

iron in conducting away the heat in the field windings, has a considerable effect on the allowable value of the constant, as is also the case in direct-current machines. If the winding depth exceeds two inches, it is likely to cause injurious heating in the inner layers. The field windings are often made of edgewise-wound coils of wide, thin copper strips. Strips of insulation are wound with the copper strips, and the edges of the copper strips are allowed to remain bare. This type of construction is not only mechanically strong but affords ready radiation of heat from the coils.

The cross-sectional area of the field cores is given by the formula, $A = \Phi \frac{v}{B}$, where Φ is the useful armature flux required from one pole, v is the leakage coefficient, and B the magnetic density desired in the field core. The pole width is determined by the mechanical and electrical conditions which fix the pitch, and only the length of the pole faces may be altered to vary B. The value of the coefficient of magnetic leakage is quite large in most types of alternators. It probably averages from 1.15 to 1.45 in standard alternators with poles set in a circle either without or within the armature. The calculation of this coefficient of alternators may be carried out upon the same methods as those used for continuous current-machines,* or it may be measured for a complete machine by the search coil method; that is, a test coil may be arranged about a pole piece and connected to a ballistic galvanometer. When the field current is then suddenly reversed, the throw of the needle is closely proportional to the magnetic flux in the pole. A coil of equal turns may then be laid on the armature surface inclosing an arc equal to the polar pitch and extending the full length of the armature coils. The throw is now proportional to the useful flux from the pole. The ratio of the first to the second reading is approximately proportional to the leakage coefficient. The coefficient varies somewhat with different armature currents and power factors on account of the magnetic reactions of the armature currents upon the poles. By likewise using test coils placed around different parts of the field core, the distribution of the magnetic flux may be measured.

* Jackson's *Electro-magnetism and the Construction of Dynamos*, p. 133.

150. Determination of the Number of Armature Conductors. — The formula for voltage induced in an alternator,

$$E = \frac{2\,KS\Phi\,Vp}{10^8 \times 60},$$

may be put into the form

$$E = \frac{2\,KS\Phi f}{10^8},$$

since $\frac{Vp}{60}$ is equal to the frequency f of the induced voltage.* Taking a proper value for K, as already explained, gives the effective value of E. In a well-designed machine of the usual American types, the value of K is about **1.1**, which is the value for a machine which gives a sinusoidal voltage and in which the differential action is negligible. The conditions of service usually fix the values of E and f in any particular case, and the equation then contains two dependent variables, Φ and S. The ratio of these is determined from the form and dimensions of the armature which it is desired to make. The number of poles is limited by constructive considerations and the importance of keeping the magnetic leakage within reasonable limits. On the latter account the poles must not come too near together. With this in view the frequency cannot be increased beyond a certain limit by increasing the number of poles, without carrying the periphery velocity of the armature beyond the safe limit. Thus, suppose a machine is designed with a twenty-pole rotating field magnet, and its field magnet is designed to be driven at the safe limit of velocity. If it should be desired to increase the frequency of the voltage by 10 per cent, two poles must be added to the field magnet and the revolutions kept constant, or else the field magnet must be speeded up 10 per cent. Suppose the latter is not permissible on account of mechanical safety; then, if the poles are already as close together as economy admits, in order to increase the number of poles the pitch circle must be increased in diameter. This again calls for an increase of the periphery velocity of the field magnet, the revolutions per minute remaining constant. Hence in any type of machine a limit of frequency may be reached which cannot be safely exceeded. The limiting frequencies in the ordinary types

* Art. 20.

of machines designed for commercial service are from 100 to 150 periods per second. In the old Mordey type the limit was much higher, since the poles of this machine may be very close together and cause no additional leakage, and structural conderations, only, set the limit. Commercial frequencies for electric lighting and power are all less than 150 periods per second, and are usually 60 or less, so that all practical types of alternators may be used in commercial service. The requirements of wireless telegraphs, on the other hand, are now making a demand for some relatively small alternators of special types designed for generating currents of frequencies as great as two or three hundred thousand cycles per second.

Fixing the frequency of a machine and the periphery velocity and revolutions of the revolving part, fixes both the diameter of the armature and the number of field poles. The diameter of the armature fixes the space for armature slots. With the slot space fixed and the dimensions of the slots determined from the conditions of tooth saturation and core reluctance desired, the value of S is determined from the number of insulated conductors of requisite area which can be properly placed in the slots. The value of S being determined, the length of the armature must be made such that the necessary total magnetization may be set up in the field cores and armature without forcing the magnetic density in the teeth and cores to too high a value. Finally, the ratio of radiating surface to I^2R loss should be checked.

It is well to consider here the effect upon the output of an alternator, of making a change in the number of armature conductors. The voltage developed in the coils due to their cutting lines of force is proportional to the number of turns in the coils. The voltage of self-induction, on the other hand, is proportional to the square of the number of turns in a coil. Hence, increasing the number of armature turns beyond a certain limit may actually decrease the output of the machine, and if carried to a sufficient degree may make it even tend to regulate for constant current instead of constant voltage. The following argument, in which is approximately determined the number of turns that will give maximum output, is of some value in giving a conception of the effect of the number of turns upon output. In the discussion the effect of armature reactions on

the field is assumed to be included in the inductive reactance of the winding. Where the load is non-reactive the assumption is justifiable.* If L_1 represents the effective or working self-inductance for each armature conductor, including its proportion of reactive effect on the field magnet, then the total effective self-inductance of the armature is approximately $L = S^2 L_1$. In the same way, if E_1 represents the voltage developed per conductor, the total voltage developed in the armature is $E = SE_1$. From the fundamental formula

$$I = \frac{E}{\sqrt{R^2 + 4\pi^2 f^2 L^2}},$$

we get $$I^2R^2 = E^2 - 4\pi^2 f^2 I^2 L^2,$$

or $$IR = \sqrt{E^2 - 4\pi^2 f^2 I^2 L^2}.$$

This may be put into the form

$$IR = \sqrt{S^2 E_1^2 - 4\pi^2 f^2 I^2 S^4 L_1^2},$$

supposing the external circuit to which the alternator is connected to be non-reactive. The term IR is the active voltage in the complete alternator circuit, and it is desirable that this be a maximum for a given armature. Differentiating the equation with respect to S and solving for a maximum, we get the following:

$$\frac{d(IR)}{dS} = \frac{2S(E_1^2 - 8\pi^2 f^2 I^2 S^2 L_1^2)}{2\sqrt{S^2 E_1^2 - 4\pi^2 f^2 I^2 S^4 L_1^2}} = 0;$$

hence $E_1^2 - 8\pi^2 f^2 I^2 L_1^2 S^2 = 0$. Or IR is a maximum when

$$S = \frac{E_1}{2\sqrt{2}\,\pi f I L_1}.$$

In this it is assumed that the armature resistance is small compared with that of the external circuit, which is always the case in efficient working.

For E_1 may be substituted its value

$$E_1 = \frac{2K\Phi f}{10^8},$$

and the expression for S at maximum output becomes

$$S = \frac{K\Phi}{10^8 \sqrt{2}\,\pi I L_1} = \frac{\Phi}{IL_1} A,$$

* Art. 155.

where A is a constant depending upon the type and dimensions of the machine. If K has a value of 1.1, the value of A is practically 25×10^{-10}.

The final form of the variable portion of the expression giving the maximum economical value of S is striking. Its numerator is the total useful magnetization due to the field magnet which passes through an armature coil, and its denominator is the magnetization passing through the coil due to the current in each of its own conductors. The fact is, however, that mechanical limitations and limitations due to heating and regulation, and the considerations of efficiency, prevent the number of conductors in modern alternator armatures even approaching the number given by this formula, when the machine is worked on a non-reactive load.

When the external circuit is inductive, as is commonly the case, the number of armature turns which gives a maximum active voltage is smaller than when the external circuit is non-reactive. If S' is the number of conductors giving the maximum active voltage when the external circuit has self-inductance L_e, and S is the number of conductors giving the maximum voltage when the external circuit is non-reactive, then

$$S^2 L_1 = S'^2 L_1 + L_e, \text{ or } \frac{S'^2}{S^2} = 1 - \frac{L_e}{S^2 L_1},$$

or

$$\frac{S'}{S} = \left(1 - \frac{L_e}{S^2 L_1}\right)^{\frac{1}{2}}.$$

In the latter expression $S^2 L_1$ is the self-inductance of the armature when wound with the proper number of turns to give a maximum active voltage when the external circuit is non-reactive. This becomes larger when there is an inductive load, since the reactions of the armature upon the field are greater.* When L_e is greater than $S^2 L_1$, the right-hand member of the expression for $\frac{S'}{S}$ becomes imaginary; that is, the ratio is an indeterminate.

It is impossible to put so many turns on commercial alternator armatures as the criterion shows would give the greatest output at unity power factor, since the question of regulation

* Art. 152.

in constant-voltage alternators demands that the fall of voltage in the armature due to resistance and inductance shall be as small as possible. Nevertheless where alternators are subject to inductive loads, the self-inductance, reactive effect upon the field, and resistance of the armature conductors are most important elements affecting regulation under the various operating conditions to be met in practice, as is shown clearly by the relations just developed.

151. Armature Ampere-turns and Self-inductance of Alternators. — The self-inductance of an alternator armature may be approximately estimated from the magnetic and electric data of the machine. The reluctance of each magnetic circuit must be calculated exactly as in the case of a direct-current multipolar dynamo, — in order to determine the field windings, — using the formula $P = \frac{l}{\mu A}$, where l is the length, A the cross section, and μ is the permeability of the part of the magnetic circuit

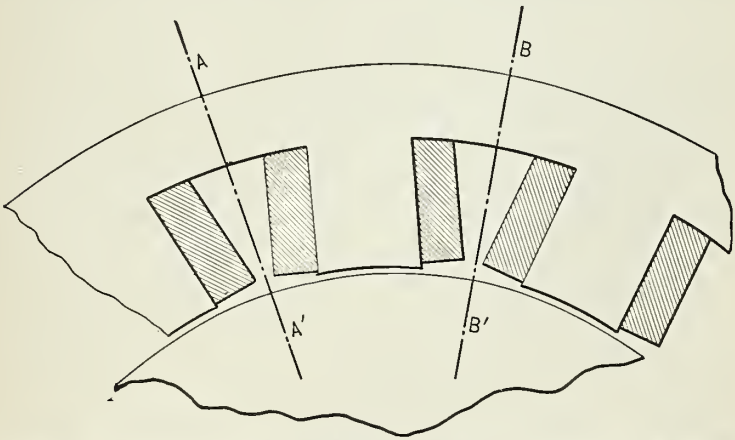


FIG. 349. — Diagram for showing the Magneto-motive force required for Each Field Core to set up the Required Magnetic Flux.

under consideration. The reluctance to be overcome by the magneto-motive force of each field core belongs to that part of the magnetic circuit which lies between the planes parallel to the armature shaft in which lie the lines AA' and BB' in Fig. 349, which is a section of an alternator with rotating armature. Machines with rotating field magnets may be treated in the

same manner. Calling that reluctance P, the ampere-turns for each field core are in number,

$$nI = \frac{\Phi_f P_f}{1.25} + \frac{\Phi_s P_s}{1.25} + \frac{\Phi_a P_a}{1.25} + \frac{\Phi_t P_t}{1.25}$$

where Φ_f, Φ_s, Φ_a, and Φ_t, and P_f, P_s, P_a, and P_t are respectively the magnetic fluxes and reluctances of the field core and yoke, air gap, armature body, and armature teeth lying between AA' and BB'.

The reluctance in the different parts of the magnetic circuit met by lines of force which are set up by the magneto-motive force of the armature ampere-turns when the field magnet is excited, may be assumed to be equal to the reluctance in the same parts of the circuit met by the lines set up by the field coils, except for the part of the flux due to armature turns which leaks from tooth to tooth in a multi-tooth armature. The number of lines of force set up in the portion of the magnetic circuit between the planes AA' and BB' by a unit current in one armature coil, the center of which is directly under a pole face, is $\frac{1.25\, S_1}{2\, P}$, where S_1 is the number of conductors in the coil and is equal to twice the number of turns in the coil, and P is the equivalent combined reluctance of the magnetic path in the field magnet, armature, and air gap. Each one of these lines of force in completing its circuit through another segment of the magnetic circuit must link another armature coil, so that we may say that the number of lines of force set up by each pair of coils is $\frac{1.25\, S_1}{2\, P}$. The self-inductance of the pair of coils, therefore, so far as relates to the magnetic circuit through the field core, is

$$L_1 = \frac{1.25\, S_1^2}{2 \times P \times 10^8},$$

since S_1 is equal to the number of turns in two coils.

The self-inductance per phase, of the whole armature due to this magnetic flux, is equal to L_1 multiplied by the number of pairs of coils, when the armature windings of a phase are connected in series, or

$$L = \frac{1.25\, S_1^2\, p}{2 \times P \times 10^8},$$

when the coils are placed in one slot per pole per phase. When the windings are distributed in several slots per pole per phase the value of L is diminished. When the armature is connected with the halves in parallel, this inductance is one fourth as great as is given by this formula; but in the case of two similar armatures built for the same voltage and output, the one connected with the halves in parallel has twice as many conductors in each coil as has the other armature, and their self-inductances are equal.

The path of the lines of force set up by the armature coils has been assumed to be the same as the path of those set up by the field magnets, and the real effect of the armature magnetomotive force upon the number of lines of force in the mutual magnetic circuits is to increase or decrease the number that would exist were the armature turns absent, rather than to set up an independent magnetic flux. The extent of this effect depends upon the lag of current in the armature, and the effects of armature reactions and of self-inductance are therefore closely related. In the case of machines with toothed armature cores, as already said, the reluctance measured around the path of the leakage lines of force set up by each group of conductors in a slot may be quite small. This is due to the effect of the leakage from tooth to tooth. Consequently, the self-inductance of the armature must be increased over the amount given in the preceding formula by an amount equal to that caused by the tooth to tooth leakage. This is made up of the self-inductance caused by this leakage for each group of conductors in a slot multiplied by the number of groups in the whole winding of a phase. It is comparatively constant in value for different armature currents, since the leakage path may be partially in air, and the teeth, because of their small cross section relative to other parts of the magnetic field circuit, are apt to be magnetized to a comparatively high degree of saturation.

152. Armature Reactions in Alternators. — The armature reactions of alternators do not cause as serious consequences in some respects as those of direct current machines, as the commutator is absent, but their effect upon regulation is of prime importance and demands first attention from the designer. Consider first the conditions which exist in single-phase machines, or in polyphasers when the armature currents are

unbalanced, which, as will be seen later in the discussion, produces much the same effect as is produced in single-phasers. Then when the current of the armature is in exact phase with the impressed voltage, the armature current has comparatively little opportunity to affect the field magnetism. When the armature conductors are directly between the pole pieces, the instantaneous current is zero, and therefore at this point the armature has no effect upon the field magnetism. When the coils have

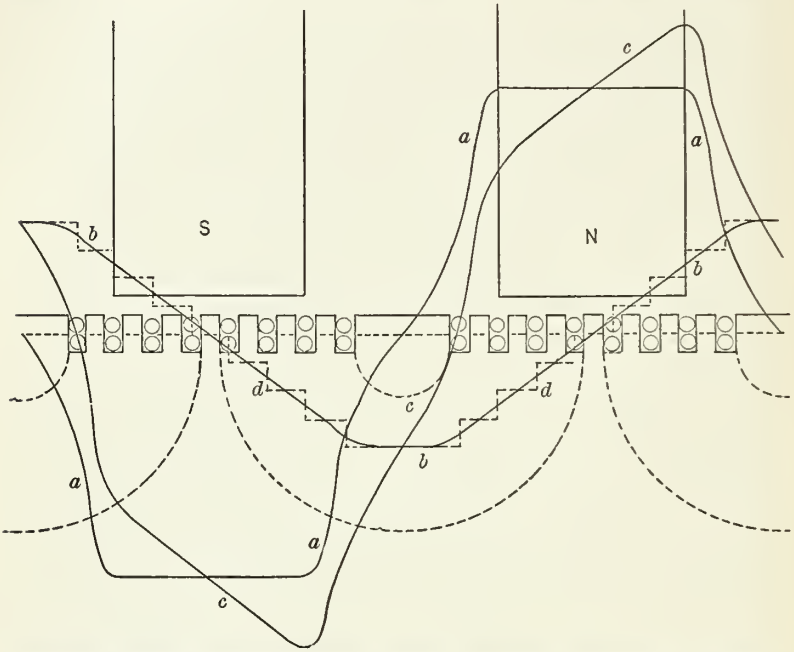


FIG. 350. — Diagram for showing the Armature Reactions produced by a Single-phase Machine when running at Unity Power Factor. (*a*) Original field flux. (*b*) Flux caused by armature current. (*c*) Resultant flux. Relative motion of the armature toward the right-hand with respect to the field poles.

moved through one half the pitch, a sheet of current at its maximum value flows directly under the pole faces. This current has such a direction that its magnetic effect tends to crowd the lines of force of the field magnet into the trailing tips of the poles (Fig. 350). Hence the field is weakened on account of the increased reluctance of the magnetic circuit. This effect is probably not very marked in the usual forms of alternators, since the armature ampere-turns are usually not large compared with the impressed magneto-motive force of the field magnets, but, to

whatever extent the effect exists, it produces a periodic shifting of the magnetic density across the pole face as the armature coils move from pole to pole. The distortion and consequent weakening of the field may be reduced by cutting a slot longitudinally across the pole faces, or by carefully designing the pole tips. In Fig. 350 the irregular line *dd* indicates the tendency of the groups of conductors in a slot to distort the curve *bb* on account of the fact that the magneto-motive forces of their currents are not evenly distributed over the armature surface. This effect is usually not large.

When the armature current is out of phase with the induced voltage, the conditions are quite different. Suppose that the phase of the current is retarded on account of self-induction. When the centers of the coils are under the poles, the current is not zero, but has an instantaneous value which depends upon the amount of retardation. This current in a generator is in such a direction that its magnetic effect opposes that of the field, as illustrated in Fig. 351, in which the dot or arrow point on the armature conductors represents the current flowing toward the observer and the cross or arrow feather represents the current flowing away from the observer. As the coils move, the opposing effect of armature reactions merges into the cross effect already indicated, and the armature reactions therefore tend to cause a periodic variation of the strength of magnetic flux. When the machine under consideration is operating as a synchronous motor, the current under the poles lagging behind the impressed voltage but being reversed in phase with respect to the induced voltage, evidently tends to strengthen the fields instead of to weaken them.

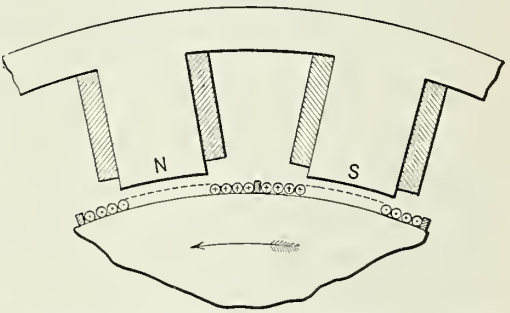


FIG. 351. — Diagram for showing the Effect of Armature Reactions of a Single-phase Generator when the Current in the Armature lags behind the Induced Voltage.

If the current is in advance of the phase of the induced voltage, the armature of a generator tends to strengthen the field

magnetism when the coils are directly under the poles, as may be understood by examining Fig. 352. On the other hand, a motor armature with the current leading the phase of the impressed voltage tends to weaken the fields. This tendency of the armature current, when in advance of the induced voltage, to strengthen the fields could be taken advantage of to make an alternating-current generator completely self-regulating, or even self-exciting, through the action of its armature current. This, however, would require the use of a condenser or other source of capacity reactance attached across the armature terminals to give the proper lead to the current, which would not be practical in application.

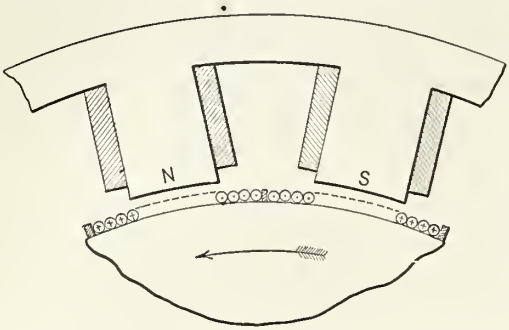


FIG. 352. — Diagram for showing the Effect of Armature Reactions of a Single-phase Generator when the Current in the Armature leads the Induced Voltage.

The quantitative effect of the armature current and the angle of lag on the results produced by armature reactions is not readily determined in single-phase machines. The effect is periodic and depends for its relative instantaneous values upon the instantaneous positions of the armature coils with respect to the poles, and the corresponding instantaneous current values. The latter depend upon the effective value of the current, the angle of lag, and the form of current wave. Doubtless the relative shapes of armature coils and pole pieces also enter the relation. Moreover, the field frames are fairly large masses of iron, and they do not respond rapidly to changes in their magnetic surroundings ; this apparent magnetic inertia being caused by the effect of eddy currents and the considerable inductance of the field circuit, which tend to suppress rapid changes of the magnetic flux. It is therefore reasonable to assume in general that the results of armature reactions are less marked than might be inferred from a scrutiny of the variation of the instantaneous values of armature magneto-motive forces acting in the polar regions of the magnetic circuit.

The instantaneous value of the back turns of each armature coil at any moment depends upon the instantaneous value of the current multiplied by a function of the instantaneous position of the coil. If the current is sinusoidal, then the current at any instant is proportional to $\sin \alpha$, and if the magnetizing effect on the pole pieces caused by the current in the coil is assumed to vary as a sine function of the position of the coil with reference to its position when the induced voltage is zero, then the magnetizing effect per ampere is proportional to $\cos \alpha$. The back turns are then equal to

$$\sqrt{2}\, nI \sin(\alpha - \theta) \cos \alpha$$

at each instant, in which θ is the angle of lag, I is the effective value of the current in the coil, and n is the number of turns in the coil. Expanding this expression reduces it to

$$\frac{nI}{\sqrt{2}}(\sin 2\alpha \cos \theta - \cos 2\alpha \sin \theta - \sin \theta).$$

If θ is zero, this is a periodic expression of twice the fundamental frequency and of zero average value. Averaging the expression from $\alpha = 0$ to $\alpha = \pi$, when θ is not zero, gives the value $-\frac{nI}{\sqrt{2}} \sin \theta$.

The armature reactions in polyphase generators are materially different from those in single-phase generators, but consist essentially of the sum of the effects of the several phases. Thus, referring to Fig. 350, it was seen that when the current of a single-phaser is in phase with the voltage, magnetism is crowded into the trailing pole tips at each time of maximum current, and resumes its initial position when the current falls to zero. In the case of polyphase machines, in which the wire, in effect, covers the entire surface of the armature, the skewing effects which the currents in the armature coils of the several phases of a balanced machine produce on the magnetism of the field magnet come into action successively so as to maintain the total effect uniform when the currents are equal and in phase with the induced voltages of their respective circuits. When the currents lag

or lead, the skewing effect decreases and the back turns become

$$\sqrt{2}\,nI\left\{\sin(\alpha-\theta)\cos\alpha+\sin\left(\alpha-\theta-\frac{2\pi}{m}\right)\cos\left(\alpha-\frac{2\pi}{m}\right)+\cdots\right.$$
$$\left.+\sin\left(\alpha-\theta-\frac{2(m-1)\pi}{m}\right)\cos\left(\alpha-\frac{2(m-1)\pi}{m}\right)\right\},$$

in which m is the number of phases.

This is zero and the back turns disappear when $\theta=0$, since

$$\sin\alpha\cos\alpha+\sin\left(\alpha-\frac{2\pi}{m}\right)\cos\left(\alpha-\frac{2\pi}{m}\right)+\cdots$$
$$+\sin\left(\alpha-\frac{2(m-1)\pi}{m}\right)\cos\left(\alpha-\frac{2(m-1)\pi}{m}\right)=0.$$

Under these circumstances the skewing effect of the armature magneto-motive force is a maximum.

When θ is not zero, the expression becomes

$$\frac{nI}{\sqrt{2}}\left[\sin 2\alpha+\sin 2\left(\alpha-\frac{2\pi}{m}\right)+\cdots+\sin 2\left(\alpha-\frac{2(m-1)\pi}{m}\right)\right]\cos\theta$$
$$-\left[\cos 2\alpha+\cos 2\left(\alpha-\frac{2\pi}{m}\right)+\cdots+\cos 2\left(\alpha-\frac{2(m-1\pi)}{m}\right)\right]\sin\theta$$
$$-m\sin\theta.$$

But

$$\sin 2\alpha+\sin 2\left(\alpha-\frac{2\pi}{m}\right)+\cdots+\sin 2\left(\alpha-\frac{2(m-1)\pi}{m}\right)=0,$$

and

$$\cos 2\alpha+\cos 2\left(\alpha-\frac{2\pi}{m}\right)+\cdots+\cos 2\left(\alpha-\frac{2(m-1)\pi}{m}\right)=0.$$

Therefore the expression for the armature magneto-motive force has the constant value,

$$-m\frac{nI}{\sqrt{2}}\sin\theta.$$

This is obtained on the assumptions that the current is sinusoidal and that each armature coil acts on the main magnetic circuit with an influence proportional to $\cos\alpha$. Neither of these assumptions is exactly met in most machines, and the back turns therefore are more or less irregularly variable through each period of the armature current. In case the cir-

cuit is unbalanced, a pulsation is introduced by the differences between the phases.

The effect of armature ampere-turns in a single-phase machine is illustrated in Fig. 353, coils a, a, connected as in Fig. 350, being indicated. (To simplify the figure, winding slots are not shown.) Curve E is the induced voltage curve of the machine plotted as a function of time beginning when the coils are located neutrally in the magnetic fields as shown in the Figure, that is, at the instant when the induced voltage is zero; and curve c is a current curve assumed to be in phase with the induced voltage. Curve d represents the magnitude of the magnetizing effect which would be produced per ampere by a continuous current flowing in the moving armature winding, which aids or opposes the field magnetization, but differs in value with the instantaneous positions of the coils, and is here plotted with space abscissas corresponding to the time abscissas of curve E.

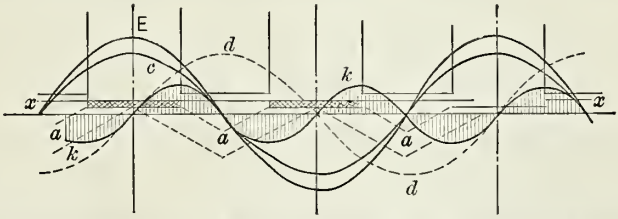


FIG. 353. — Curves showing the Magnetic Activity of the Armature of a Single-phase Generator when the Current is in Phase with the Induced Voltage.

This magnetizing effect or activity with a uniform current flowing in the armature winding must evidently have a maximum value when the centers of the coils are directly under the centers of the poles, *i.e.* when the coils are at the point shown in the figure. When the centers of the coils are ninety electrical degrees from the position shown in Fig. 353, only a skewing effect results from the armature ampere-turns acting on the field magnet, as is illustrated in Fig. 350. Multiplying the ordinates of the current curve with the corresponding ordinates of the curve of magnetic activity, the actual effect of the alternating current in the armature coils may be obtained, as is shown by the shaded areas outlined by curve k. It is seen that the sum of the positive and negative effects is zero, but that they produce the periodical skewing effect caused by the armature

current upon the field magnetism as already explained. In the same manner it is shown in Fig. 354 that if the current lags behind or leads the voltage, alternate loops of the curve k are greater than the others, and the field magnets are periodically either weakened or strengthened. In this case, the effect of the back turns is larger while the effect of the cross turns is smaller.

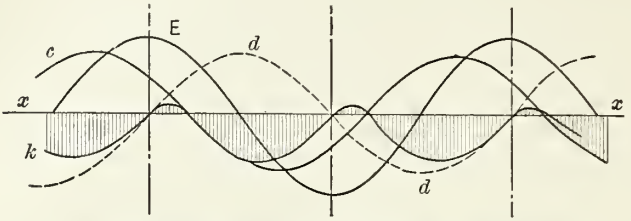


Fig. 354. — Curves showing the Magnetic Activity of the Armature as in Fig. 353, but with the Current out of Phase with the Voltage.

Now, in the case of a balanced quarter-phase alternator, a second curve of reaction exactly similar to k and one-fourth the pitch, or 90 electrical degrees, from k, may be drawn as shown in Fig. 355 to represent the action of the second set of coils when the current is in phase with the induced voltage. The vertical lines crossing Figs. 353, 354, and 355 are located 180 electrical degrees apart, as guides. The magnetizing effect of one phase gradually increases while the magnetizing effect of the other phase decreases, and the skewing effect becomes nearly neutralized when the armature current and voltage in each branch are in phase with each other. Similarly, when the current lags or leads in the two branches of the two-phase circuit, the armature magneto-motive forces add together so as to give a more or less constant weakening or strengthening of the field magnets, as shown in Fig. 355, where the ordinates of the hori-

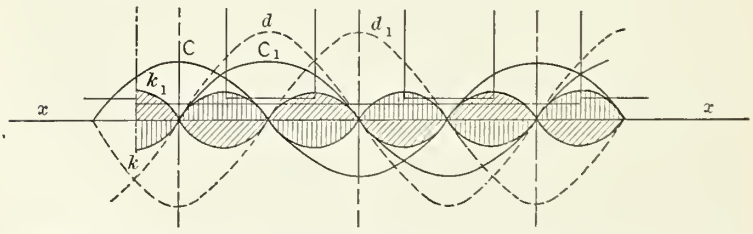


Fig. 355. — Curves showing the Magnetic Activity of the Armature of a Quarter-phase Generator. Currents in Phase with Induced Voltages.

zontal line k'' represent this effect. In Fig. 355 curves c and c_1 represent the currents of the two phases; d and d_1 are the curves of activity of the coils of the two phases respectively; and k and k_1 are the reaction tendencies of the separate phases. The poles are assumed to lie midway between the vertical broken lines as in the previous two figures. In case the current leads, the fields are strengthened, and in case it lags the fields are weakened, as illustrated in Fig. 355 a. By the same process it may be shown that the armature reaction in any polyphase machine is practically constant when the system is balanced.

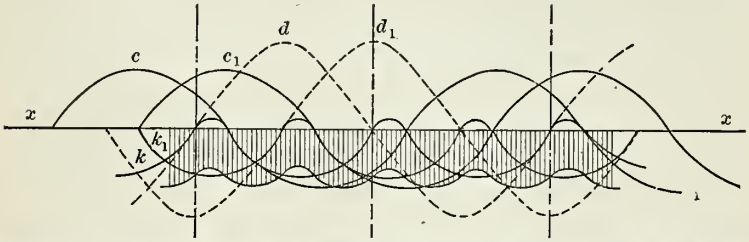


FIG. 355 a. — Curves showing the Magnetic Activity of the Armature as in Fig. 355, but with the Currents out of Phase with the Induced Voltages.

The effect is possibly more simply explained by considering the armature resultant magneto-motive force of a polyphaser as producing a **Rotating magnetic field**, relatively to the armature core, which core may either be stationary or rotate. This is evidently permissible, as the armature windings and currents of a balanced polyphase armature bear the same relations as those in the primary circuit of a rotating field induction motor. The creation of a rotating field is explained fully in the chapter following.* It will suffice to say here that in a polyphase armature the magnetic fluxes of the phases grow successively to their maximum values in a direction opposite to the direction of rotation of the armature. Thus in a tri-phase armature a coil of one phase, then the next phase to it against the direction of rotation, and then the third in the same direction around the armature rise to maximum flux, thus giving an effect as though the flux set up by the armature windings moved 360 electrical degrees with reference to a point on the armature during the period of a cycle. Taking this view, and supposing first that the armature is stationary and the field magnet revolves, it is

* Art. 185.

evident that a rotating magneto-motive force may be considered to be set up by the armature current having a speed equal to, or in synchronism with, that of the mechanically rotating alternator field magnet. This must be so since, when the alternator field magnet moves through 360 electrical degrees of rotation, it creates a complete cycle of currents in the several armature phases and hence the position of the resultant armature magneto-motive force for the coils of each phase also moves 360 electrical degrees during the same period of time. When the power factor is unity, the magneto-motive forces for the several balanced phases substantially neutralize each other. When the armature currents lag, their combined rotating magneto-motive force lags in space with reference to the alternator field magnet, causing a weakening of the field magnet by back magnetization. When the currents lead, the armature magneto-motive force steps forward in its space phase relation with the alternator field magnet, and causes a strengthening of the latter.

Similar reasoning applies when the armature rotates and the field magnet is stationary; since, in that case, the magneto-motive force set up by the armature currents may be considered to stand in a fixed position with respect to the stationary field magnet as the armature revolves, which fixed position depends on the lag of the currents with respect to the induced voltages in the branches of the polyphase windings.

The resultant magneto-motive force always rotates in relatively the same direction about the armature as the relative motion of the alternator field magnet, or against the direction of rotation of the armature, as said before. This may be proved by drawing the curves of phase currents and laying out the space phase positions of the resultant armature magneto-motive force and alternator magnet for a series of angular advances of the armature or field.

The opposing and cross-magnetic effects of lagging armature currents in alternating current generators, when operating under usual conditions, cause the external characteristics* to slope toward the horizontal axis. This effect must be added to the slope of the characteristic caused by resistance and reactance in the armature. When the current leads, the voltage tends to rise due to the strengthening of the field, a character-

* Art. 155.

istic that is made valuable use of in maintaining uniform voltage at the receiving ends of transmission lines through the use of synchronous apparatus which is regulated to furnish a leading current.* The rise, however, may be made excessive by the same effect when the conditions of the line and load are abnormal.

It is evident that the effect of the armature reaction depends for its relative instantaneous values upon the instantaneous positions of the coils with reference to the poles, as already pointed out; and that it depends further for its actual instantaneous and average values on the current strength, the angle of lag, the shape of the current curve, and the conditions of balance in polyphase systems. The relative shapes of armature coils and pole pieces also enter the relation, as do also the air space and tooth reluctances, as well as the total reluctance of the magnetic circuits. Since the effect of the reactions is periodic in the case of single or unbalanced polyphase machines, it is difficult to determine accurately its exact result in any particular case, by any means except that of experiment. By taking the instantaneous value of the single-phase current or unbalanced polyphase current for a number of angular positions of the armature, the average skewing and direct magnetizing or demagnetizing effect of the armature reaction can, however, be approximately determined. The reactions of polyphase machines, however, may be determined with a greater degree of accuracy and ease for any load power factor when the system is balanced, as has been shown by the previously given equations.

153. Alternator Characteristics. — There are four curves which exhibit particularly important relations between the functions of alternators. These curves, which are called characteristics, may be enumerated as follows :

1. The **Saturation Curve** or Curve of Magnetization.
2. The **External Characteristic.**
3. The Curve of **Synchronous Impedance.**
4. The Magnetic Distribution Curve and Voltage Curve.

154. Curve of Magnetization. — The curve of magnetization shows the relation between the ampere-turns in the field windings and the total voltage induced in the armature. From the

* Art. 175.

voltage induced in the armature, the value of Φ may be deduced by means of the formula $E = \frac{2\,KS\Phi f}{10^8}$, provided the value of K is known.* The value of K cannot be determined exactly by calculation, but may be ascertained by means of the fourth curve named in the preceding article. The experimental determination of the curve of magnetization is carried out exactly as in the case of direct-current machines, substituting for the direct-current voltmeter an instrument which is capable of measuring alternating voltages. It is desirable that the instrument used shall indicate the effective value of the voltage; hence, the measurements should be made by a voltmeter built upon the principles of either the hot-wire instrument, the electrostatic instrument modeled after the quadrant electrometer, or the non-inductive form of high resistance electrodynamometer, arranged preferably for direct reading of voltage.† All instruments used in alternating-current measurements which depend for their indications upon electrodynamic action, must be constructed with no masses of conducting metal about them, or their constants will depend upon the frequency of the current measured. This is due to the dynamic effects which eddy currents, circulating in metallic masses, have on the currents in the moving parts of the instruments. If a voltmeter has an appreciable inductance, its reading will also depend upon the frequency, since the current flowing through it is inversely proportional to $\sqrt{R^2 + 4\,\pi^2 f^2 L^2}$. From this it is readily seen that if fL is not negligible in comparison with R, the current flowing through the voltmeter when it is connected to a circuit, and hence its indications, will be dependent upon the frequency. The indications of an inductive voltmeter will always be less when it is connected to an alternating-current circuit than when it is connected to a continuous-current circuit of equal effective voltage. The self-inductance of an electrodynamometer arranged for use as an amperemeter is usually quite small, but in some cases may reach a millihenry. An electrodynamometer which is arranged to be used as a voltmeter, usually has a considerable non-inductive resistance in series with the inductive working coils, so that its time constant is quite small.

* Art. 20. † Art. 24.

If the alternator under examination is a special one and was designed to be partially or entirely a self-exciting one, there is some question of the comparative magnetizing effects of continuous and rectified currents. In general, however, the rectified current in a self-excited field is doubtless always a wavy one. The effective value of this, as indicated by an electrodynamometer, is very nearly the same as the average value indicated by an ordinary amperemeter. The average magnetizing effect of the current is also practically equal to that of a continuous current which gives the same indications on the instruments, though the actual magnetic flux set up is always proportional to the instantaneous value of the current divided by the magnetic circuit reluctance, after due correction has been made for hysteresis and the magnetizing effects of any eddy currents induced in the field magnets by a pulsating excitation. A wavy current tends to set up eddy currents in the iron of the magnetic circuit and thus cause heating, but this result is not marked.

The exciting current of a separately excited alternator, the type used almost exclusively in modern practice, may also be caused to become wavy if the armature reactions are very large. It has already been shown that the effect of the armature reactions of a single-phase or unbalanced polyphase machine is a periodic one ; and when the periodic effect becomes of sufficient magnitude, it causes fluctuations in the field magnetism, which react upon the windings and throw the exciting current into pulsations. On account of this tendency of the exciting current to become wavy and of the magnetic flux to vary locally in the pole pieces, it is common to build the pole cores of laminations in order to prevent the generation of excessive eddy currents in them.

The general form of the curve of magnetization for an alternator is similar to the form of the curve for a direct-current dynamo. As it is not uncommon for alternators to have a somewhat larger reluctance in the air space than have direct-current dynamos of the same size, the knee in the alternator curve is sometimes not so abrupt as it is in the case of direct-current machines (Figs. 356 and 357). For studying the details of the design of the magnetic circuit, the curve may be resolved into component curves representing the air space, frame, and armature. In this case the excitation compared to

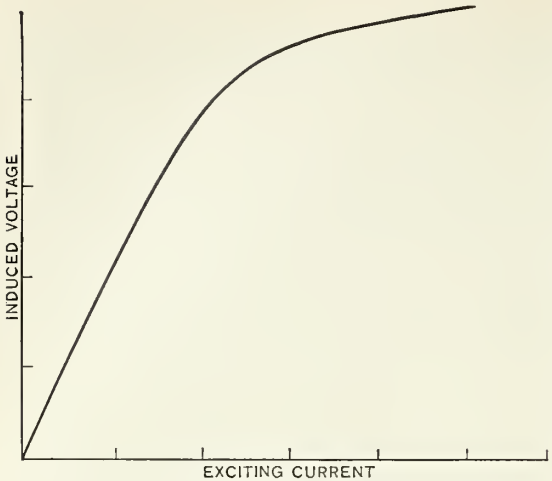


FIG. 356. — Type of Curve of Magnetization of an Alternator.

the magnetic flux is a straight line for the air space, since the reluctance of that part of the magnetic circuit is constant. Figure 357 shows the relation between exciting current and terminal voltage in a particular alternator when current flows in the armature.

155. External Characteristic. — An external characteristic is a curve showing the relations between the armature current and the terminal voltage which exist for a stated condition of field excitation. To experimentally determine an external characteristic of an alternator, it is excited by the method for which it is designed, so as to give its normal voltage on open circuit. The volts at its terminals, and the current and kilowatts in the external circuit, are measured with various loads in the external circuit of various magnitudes but fixed power factor. The observations may be plotted in a curve,

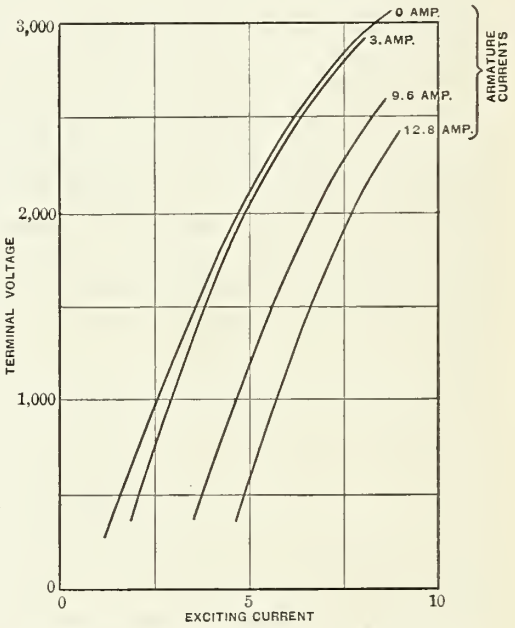


FIG. 357. — Relation of Terminal Voltage to Exciting Current with Different Currents in the Armature.

using volts as ordinates and kilovolt-amperes or kilowatts as abscissas. In separately excited alternators, the curve cuts the vertical axis at its highest point at no load, and then gradually falls, the exciting current being held constant. The decrease of the ordinates (drop in voltage) is caused by the effects of armature resistance, armature self-inductance, and armature reactions.

The magnitude of the armature reactions is changed on account of the lag or lead of the current, when there is reactance in the external circuit. If it is desired to quantitively determine the effect of lag or lead on armature reactions, curves may be taken with different values of reactance inserted in the external circuit. The portion of the drop which is caused by self-inductance and armature reactions when the machine is supplying current to a load of unity power factor may be separated from that caused by resistance by the formula, $E_i^2 = E_a^2 + E_s^2$. In this case E_i is the open-circuit voltage, E_a is equal to the terminal voltage for the current flowing plus IR_a where R_a is the armature resistance, and E_s is the quadrature component of the drop which results from self-inductance and armature reactions. By taking the characteristics of an alternator when worked on circuits of different known resistances and reactances, the effect of armature reactions may be determined for different values of the angle of lag, and such tests are quite valuable for determining the purpose for which the machine is best adapted. The self-inductance and reactions of some of the early alternators were so great that the external characteristic drooped to the X axis at a current not greatly exceeding full load. Figure 358 shows the characteristics of two alternators plotted to show the relations of terminal volts to armature current when the load is non-reactive. One (Curve A) is of a standard type of alternator, having small armature self-inductance and reactions, and the second (Curve B) of a special machine having large armature self-inductance and reactions. Curves showing the relation of output and armature current for the two machines are also shown. These are especially enlightening as showing relatively how much lower the output of the highly inductive machine is for the same proportions of full load current. The two curves of machine B illustrate the difficulty of obtaining good regulation or the economical utilization of the copper and iron in the machine when the machine

is highly reactive. As inductive reactance in the load has the same effect upon these characteristics of operation as has reactance in the armature itself, it is evident from the figure that inductive reactance should be avoided in both the machine itself

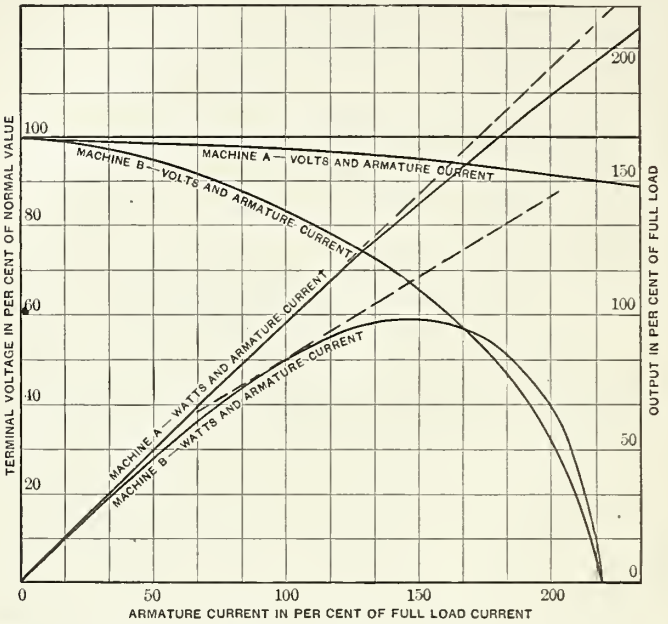


FIG. 358. — External Characteristics. Machines worked on Non-reactive Loads. Machine *A* with Small and Machine *B* with Large Armature Reactance. The Relations of Output to Current are also shown.

and its load so far as possible. The three following figures (Figs. 359, 360, and 361) graphically indicate the effect of an inductive load. Figure 359 shows the relation between voltage drop and load power factor upon a generator of normal design connected to inductive loads of different power factors. In this case, the load was at the end of a transmission line. The drops in the step-down transformers, line, and generator, and their total are given for power factors ranging from 100 per cent to 20 per cent, inductive, when the full load rating of 250 kilovolt-amperes are being delivered to the transmission line. Figure 360 shows the reduction in the kilowatts of capacity available in this particular generator when used on the same range of power factors and with a kilovolt-ampere output for the various

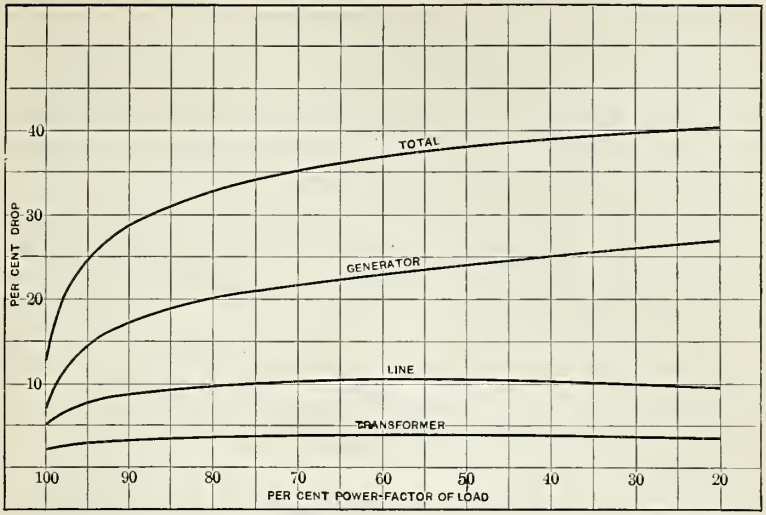


FIG. 359. — Curves showing the Effects of the Power Factor of Inductive Loads on the Regulation of an Alternator.

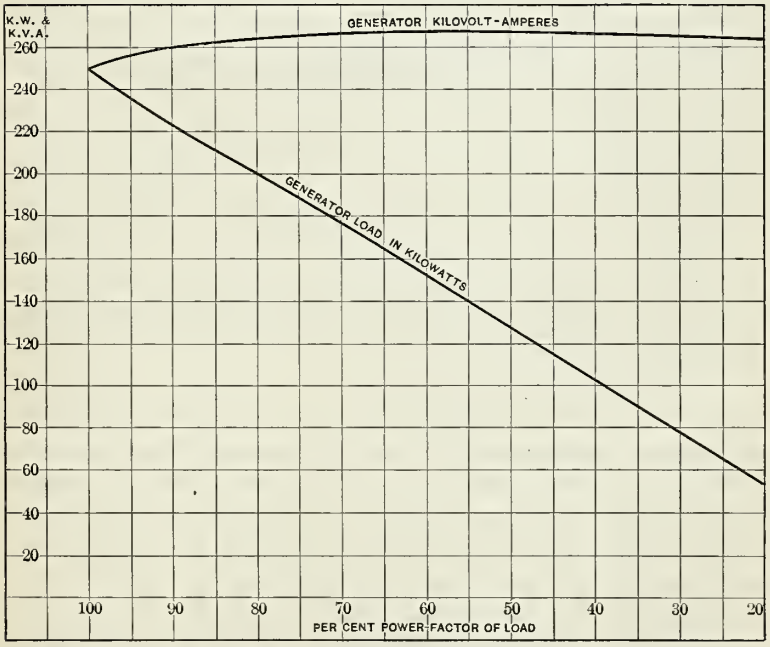


FIG. 360. — Curves showing the Relation of Inductive Power Factor to Kilowatts and Kilovolt-amperes for the Alternator referred to in Fig. 359.

power factors as shown by the upper curve. The lessening of the capacity is due to the fact that the output of the generator is limited by the heating of the armature, and therefore when the current contains a large quadrature component, the active component must be relatively low. Figure 361 shows the field exciting currents required by the alternator to supply normal voltage to the transmission line at the same range of power factors, when the alternator is operated so as to furnish 250 kilovolt-amperes, and it also shows the corresponding armature core losses.

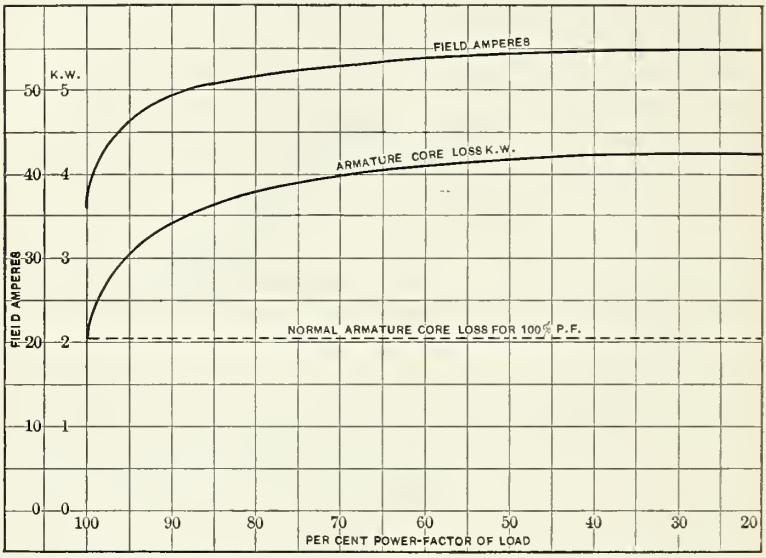


FIG. 361. — Curves showing the Variation of Field Exciting Current and Armature Core Losses with the Power Factor of an Inductive Load for the Alternator referred to in Figs. 359 and 360.

When the load has the effect of electrostatic capacity the quadrature leading current tends to strengthen the alternator field magnet and neutralize the effect of armature self-inductance. It may, therefore, reduce or neutralize the drop in the external characteristic, or cause it to rise, depending upon the amount of the quadrature current flowing.

The external characteristic of a special, self-excited, shunt-wound alternator is shown in Fig. 362. The droop in the curve is a little greater than would occur for the same machine separately

excited. This is due to the loss of exciting current as the armature drop increases, and the terminal voltage decreases accordingly.

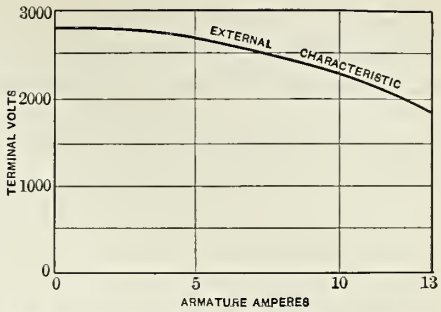


FIG. 362. — External Characteristic of a Special Self-excited Alternator.

The effect of armature resistance and true self-inductance can be shown by drawing the loci of the alternator terminal voltage — maintained constant by proper variation of the field excitation — when the power factor of the load is varied, but the load current is maintained constant. The construction is shown in Fig. 363.

In this figure the circular arc $EE_1E_2E_3E_4$ with the center at O is the locus of constant terminal voltage when the constant current OI flows but the power factor is varied. The triangle OXR represents the voltage drops in the impedance of the armature by reason of the flow of current OI. Then the total voltage generated, $\bar{E}'$, must be equal to the vector sum of OX and the terminal voltage $\bar{E}$. By geometrical construction it is seen that the locus of $\bar{E}'(= OE'$, etc.) for various angles of lag θ in the external circuit must be the circular arc $E'E_1'E_2'E_3'E_4'$ with its center at X. In the figure, when OI lags by the angle θ with respect to the terminal voltage, the latter is represented by OE, the total voltage generated by OE', and the impedance drop by EE', of which Ea is used in overcoming self-inductance and aE' in overcoming resistance. The figure shows the vector diagrams for lagging angles θ and θ_1, zero lag θ_2, and leading angles θ_3 and θ_4, where the thetas represent the phase differences of the current and voltage in the external circuit. It will be noted, in order that constant current may flow with varying power factors, the load must be assumed to change in its components of resistance and self-inductance or capacity in such a way as to keep the impedance Z constant. The line RX in the figure is assumed to be constant, which is not absolutely true, as the reluctance of the self-induction paths around the armature conductors will change with varying values of theta. In this figure the values of total voltage in-

duced, OE', OE_1', etc., are due to cutting the total flux in the air gap, which in turn is created by the vector sum of the field ampere-turns and effective armature ampere-turns. The latter

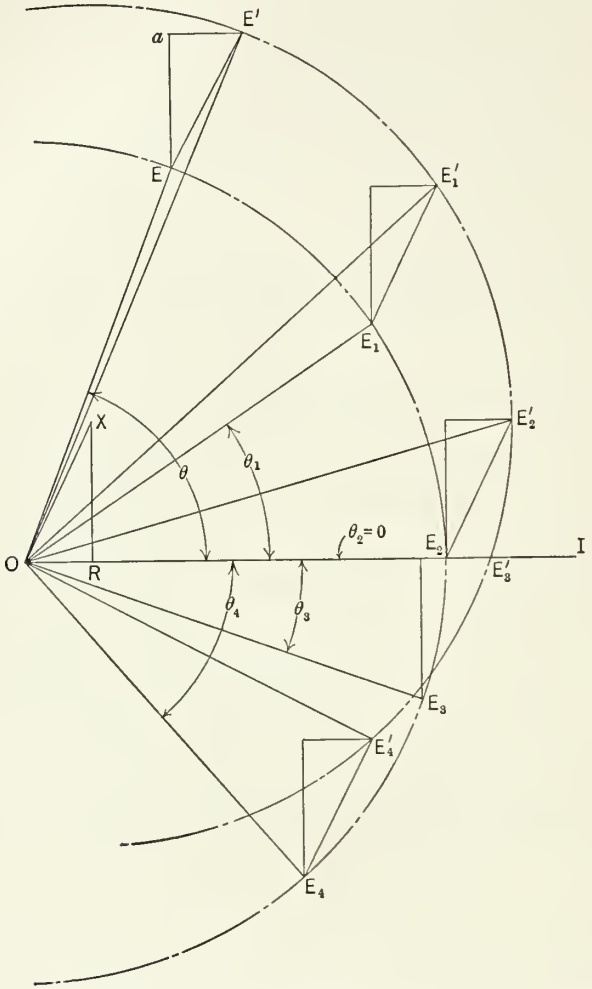


FIG. 363. — Loci of Terminal and Total Voltages generated in an Alternator when the Power Factor of the Load is varied but the Numerical Values of Terminal Voltage and Current are maintained constant.

vary with the positions of the centers of the armature coils at the instant they are carrying maximum current, as shown in an earlier article, or approximately with $\sin \theta$.* This shows that

* Art. 152.

when θ is positive the field ampere-turns must not only create sufficient flux to create the total voltage required, but must also neutralize the effective armature-turns, but that when θ is negative the field ampere-turns directly add to the effective armature ampere-turns to set up the required flux. A diagram which expresses graphically the effect of armature-turns upon the voltage generated is shown in Fig. 364. Triangle OFC represents the voltage triangle of the external circuit, when the generator terminal voltage is OC, the angle of lag θ, and the current OI. Triangle CDA represents the internal drops in the machine due to self-inductance and resistance, the total drop being CA. OA is the voltage that would be required to drive the current OI through an impedance equal to $\frac{OA}{I}$. This voltage, however, is less than would be generated by the same field excitation on open circuit by reason of the demagnetization caused by the current flowing in the armature coils. Representing the loss of voltage from this source by AB, then the open circuit voltage is represented by OB. Or, a field excitation which would create an armature voltage OB is required to furnish a terminal voltage OC when current OI flows at an angle of lag θ behind OC. The triangle OAB may then be considered proportional to the ampere-turns of the machine, in which OB is proportional to the useful field ampere-turns, AB the armature ampere-turns, and OA the net ampere-turns effective in producing useful flux in the armature. It will be noted that the total field ampere-turns must include the ampere-turns

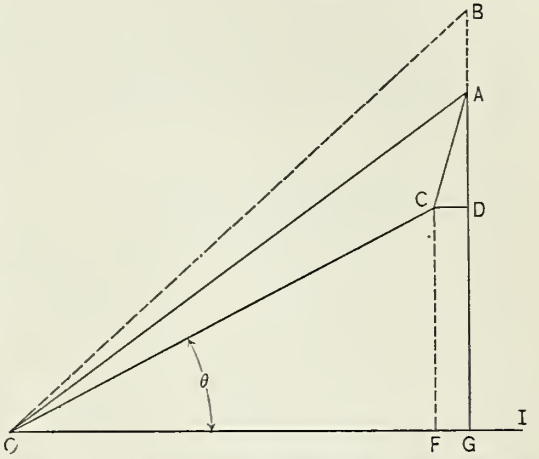


FIG. 364.—Vector Diagram showing the Effect of Armature Magnetizing Ampere-turns upon the Voltage generated in an Alternator. Armature Current lagging behind Terminal Voltage.

which set up leakage flux ; and these vary, since the leakage increases with a positive lag of the current and decreases with a negative lag. Since OB does not include ampere-turns necessary to set up leakage flux, the figure is only approximate. Figure 365 is a diagram showing the effect of armature magnetization when the current in the external circuit leads the terminal voltage.

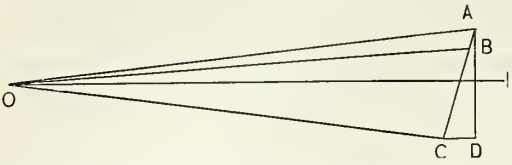


Fig. 365. — Diagram showing the Effect of Armature Magnetizing Ampere-turns upon the Voltage Generated in an Alteruator. Armature Current leading the Terminal Voltage.

156. The Short-Circuit Current Curve and Synchronous Impedance of an Alternator. — From the discussion preceding it is evident that there is difficulty in separating the effects of true self-inductance and armature reactions in an alternator, but the combined effect of the two in connection with the armature resistance may be studied to advantage by the use of the **Short-circuit current curve.** This curve shows the relation between the ampere-turns of the field magnet and the armature current flowing when the armature terminals are short-circuited. The curve may be obtained by placing an amperemeter in the field circuit and short-circuiting the armature directly across its own terminals except that a very low resistance amperemeter should be placed in at least one of the leads. The field magnet may be then excited by successively increasing values of exciting current and the readings of the amperemeters observed and plotted, this process being continued until the maximum safe value of the armature current is reached. The machine should be driven at normal speed. As in the case of the saturation curve, the curves of rising and falling field excitation will differ on account of the effect of hysteresis in the field magnet. In order to obtain all the information desirable, a saturation curve for the same excitations should also be obtained and should be plotted on the chart with the short-circuit current curve. Figure 366 shows a short-circuit curve of an 1800-kilowatt, tri-phase machine, marked A. Curve B is the saturation curve of the same machine, and the points R and S are the points for full load rated voltage and current respectively.

From these two curves an imaginary impedance may be obtained corresponding with each armature current, which is approximately equal to an actual impedance calling for the same loss of voltage as is jointly caused by the armature resistance, armature self-inductance, and armature reactions. This is called the **Synchronous impedance**, and is sometimes defined as the numerical ratio of the open-circuit voltage generated at normal speed for a given field excitation to the short-circuit current at the same speed and excitation. The term synchronous impedance, however, can be better considered as designating the

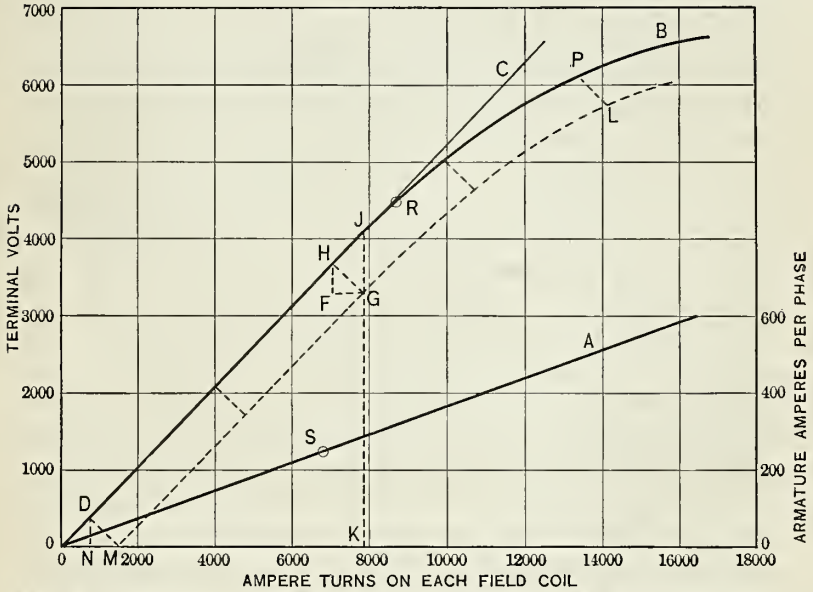


FIG. 366. — Short-circuit Current Curve of an Alternator. *A*, Short-circuit Curve; *B*, Magnetization Curve.

actual vector loss of voltage, due to armature resistance, self-induction, and reactions, divided by the armature current flowing. Its value, therefore, changes with the angle of lag of the armature current on account of the resultant change in effective armature magnetizing turns.

A curve which is useful in determining the excitation required to give a desired terminal voltage or *vice versa*, when a given current flows at a given angle of lag, is marked *MGL* in Fig. 366 and is sometimes called the wattless current characteristic. The curve is constructed in this way: assume some

armature current, say 50 amperes, and lay off the distance OM equal to the number of ampere-turns per pole that are shown by the short-circuit curve to be necessary to cause a short-circuit current of the chosen value to flow through the armature circuit. Then lay off MN, which is the back-magnetizing turns per pole caused by the armature current of 50 amperes when flowing at an inductive lag angle of 90° and is computed for a polyphase machine by the formula on page 616 and for a single-phase machine by the formula on page 615. To do this the armature resistance must be considered negligible, which for ordinary machines is admissible. Then ON represents the ampere-turns per pole in the field windings which are effective in setting up useful flux, and DN (which is the induced voltage corresponding to field ampere-turns ON) represents the inductive drop in the armature. The field reaction of a constant quadrature-lagging current in the armature, and the self-inductance set up thereby, are constant whatever the terminal voltage may be, provided the effect of saturation may be neglected, and therefore if the machine is loaded with pure inductive reactance and the field current modified so as to keep the armature current constant, the relation between terminal voltage and field current will be expressed by curve MGL of Fig. 366, the offsets HG, PL, etc., being all parallel and equal to DM. Then, if the reactive load is such that the terminal voltage is GK and the ampere-turns per field spool are OK, G will be one point on the wattless current characteristic and GH must equal and be parallel to DM. The curve MGL is evidently constructed by drawing it through the extremities of lines drawn from curve B equal and parallel to DM. For greater currents MGL is further removed from B.

The line OC represents approximately the ampere-turns required by the air space.

157. The Magnetic Flux Distribution Curve of an Alternator, and Curves of Voltage and Current. — These curves may be experimentally determined by various methods. They consist of a series of curves which are closely interrelated, but may be, and in fact are likely to be, of quite dissimilar forms. The form of the curve representing the wave of impressed voltage, or total voltage induced in the armature of an alternator, is dependent upon the distribution of the magnetic flux over the pole faces, and also

on the arrangement of the armature windings. By design, either of these may be given a controlling influence to the exclusion of the other. A two-pole machine with closed-circuit, evenly distributed, Gramme ring armature winding, as shown in Fig. 30, gives an excellent illustration of this, when it has two connections 180 degrees apart to two collector rings, thus connecting the two halves of the winding in parallel between the rings. The differential action, which occurs in the coils of such an armature, makes its curves of voltage almost independent of the distribution of the magnetism over the pole faces, provided the same total number of lines of force is cut by each conductor per revolution; and the maximum value of the voltage is entirely independent of the magnetic flux distribution. This is shown by Fig. 367, where the full line in the left-hand drawing shows the curve of voltage developed in such

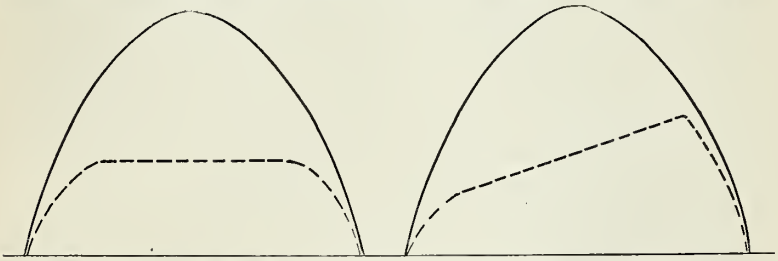


FIG. 367. — Voltage Curves produced by a Two-pole, Gramme Ring Armature Winding having Two Coils in Parallel and each Coil covering 180 Electrical Degrees, when subjected to Magnetic Fluxes of Differing Distribution. Full Lines represent Voltages and Dotted Lines Magnetic Distribution under the Pole Faces.

an armature winding with a uniform distribution of the magnetic flux of the field, and the full line in the other drawing shows the curve of voltage with the same total field greatly distorted. The dotted lines in the drawing indicate the distributions of the magnetism over the pole faces. The unidirectional voltage delivered by the armature when the windings are connected to a commutator and the machine is operated as a direct-current dynamo, is not affected by the distortion, provided the brushes are always placed on the neutral plane and the total magnetism passing through the armature remains constant. When the machine has been converted into an alternator by the addition of collector rings as stated, the maximum instantaneous voltage is equal to the voltage developed in the direct-current machine, and is therefore independ-

ent of the distribution of the magnetism,* and the form of the curve representing the wave of voltage is but slightly altered, as shown in Fig. 367. Now suppose the same machine to be arranged with a single narrow coil on the armature. The change in the magnetic distribution now not only changes the form of the voltage curve proportionally, but also changes in a marked manner the maximum value of the instantaneous voltage. The difference between the curves of voltage developed by the broad coil and narrow coil armatures is due to the effect of differential action in the broad coil armature.

It has already been shown that differential action occurs to some degree in commercial alternators, but it does not occur to a sufficient degree to make the form of the voltage wave independent of the magnetic distribution. It is therefore true that the magnetic distribution largely influences the form of the voltage wave, and the magnetic distribution should therefore be carefully studied during the development of a type of alternators. A proper study of the magnetic flux distribution and of the arrangement of the armature windings makes it possible to so design an alternator that it will produce any desired form of voltage wave.

The angular relation between the curves representing the magnetic flux distribution and the impressed voltage is interesting. The ordinate of the curve of voltage at any instant is proportional to the rate at which lines of flux are cut by the armature conductors at that instant, the rate being taken algebraically. Consequently, the voltage is zero when the magnetization is all symmetrically threaded through the coils; that is, when the algebraic net rate of cutting flux by the conductors is zero. The voltage is a maximum when the rate of cutting flux is the greatest; that is, when the algebraic summation of the number of flux lines threaded through the coils is a minimum. A curve which represents the algebraic summation of the number of flux lines threaded through the coils at each instant, therefore, has an angular position which is 90° from that of the voltage curve (Fig. 368). The form and dimensions of this curve of magnetic linkages evidently depend upon the actual distribution of the magnetic flux and the arrangement of the armature windings. When the voltage curve is irregular, and the angular

* Art. 20.

relation is not made evident from the curve, as in Fig. 369, the point at which the curve of magnetic linkages cuts the X-axis may yet be easily found, since it is directly under the center of gravity of the voltage curve. That is, since as many lines of flux must be withdrawn from the coils as are inserted, for each loop of the curve, the summation of the ordinates of the voltage curve on each side of the crossing must be equal.

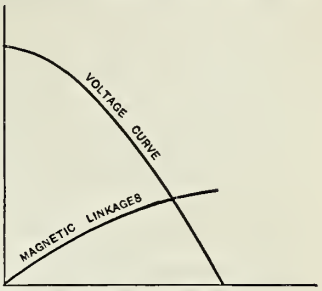


FIG. 368. — Curves showing Relation of Magnetic Flux Linkages and Induced Voltage in an Alternator.

This curve, which shows the algebraic number of magnetic lines of flux which are threaded at each instant through the coils, may be deduced from a known curve of voltage. Erect an ordinate to the voltage curve which bisects the area as in Fig. 369; then by means of ordinates divide the half areas into a number of small areas. The magnetism threaded through the armature coils is algebraically equal to zero at the instant represented by the bisecting ordinate, and the algebraic value of the magnetic leakages through the armature coils at any other instant is proportional to the area inclosed by the voltage curve between the corresponding ordinate and the bisecting ordinate, since $e = \frac{d\phi}{dt}$ and $\phi = \Sigma e dt$, where e is instantaneous voltage, ϕ magnetic flux linkages, and t time. Therefore, the ordinates of the curve representing the magnetic linkages through the coils are, at the instants represented by the ordinates which divide the

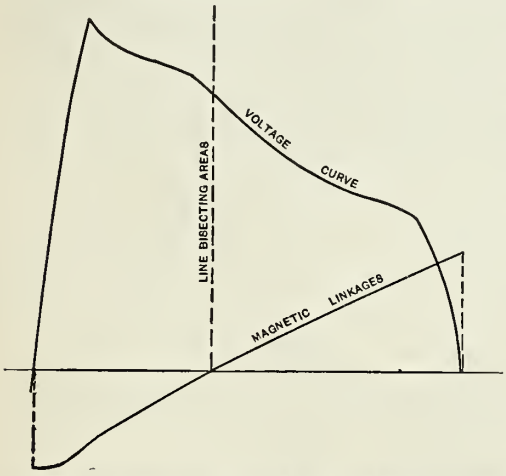


FIG. 369. — Alternator Voltage and Magnetic Linkage Curves. Area of Voltage Curve is bisected.

voltage curve into small areas, proportional to the area between the corresponding instantaneous voltage ordinate and the bisecting ordinate. The full process is, therefore, as follows: lay off on each ordinate a length from the X-axis proportional to the area inclosed by the voltage curve between the respective ordinate and the bisecting ordinate. The points thus found are points on the desired curve. It is evident that the maximum ordinate of the curve comes at the instant when E is equal to zero. The length of this ordinate is equal to Φ, the maximum value of the magnetic flux linkages, and the scale of the curve may thus be conveniently fixed. The loops may not be symmetrical, but, with a fixed value of Φ and a fixed armature winding, the successive loops must always be exactly alike, though they may be looked upon as alternately positive and negative, since the magnetic flux is alternately threaded through the coils in opposite directions.

The corresponding curves of voltage and magnetic linkages for various forms of voltage curves are shown in the accompanying figures. In Fig. 369 the voltage curve is one experimentally determined from a special alternator when working on a series load of arc lights.* In Fig. 370 the voltage curve is an

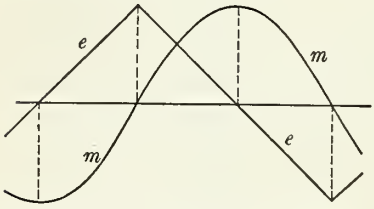


FIG. 370. — Curve e is a Triangular Voltage Curve; Curve m, Magnetic Flux Linkages.

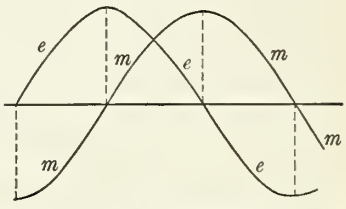


FIG. 371. — Curve e, Sinusoidal Voltage Curve; Curve m, Magnetic Flux Linkages.

isosceles triangle, in Fig. 371 it is a sinusoid, and in Fig. 372 it is a rectangle. The voltage curves of Figs. 373 and 374 are respectively a flat-topped curve and a parabola.†

Since the induced voltage is proportional to the rate of change of the number of lines of flux threaded through the

* Tobey and Walbridge, Stanley Alternate-Current Arc Dynamo, *Trans. Amer. Inst. E. E.*, Vol. 7, p. 367.

† Emery, Alternating Current Curves, *Trans. Amer. Inst. E. E.*, Vol. 12, p. 433.

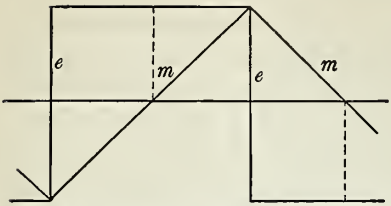


FIG. 372. — Curve e, Rectangular Voltage Curve; Curve m, Magnetic Flux Linkages.

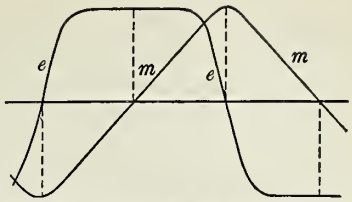


FIG. 373. — Curve e, Flat-topped Voltage Curve; Curve m, Magnetic Flux Linkages.

armature coils, the ordinates of the voltage curve are proportional to the tangents of the curve representing the number of magnetic linkages through the armature coils. Figure 375 shows a graphical construction for determining the voltage curve from this magnetic flux linkage curve. Ordinates Oa', Ob', and OA are, by construction, made proportional in length to the tangents of the angles made with the X-axis by lines tangent to the magnetic curve at p_1, p_2, and O. The point O' is a point taken at any convenient position, and the lines $O'a'$, $O'b'$, $O'A$ are drawn parallel respectively to aa, bb, and the tangent to the

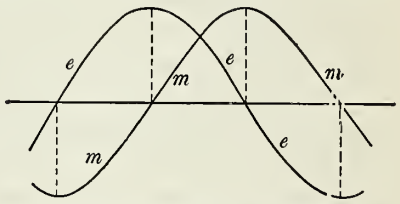


FIG. 374.—Curve e, Parabolic Voltage Curve; Curve m, Magnetic Flux Linkages.

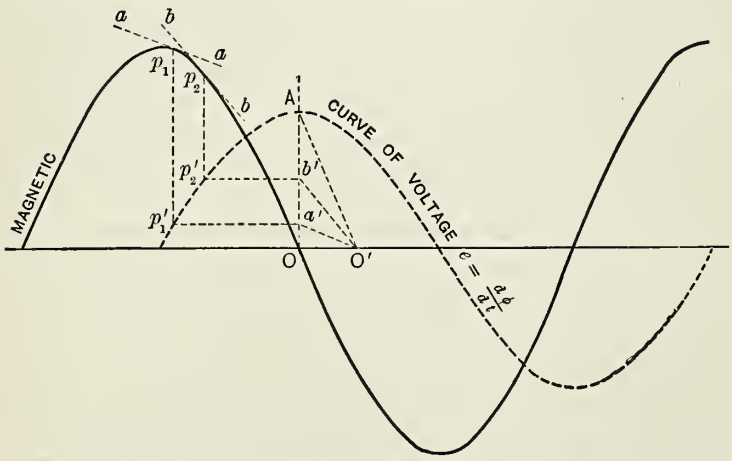


FIG. 375. — Construction showing Method of Determining Voltage Curve from the Magnetic Flux Linkage Curve.

magnetic curve at O. The points of intersection of horizontal lines drawn from a', b', etc., and vertical lines drawn from p_1, p_2, etc., are points p_1', p_2', etc., which are located on the required voltage curve.

Another directly useful magnetic flux distribution curve is one showing the distribution of the lines of flux over the pole faces. This curve is analogous to the magnetic flux distribution curve of direct-current machines. It may be experimentally determined by using a narrow test coil connected to a ballistic galvanometer and fastened to the armature surface. By rotating the field magnet or armature by equal small increments through 360 electrical degrees, the galvanometer readings plotted in the form of a curve will have ordinates proportional to those of the curve of magnetic flux distribution. If the number of turns of the test coil and the constants of the galvanometer are known, the number of flux lines cut by the coil are at once determinable. When this method is used, it is convenient to wrap the test coil across the face of the armature and around the core like a ring winding when the armature construction permits this arrangement. The number of turns needed in the test coil depends upon the density of the magnetic flux and the constants of the galvanometer. It can readily be determined by trial. Another method is to make a narrow test coil of pancake shape as long as the armature coils. This is laid upon the armature surface. The field current is then reversed and the galvanometer throw read. The field magnet or armature is then rotated a distance equal to the width of the coil and another reading taken, and so on through 360 electrical degrees. The sides of the coil in this case should be very narrow so that all of the turns will inclose essentially all of the magnetic flux passing within the boundary of the coil.

If the average distribution of magnetic flux over the pole faces of a machine is known, it is evidently possible to approximately determine the form of the voltage curve which will be produced by any particular arrangement of the windings. It is also equally possible to determine the arrangement of the winding required to give any desired form of voltage curve. Again, if a particular form of winding is desirable, the magnetic flux distribution which is necessary to give a desired voltage curve may be determined. This distribution may then be used as a

guide in designing the width and shape of the pole faces. The application of the magnetic flux distribution curve is illustrated in Fig. 376. The dimensions and form of the pole pieces and of an armature coil belonging to an alternator are indicated in the figure. The ordinates of the line $ABCD$ represent the magnetic density in the air space. When the coil is in the instantaneous position represented, the value of the voltage is zero. As the coil moves, each conductor cuts lines of flux. Suppose that in a fraction such as a twelfth of a period the coil has moved from the position indicated by the letters x, y, to x', y'. During this motion each conductor has cut a certain number of lines of force, and the number cut by all the conductors is approximately proportional to the sum of the areas of the curve $ABCD$ taken from x to x' and from y to y'. The shorter the step taken, the more accurate this becomes. The average voltage developed during this interval is also proportional to the same area. Consequently an ordinate which is numerically equal to the area may be erected at the point corresponding to the intermediate position of the coil during the increment motion, to approximately represent the voltage at that point in the revolution of the armature. This proceeding may be repeated step by step through the half period, taking the algebraic summation of the voltage developed in the two halves of the coil, and the outline of the voltage curve which corresponds to the particular arrangement of armature conductors and flux distribution is thus determined.

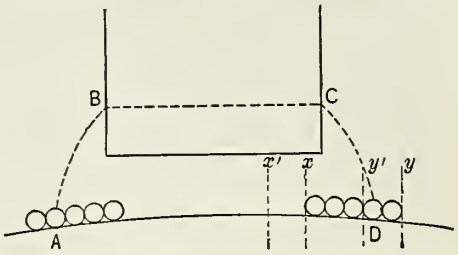


FIG. 376. — Diagram indicating Use of Magnetic Flux Distribution Curve for determining Form of Voltage Curve.

The curve representing the current wave also represents, when taken to the proper scale, the curve of active voltage. From preceding pages it is evident that the curves of active and impressed voltages have the same forms and are superposed, if the circuit in which they act is non-reactive. They have the same form, also, when the circuit is reactive if the impressed voltage is sinusoidal, provided the inductance or capacity

causing the reactance is independent of the instantaneous values of the voltage and the current and the armature reactions are approximately uniform. The condition of a uniform inductance can only hold when no iron is inclosed in any portion of the circuit. In general this condition is not found in commercial service. Even with a uniform inductance the curves of impressed and active voltages will not coincide in phase, since phase coincidence between them can occur only when the circuit is without either inductance or capacity, or these exactly neutralize each other. As the latter is also a condition not often found in commercial service, it may be said that, in general, curves of impressed and active voltages are neither similar in form nor coincident in phase; but they are always, perforce, of the same frequency, though usually composed of more than one harmonic.

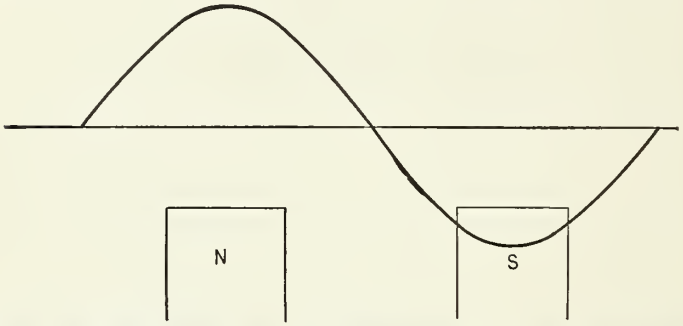


Fig. 377. — Diagram arranged to show the Angular Space Position of Alternator Pole Cores with the Angular Time Value of Voltages or Currents.

The curves are usually plotted to rectangular coördinates; magnetic densities, magnetic linkages, instantaneous voltages, or instantaneous currents being plotted as ordinates, and electrical degrees or time as abscissas. To more definitely locate the phases it is not unusual to indicate the position of the pole pieces by laying them off at the top or the bottom of the plot (Fig. 377), a method used heretofore in this text.

To make a complete study of the magnetic flux distribution in an alternator, the machine should be worked at various loads under different conditions of current lag.* Since, at any instant, the effect of the armature current on the magnetization

* Art. 155.

of the pole pieces depends upon the positions of the armature conductors as well as the strength of the current, the effect of the magneto-motive force of the armature is evidently a variable, and consequently the distribution of the magnetic flux will not be constant in a single-phase machine. In other words, both the cross and back magneto-motive force of the armature winding for any load vary continuously during each period, and therefore the magnetic flux distribution varies with the position of the armature. The variation due to this cause is probably not very great in machines with multitooth armatures, and an average distribution may be assumed as satisfactorily representing working conditions. In polyphase machines where the system is balanced, the magneto-motive forces of the windings of the several phases of the armature combine to form an approximately constant reaction; unbalancing, however, destroys the uniformity of the resultant and the reactions vary in value, depending upon the amount of unbalancing.* In machines with only a few teeth, the effect of armature reactions and the movement of the teeth across the pole faces is often sufficient to cause regular pulsations in the field flux and extremely marked distortions in the voltage curve. The pulsations are sometimes sufficient to materially affect the exciting current. Figure 378, Curve II, shows an experimentally determined curve of the field current of a single-phase alternator having a very deeply toothed armature core with one tooth per coil and a large self-inductance in the armature. The machine was excited by a

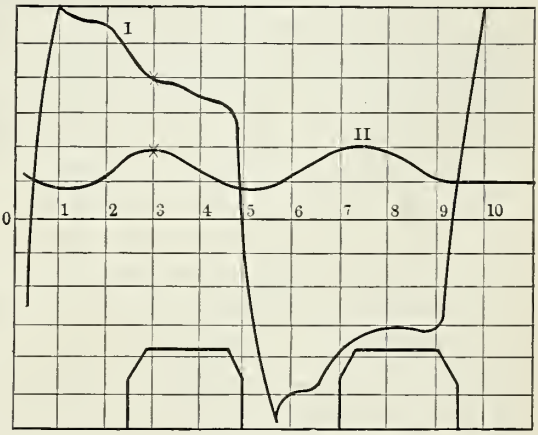


FIG. 378. — Curve illustrating the Variation of Field Current caused by Poorly Designed Armature Teeth. Curve I is the Irregular Voltage Wave, and Curve II shows the Pulsations in the Field Current due to the Reactions of the Teeth and the Armature Current.

* Art. 152.

small shunt-wound exciter.* The very irregular wave of voltage is shown by curve I.

Various methods may be used for experimentally determining the form of electric voltage or current curves.

1. *The Oscillograph.* — Much the most efficient method of tracing alternating current and voltage curves is by means of the Oscillograph, as originally worked out by Blondel and Duddell, but now become one of the important measuring instruments for alternating current phenomena. This instrument

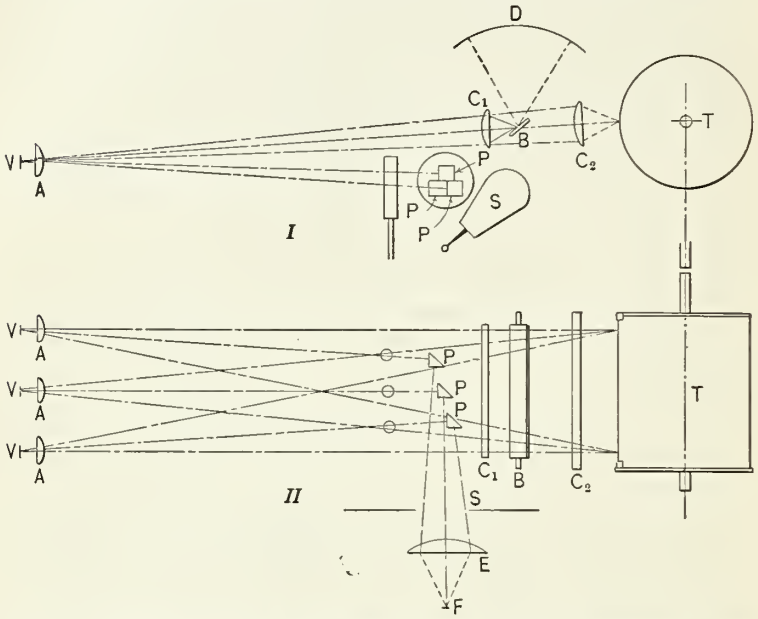


Fig. 379. — Diagrams showing the Essential Features of a Three-element Oscillograph. I, View from the Side. II, View from Above.

in essential parts consists of a loop of wire ribbon placed in a strong field of constant magnetic flux with the plane of the loop parallel to the lines of force of the field. Figure 379 is a diagram of the optical system of an oscillograph having three independent loops placed at *V*, *V*, and *V*. Sketch I is a side view and II a view looking from above. The curve desired may either be made visible upon the screen *D* or be photographed on a sensitized film wrapped upon the drum *T*. First considering the visible record: an arc light at *F* throws a beam of light

* See *Trans. Amer. Inst. E. E.*, Vol. 7, p. 374.

through the condensing lens E to the prisms P, P, P, where it is deflected in three beams and passes to mirrors fastened to the loops at V, V, V, each mirror being fastened at one edge to one strand of a loop and at the other edge to the other strand of the same loop. The loops are so designed that the two wires of the sides have a very rapid natural rate of vibration — sometimes as much as 10,000 vibrations per second, though more frequently about one half that number. The beams of light thrown upon the mirrors V, V, V pass through a long cylindrical lens at C_1 and fall upon a long mirror shown at B which oscillates on a horizontal axis. At the focus of C is placed a semiopaque screen D. When an alternating current, the curve of which is to be determined is passed through a loop, the magnetic reactions between the current in the two sides of the loop and the constant field flux tend to turn the mirror at V to the left or right, depending upon the instantaneous direction of the current. Since the natural rate of vibration of the loop is much higher than that of the current or its harmonics, the curve of which is to be traced, the mirror changes its angular position substantially proportionately to the instantaneous values of the current.

The beams of light from V, V, V move over the length of mirror B. Mirror B is oscillated at one half synchronous speed compared with the frequency of the current through the loop wires. The beams of light are cast by B upon the semiopaque screen at D. Each is then subjected to the resultant of the two mirror movements, *i.e.* a movement by B which is uniform and is proportional to the angular advance of the voltage of the circuit, and a movement at right angles thereto caused by the mirror at V, V, V, which is at each instant proportional to the strength of current in the loop wires. The beams of light thus trace upon the screen curves in which the abscissas caused by B are always proportional to the angular advance during a cycle of the impressed voltage and in which the ordinates are proportional at each instant to the current strength in the loop wires. The curves are, therefore, of the same character as those obtained by the ordinary method of plotting alternating current waves.

By using the several oscillograph loops at once and casting the beams of light from their mirrors upon the same mirror B, curves of three currents may be traced upon the screen at the same time and their relative phase positions and forms compared.

The wave cycles are repeated over and over at the frequency of the impressed voltage, and for ordinary frequencies appear like fixed curves on the semiopaque screen. To prevent the curve from being shown double, the shutter S is driven in synchronism with the vibrating mirror B in such a way as to cut off the light from F during every other half oscillation of B.

When the photographic apparatus is to be used the beams of light from V, V, V pass through the long cylindrical lens C_2 and strike the photographic film placed upon T, C_1 and B being moved aside. The drum T is rotated at a desirable speed and the shutter S opens and closes at the beginning and end of one of its revolutions. The movement of the drum obviously describes the abscissas of the curve.

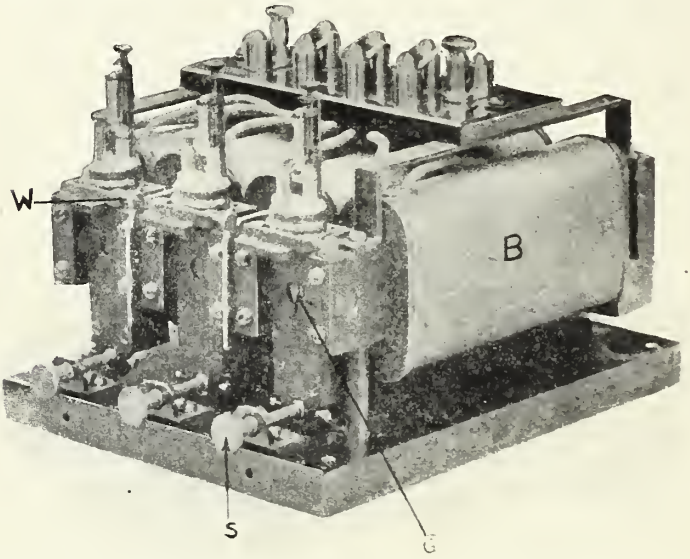


Fig. 380. — Vibrating Mechanism and Electro-magnets of One Type of Oscillograph — Cover removed.

Figure 380 shows the vibrators and magnets of a commercial type of three element oscillograph. In this figure, G is one of the openings for admitting the three beams of light to the galvanometer or loop mirrors just inside. B is one of the electromagnets. The cells holding the loops, sometimes called vibrators, one of which is shown at W, are immersed in a liquid to deaden the natural vibrations.

Figure 381 shows the vibrator element at A, which is made of fine silver or bronze ribbon. The part between the bridges B and B_1 is between the magnet poles and to that part is attached the mirror.

To obtain curves of current, current transformers may be required, or sometimes the loop is shunted around a non-reactive resistance device through which the current to be measured flows. In measuring voltages it is in ordinary cases necessary to use a constant voltage transformer to cut down the voltage impressed upon the loop as well as to place non-inductive resistance in series in the loop circuit.

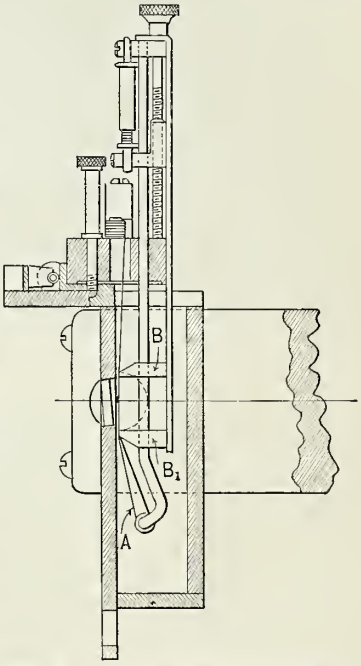


FIG. 381. — An Oscillograph Vibrator.

A forerunner of the oscillograph is due to Gerard and may sometimes be used for tracing the voltage curves of an alternator when no current flows in the armature. The machine to be tested is rotated at a very slow speed, the field being excited in the usual manner. The terminals of the machine to be examined are connected to a shunted d'Arsonval galvanometer. The natural rate of oscillation of the galvanometer bobbin is made quite rapid, as compared with the period of the voltage supplied by the alternator at its slow speed. Then the deflection of the needle at each instant will be proportional to the instantaneous voltage. By moving a sheet of sensitized paper before the galvanometer mirror which throws upon it a beam of light, the paper being moved at right angles to the mirror movement, the curve of voltage may be permanently recorded.*

Figure 382 is a current curve traced by an oscillograph of a current containing a marked peak caused by highly saturated iron cores in coils in series with the circuit. Figure 383 is a

* See Gerard's *Leçons sur l'Electricite*, Vol. 1, p. 565, 3d ed.

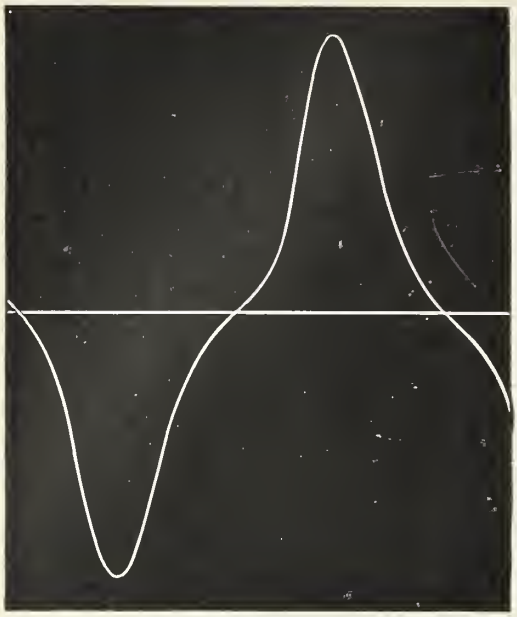


FIG. 382. — Peaked Current Curve traced by an Oscillograph.

voltage curve of an alternator which is almost sinusoidal in form. In both cases the waves were taken by means of the photographic attachment to the instrument.

2. *Methods using Contact Makers.* — A large number of methods require the use of a revolving contact maker of some kind, and they therefore have much in common. These were much used in the past and are useful for special purposes or where an oscillograph is not available. The principal differences in the methods relate to the types of instruments used to give the indications, and the convenience with which the manipulations may be made. Whether current or voltage curves are to be obtained, only instantaneous voltage measurements are made. For the former, the instantaneous voltages are taken at the terminals

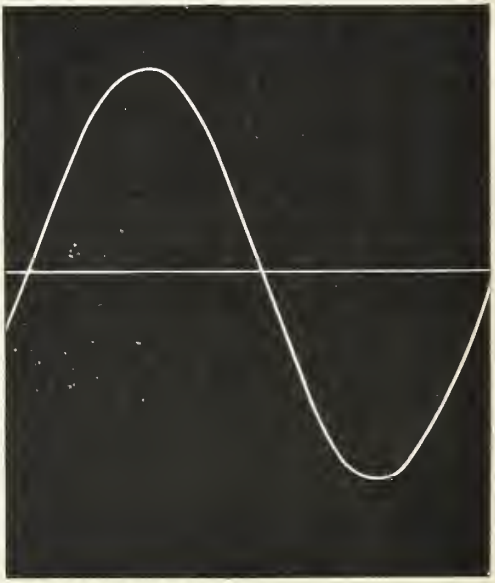


FIG. 383. — Sinusoidal Alternator Voltage Curve traced by an Oscillograph.

of non-reactive resistance, and the instantaneous currents are readily deduced.

In one of the earlier methods, advanced and used by Joubert, the terminals of the alternator armature, or of one bobbin of the armature, are recurrently connected to a condenser in the following manner: One armature terminal is connected permanently to one terminal of the condenser; the other armature terminal is connected to a rotating point which may be put into brief connection with the free terminal of the condenser when the armature is at any desired point in its rotation. At the instant this contact is made, the condenser receives a charge which is proportional to the instantaneous voltage between the terminals. The charge may be measured by discharging the condenser through a ballistic galvanometer and the voltage computed from the amount of charge and the capacity of the condenser. By successively setting the contact maker so that the instant of contact corresponds with various points in the revolution of the armature, the corresponding instantaneous voltages may be thus measured and the curve of voltage may be plotted (Fig. 384). The contact maker used by Joubert was an insulated pin set in the armature shaft, against which a brush could be made to bear at any point in the revolution. A quadrant electrometer may be used in place of the condenser and ballistic galvanometer; in which case it is desirable to introduce a condenser permanently in parallel with the electrometer, to neutralize the effect of leakage in the test circuit.

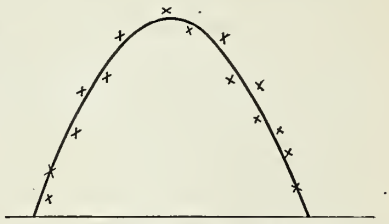


Fig. 384. — Approximate Curve of Voltage obtained by the Use of a Contact Maker.

Joubert's investigations made in 1880 resulted in the first determination of the curve of voltage of an alternator. The investigations of Duncan, Hutchinson, and Wilkes probably produced the earliest series of experimental curves showing the relations between the waves of voltage and current in circuits of different kinds. The investigations of Searing and Hoffman were probably the first made upon an alternator with iron in the armature core. Their results showed the curve of voltage developed in a smooth-core drum armature with wide coils to

approach a sinusoid. Ryan and Merritt, in 1889, did much valuable work in studying the current and voltage relations in a transformer. Various methods were used and desirable data obtained by Blondel, Searing, Hoffman, Mershon, Duncan, Ayrton, Pupin, and others. The last named used a method for determining harmonics by means of resonance, while Professor Ayrton proposed obtaining the harmonics by means of the vibrations of a stretched wire.

The earliest and simplest contact maker was, as already pointed out, simply an insulated pin set in the shaft of the alternator furnishing the current for the test. With this was a brush so arranged as to make contact with the pin at any desired point in the revolution. This arrangement is often inconvenient of application, and is likely to give rather irregular results. The contact is likely to be variable in resistance and, as the brush wears, the duration of contact varies. Each of these variables introduces errors of greater or less magnitude, depending upon the conditions of the test. Various refinements of construction have been introduced by experimenters in order that the defects of the contact makers may be eliminated. If a contact of absolute uniformity were assured, special instruments would not be necessary for taking the indications in determining voltage and current curves, because the indications of a sensitive electrodynamometer might then be directly used.

The contact maker was originally arranged for a single contact, but it is frequently desirable to make simultaneous observations of several curves. This may be readily accomplished by using a contact maker with the appropriate number of contact disks on the same spindle. Then a satisfactory instrument, such as an electrostatic voltmeter, may be used in each circuit.

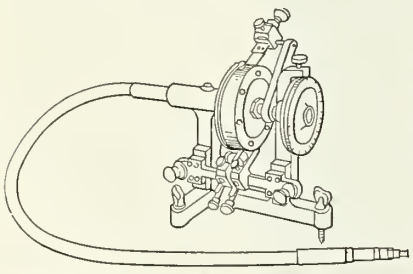


Fig. 385. — Contact Maker with a Flexible Shaft.

Sometimes it is not convenient to have the contact maker attached to the dynamo shaft, in which case it may be attached to a short length of flexible shaft (Fig. 385), which may in turn be attached to the dynamo shaft, but the flexible shaft must possess uniform tor-

sional rigidity or the rate of rotation of the contact maker may be unsteady, or the instrument may "hunt," and thereby introduce error by causing variations of the contact duration. When connection to the alternator cannot be conveniently made, the contact maker may be driven by a synchronous motor as has been done by Blondel, Siemens and Halske, and Fleming. In this case, "hunting" by the synchronous motors is likely to cause errors.

The contact maker shown in Fig. 385 indicates the attachment of the contact brush to a scale, whereby it can be moved to any desired angular position and there make contact once in a revolution with the contact button on the revolving drum.

The plan of charging and discharging a condenser to measure the instantaneous voltage is not convenient, and various other expedients have been proposed and more or less used. A galvanometer with a sufficiently great natural rate of vibration will be steadily deflected by the succession of impulses which it receives when connected in circuit with a contact maker. This deflection may be balanced by a steady voltage which is introduced in the circuit in series with the galvanometer and contact maker as indicated in Fig. 386. A telephone put in the place of the galvanometer gives oral instead of ocular notice of a balance. It is desirable to place a condenser in parallel with the galvanometer or telephone when this arrangement is used. In Fig. 386, *cd* represents a graduated rheostat with a moving contact post *c*, *V* is a voltmeter, and *B* a battery or direct current dynamo, to give a steady voltage which is greater than the maximum instantaneous voltage to be observed. The contact post *c* is moved along the rheostat until a balance is produced, and the voltmeter reading is then equal to the instantaneous voltage corresponding to the particular setting of the contact maker. The reversing key is introduced in the circuit to make it equally convenient to explore the positive and negative loops of the alternating voltage.

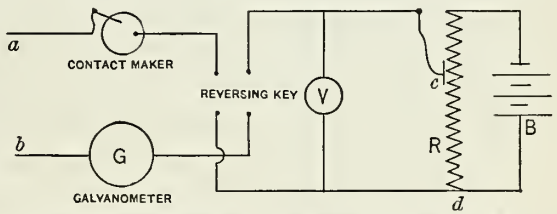


FIG. 386. — Potentiometer Arrangement for Contact Maker.

Any method enabling the use of a reflecting instrument in circuit with the contact maker may be made continuously self-recording by a proper disposition of the apparatus. In this case, a beam of light is thrown upon the mirror, and its deviation is recorded by means of a moving photographic film. In order that the complete curve may be thus recorded, the contact points must be caused to rotate continuously around the spindle of the contact maker. Since the needle of the galvanometer or electrometer which is used with the contact maker in this case must rigidly follow the intensity of the current impulses, the instrument must have little inertia and be truly deadbeat. The vibrations of a telephone diaphragm with a mirror mounted on it have been used to replace the deviations of a galvanometer or electrometer needle.

158. Areas of Successive Curves of Alternating Currents and Voltages. — In general, observations which cover one complete period entirely define the curves of commercial alternating currents and voltages. Since there is no continuous transference of electricity in one direction, the areas of successive loops of the curves should be equal. In the voltage curves produced by an alternator, for instance, $e = \frac{d\Phi}{dt}$, and $\Phi = \int e dt$, where Φ is the total number of lines of flux passing into the armature core and T is the time occupied by a cycle. If Φ and T are constant, as would be the case for a symmetrical alternator with fixed field magnetism and a rigid armature shaft, which is driven at a uniform speed, the areas of the successive loops of the voltage curve must be equal. On account of various irregularities in the construction and working of alternators, experimentally determined curves are not always uniform. In fairly large commercial machines the differences are usually not greater than might be caused by the errors of observation due to the experimental determination; and appreciable differences in the areas of successive loops of the curves produced by mechanically rigid machines, driven at a uniform angular velocity, are not to be expected, except possibly when the machines have armatures with their halves connected in parallel, and then only when the magnetic circuits lack symmetry to a considerable degree. In the case of certain small eight-pole alternators, Dr. Bedell found differences in the areas of the consecutive

loops which are not explainable upon the ground of errors of observation or of variable speed.* The curves given by two of these machines in one complete revolution (four complete periods) are shown in Fig. 387. The individual areas of the

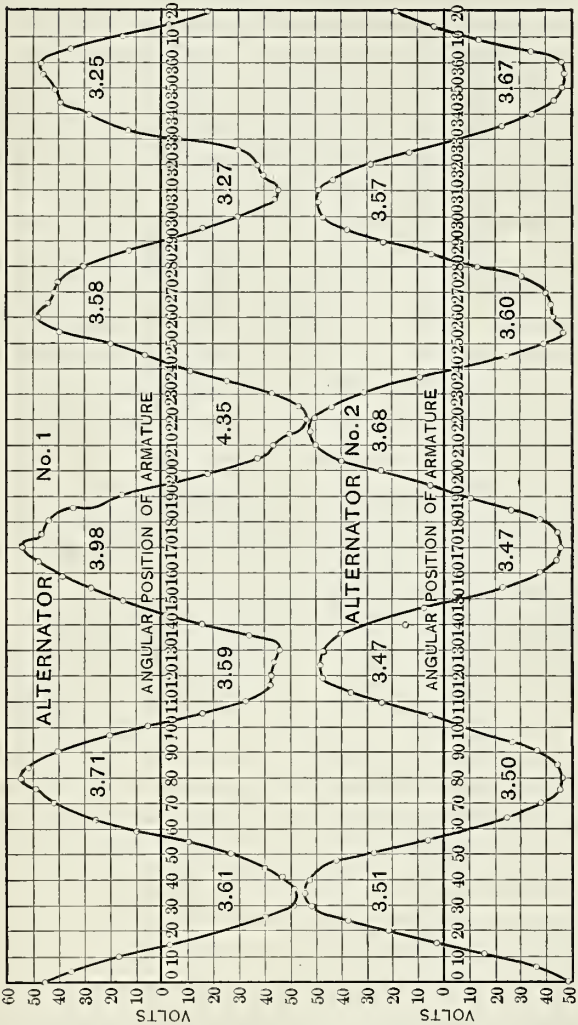


FIG. 387. — Irregular Curves of Alternators of Poor Mechanical Construction.

loops are marked upon the figure. While these differ as much as 25 per cent amongst themselves, the sums of the positive and negative areas differ by no more than might be caused by

* *Physical Review*, Vol. 1, p. 218.

experimental errors, while the angular speed of the prime mover or of the exciter may cause similar deviations in the curves. This apparently shows that irregularities in the magnetic circuits and in the armature windings may in some cases cause differences in the successive loops of the curve developed in 360 electrical degrees, but the algebraic summation of the areas due to each revolution is zero. The latter must be true, or there would be a continuous flow of electricity in one direction. The fact that the machines tested by Dr. Bedell had notable structural weaknesses leads to the probability that the springing of the shaft or other parts of the machine may have caused the unusual result which he found. Indeed, it is not uncommon where an alternator is badly aligned or the shaft is slightly bent, in either the alternator or its exciter, to have such irregularities set up in the voltage wave. Irregular angular speed of prime movers or of the exciters may cause similar variations of the areas. These may become so pronounced in certain voltage loops occurring during the alternator revolutions, as to be manifested to the eye by flickering of incandescent lamps attached to the alternator circuit. Variation between the areas of the successive loops of current curves may be introduced by unsteady resistance or reactance, even when the voltage curve is perfectly uniform.

159. Regulation of Alternators for Constant Voltage. — A separately excited alternator has, as already intimated, no inherent tendency towards regulation. The regulation is usually effected by means of a rheostat in the field circuit of the shunt or compound-wound exciter, a rheostat in series with the alternator field winding, or both. The adjustment of these rheostats may be performed by hand or through devices actuated by a relay placed in shunt to the main circuit. In Great Britain and Europe automatic devices have been used in large plants for many years, and of recent years they have come into very common use in this country.

A type commonly used in this country for controlling alternator voltage is arranged to make brief successive periods of short circuit around a part or all of the field rheostat of the exciter, the periods of short circuit being of greater or less duration depending upon whether the alternator voltage should be increased or decreased. The short-circuiting periods repeat them-

selves in rapid succession, varying the field current and giving it a wavy form having a magneto-motive force of the value needed to set up the required exciter voltage. The exciter field rheostat is usually set so that when the short circuit around the rheostat has been opened the alternator voltage will fall in from six to eight seconds to about 80 per cent of normal full load value when compound or interpole exciters are used, and to the no-load alternator voltage when shunt exciters are used. The delay with which the voltage falls to its lower value when the rheostat comes into the exciter field circuit by the removal of the short circuit is caused by the eddy currents and the self-inductance causing an apparent electro-magnetic inertia in the exciter and alternator field magnets.

At other than no load the short-circuiting contact of a regulator of this type seems always rapidly moving. The smaller the load and the greater the resultant tendency for the line voltage to rise, the shorter must be the periods of the short circuit of the exciter field rheostat. Figure 388 shows a dia-

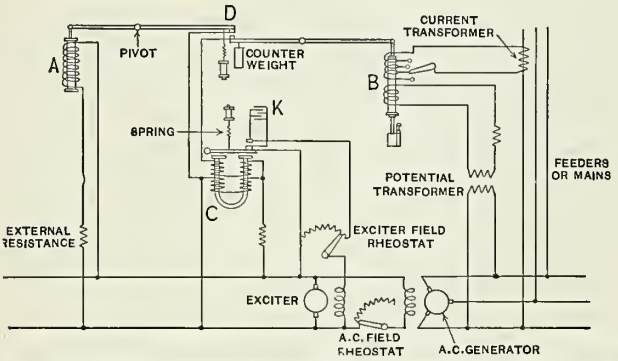


FIG. 388. — Voltage Regulator which short-circuits a Single Section of the Exciter Field Rheostat and has One Set of Relay Contact Parts.

gram of one of the commercial types of such a regulator. The regulator has a direct current control magnet *A*, an alternating current control magnet *B*, and a relay *C*. The winding of the magnet *A* is connected across the exciter circuit, and its movable core is attached to one end of a pivoted lever which carries at its opposite end a flexible contact part *D* which is pulled downward by springs. The magnet *B* has two windings, one being connected to a potential transformer and the other being a

compensating winding connected to a current transformer and giving an opposing magnetizing effect proportional to the current flowing in the alternator circuit. The movable core of magnet *B* is attached to one end of a pivoted lever which is counterweighted at its other end and carries a contact part which coöperates with the contact part *D* already referred to, securing what may be called a floating contact arrangement. The relay *C* consists of a U-shaped magnet core carrying a differential winding and provided with a pivoted armature which controls a circuit contact arranged to close and open the branch which short-circuits the exciter field rheostat. One half of the differential winding is permanently connected across the exciter circuit. The other half of the differential winding is connected to the floating contact points, and is also connected across the exciter circuit and neutralizes the magnetizing effect of the first half when the floating contact is closed. A condenser *K* is connected between the contact parts at the relay armature, to prevent serious sparking when the contact opens.

Now, assuming the floating contact to be open, the relay armature is drawn down and the field rheostat is introduced in the exciter field circuit; this lowers the voltage of the exciter and thereby lowers the voltage of the alternating current generator to which the exciter is connected. This weakens both of the control magnets, and the floating contact closes, whereupon the second half of the differential winding of the relay is energized and neutralizes the relay excitation, and the relay armature is released. This results in closing the relay contact and short-circuiting the field rheostat. That increases the excitation of the exciter and therefore of the alternator, so that both exciter and main voltages increase. Both control magnets are strengthened, the core of *A* is attracted downward and the core of *B* is attracted upward so that the floating contact opens, interrupting the circuit through one half of the differential winding on *C*, the relay armature is attracted, and the exciter field rheostat is again short-circuited. As the voltages rise in consequence of the last condition, the cycle of operation begins again, and it is repeated over and over again at a fairly high rate of vibration which maintains the alternator voltage with reasonable precision at the value desired for all

loads. The actual variation of voltage required to produce the cycle of operations is very slight.

It is often desirable to have the regulator control the alternator so as to keep the voltage constant at the point of consumption rather than at the generating plant, and this may be reasonably accomplished by the use of the combination of current transformer and potential transformer indicated in Fig. 388. This obviates the use of "pressure wires" (that is, wires which run from the center of consumption to voltmeters at the generating station for the purpose of indicating the voltage at the consumption center), which were much used in the early days of electric lighting.

Where several exciters are used in parallel on the exciter bus bars they may be connected as shown in Fig. 389. In this case,

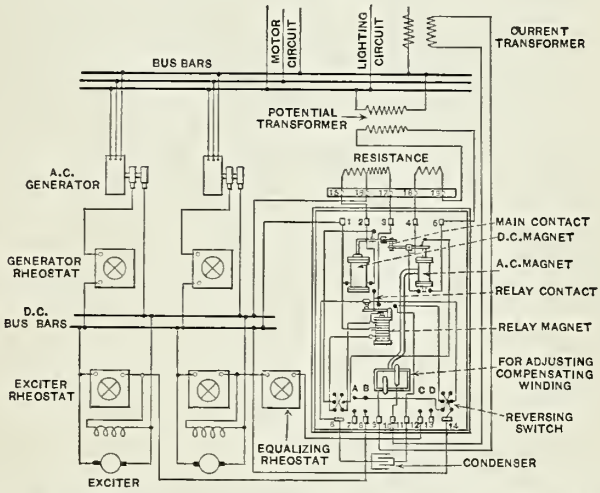


FIG. 389. — Several Exciters connected to a Voltage Regulator of the Type in which Part of the Field Rheostats are short-circuited.

the alternator field windings are connected in parallel, and care must be taken that the regulation affects the voltages of all the exciters alike, or the exciter loads will not divide equally. This requires care in the adjustment of the rheostats, and of the exciter field compounding where the exciters are compound wound. When the exciters are of large capacity, the relay magnet controls several sets of contact parts so that each exciter rheostat can be operated by an independent set, or two or more rheostats con-

trolled by separate contacts can be used for the field winding of a single exciter. By such an arrangement the values of the currents and voltages can be reduced suitably to be handled by the contacts without injury.

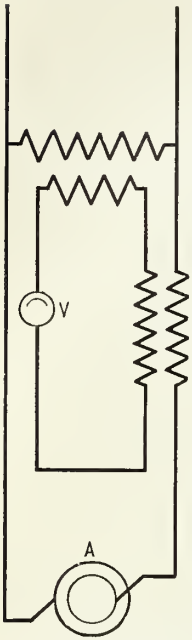


FIG. 390. — Diagram of Connections for a Compensated Voltmeter.

The device used for obtaining at the terminals of the regulator a voltage which is proportional to $\bar{E} - \bar{I}Z$, where $\bar{E}$ is the generator terminal voltage and $\bar{I}Z$ is the line drop, is similar in operation to the arrangements used for obtaining compensated voltmeter readings. It is often desirable to show at the switch-board the voltage at the point of consumption. The station voltmeter is then connected in series with the secondary windings of a voltage transformer and a current transformer which act in opposition (Fig. 390). The transformers being properly adjusted, the voltmeter shows at all times the voltage at the center of consumption. In such an arrangement several terminals may be brought out from the secondary winding of the current transformer to a switch like that shown in Fig. 391, and the adjustment of the apparatus to compensate for any given line drop may be made by changing the secondary connections by moving the switch lever and so changing the number of effective secondary turns in the voltmeter circuit. The secondary winding of the current transformer may be shunted by resistance as at R_2 in Fig. 392, and the adjustment of this resistance serves a corresponding purpose. To get exact compensation in the voltmeter circuit, R_2 should be replaced by resistance and reactance in the same proportions as the resistance and reactance of the line for the drop over which compensation is to be made. If the ratios of transformation of the potential and current transformer T and T' are alike, the ratio of the line impedance to the impedance substituted in the position of R_2 should be equal to the ratio of transformation, and the voltage at the far end of the line will then bear the same

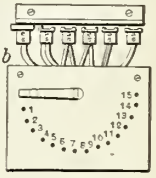


FIG. 391. — Hand-regulating Switch for Controlling Number of Turns in Transformer Secondary.

ratio to the voltage measured by the voltmeter. If the ratio of transformation of the current transformer differs from the ratio of the potential transformer, then the shunting impedance should be changed in a corresponding degree. The secondary winding of the current transformer does not necessarily have to be connected into the circuit of the regular voltmeter coil, but can be connected to an auxiliary coil which is wound alongside of or over the main coil.

When a regulator without a compensating coil is used, the voltage acting in the circuit of the automatic regulator must be caused to remain constant as the alternator current increases, while at the same time the alternator voltage increases by a sufficient amount to compensate for the fall of voltage in the feeders, in case the regulator is intended to maintain constant voltage at the center of consumption. In other words, $\bar{E} - \bar{I}Z$ must be kept constant, $\bar{E}$ being the alternator voltage, $\bar{I}$ the current, and Z the impedance of the feeders. This may be effected approximately as illustrated in Fig. 388, or as follows (Fig. 392). The regulator is connected to the secondary circuit of potential transformer T, which is connected in parallel across the feeders. The voltage of the secondary of this transformer is proportional to the alternator voltage E. A current transformer T' is connected with its primary winding in series with the feeders. The voltage developed in the secondary winding of this transformer can be adjusted so as to be practically proportional to $\bar{I}Z$ for all values of the current. The secondary winding of this transformer is connected in series with the secondary circuit of the first transformer, and in such a way that their voltages are in opposition. Hence a voltmeter V connected across the terminals of the two secondaries indicates a voltage which is proportional to the voltage at the terminals of the feeder, or $\bar{E} - \bar{I}Z$. If the automatic regulator is also connected across

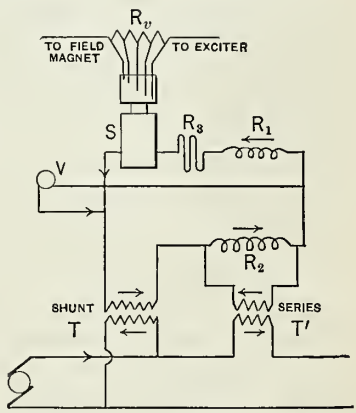


FIG. 392. — Connections of Voltage and Current Transformer to Alternator Regulator to maintain Constant Voltage at the Center of Distribution.

the terminals of the two secondaries, it adjusts the excitation of the alternator so that $\bar{E} - \bar{I}Z$ is kept fairly constant regardless of the value of I. In the figure, S is the solenoid of the regulator, R_v is the resistance automatically controlled by the solenoid to vary the excitation of the alternator, and R_1, R_2, R_3 are impedances in circuit with the regulator which are used for adjusting it to give proper indications for various values of I and Z.

Regulation of generator voltage by means of compound winding is difficult, as the number of series turns that give regulation on a non-reactive load evidently may fail for a reactive load. It is evident, therefore, that a compositely excited alternator will only regulate on loads having the power factor for which regulation of the machine was adjusted. The ratio of series ampere-turns per field-magnet pole to armature ampere-turns per coil which is required to give regulation for one alternator of a fixed type will doubtless give equally satisfactory results on machines of different capacities but of the same type; but the marked differences in the magnitude of the effects of self-induction and armature reactions in alternators of different types make it impossible to fix any ratio that would even approximately cover all types of machines. The regulation of ordinary alternators which are self-excited, or otherwise, can only be satisfactorily effected by means of variable resistance regulators placed in the exciting circuit, or by varying the exciter voltage. This statement includes the self-excited alternators now on the market in which the rectified current for the fields is obtained from a special independent winding placed in the slots with the ordinary windings of the armature. The regulation might be effected by moving the brushes on the rectifying commutator, if the machine was self-excited, but only at the expense of prohibitive sparking and wear.

If it is desired to have an alternator give constant current, the regulation may be made inherent by designing the armature reactions and self-induction to be so great that the current cannot rise much above its normal value. The armature should be wound to generate a voltage upon open circuit much greater than that required for full load, and hence the current remains near its full normal value up to, and even considerably beyond, full load. Such machines can be worked on short-circuit with-

out injury; but if the circuit is opened, they are liable to injury on account of the excessive open circuit voltage breaking down the insulation. These machines are really worked on a part of the characteristic which is caused to be almost vertical on account of the large self-inductance and reactions of the armature; that is, the number of armature conductors is many times greater than comports with obtaining the maximum output per pound of copper and iron in the machine. Constant-current alternators have sometimes been used in the past for arc lighting and are referred to here to show the effect of their extreme design on regulation. The external characteristic of such an alternator is given in Fig. 393. This shows how the terminal voltage increases as the current decreases relatively little, while the external circuit is changed from short circuit to an equivalent resistance of between two and three hundred ohms, the exciter voltage impressed on the field windings remaining constant. The curves are given for three different excitations.

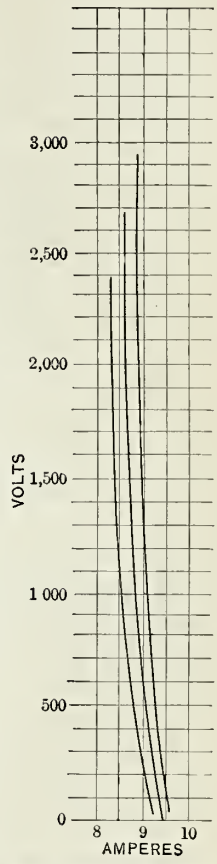


FIG. 393. — External Characteristics of an Alternator designed to Produce Constant Current.

160. Feeder Regulators. — Feeder regulators for use in plants where several circuits are fed from one alternator or set of bus bars are in quite common use, as already intimated. The regulator may be a special transformer (Fig. 394), with the secondary windings *CD* in series with one feed wire, and the primary winding *AB* connected across the main bus bars. The voltage induced in *CD* may be made to either aid or oppose the alternator voltage by means of a reversing switch, *X*. The strength of the voltage introduced into the feeder circuit by *CD* may be varied by changing the number of turns of *CD* in the circuit, by means of movable contact parts. For convenience, such contact parts are usually arranged on the arc of a circle or upon a drum. The contact slider at *F* may be divided into two parts and reactance

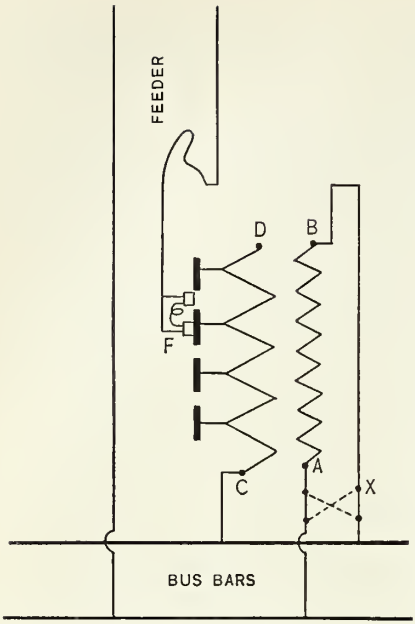


FIG. 394.—Diagram of a Feeder Regulator, consisting of Transformer with Secondary Winding with Variable Turns connected in Series with the Feeder Circuit, and Primary Winding in Parallel with the Feeder Circuit.

inserted between the parts to prevent excessive current from flowing when a transformer coil is short-circuited. Other arrangements are used for this purpose and for reducing sparking.

Devices in which the regulation is effected by varying the position of the primary and secondary coils with respect to each other, or of the core with respect to both, are also used. Thus, a regulator with armature and field windings arranged like those of an induction motor * (Fig. 395), with independent coil-wound secondaries of the required number of phases, may be used for polyphase feeder regulation. Each set of secondary coils of a phase is connected in series with a line conductor of the feeder to be controlled. The primary windings are connected in multiple arc on the main polyphase circuit. The progressing or rotating magnetic field induced by the primary windings, caused by the magnetic fluxes of the several phases rising successively across the air space and at angles advancing around the polar circle, sets up a voltage in the secondary windings. This voltage is constant

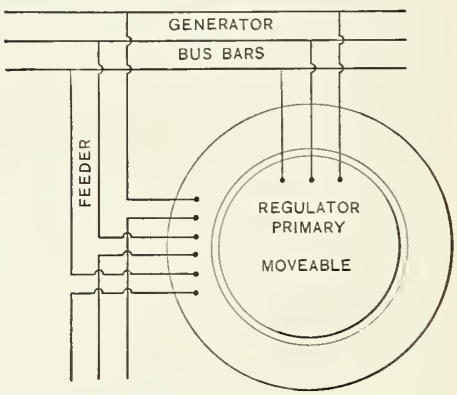


FIG. 395.—Diagrammatic Representation of a Polyphase Induction Regulator.

* Chap. XII.

in strength, but has a phase relation with the feeder voltage dependent upon the angular position of the field magnet with respect to the armature. Thus the voltage may directly add to the feeder voltage, be subtracted from it, or combine in an intermediate angular position, as shown in Fig. 396, where AO is the generator voltage of one phase and AB, AB', AB'', AB''' are the feeder voltages beyond the regulator for one phase for various positions of the regulator field magnet with respect to its armature. One core carrying either the primary or the secondary windings in such a regulator is mounted in a fixed position, and the other is mounted in such a manner that it may be rotated such part of a revolution as is needed for sufficiently influencing the feeder voltage. The core carrying the primary windings is usually made the movable core.

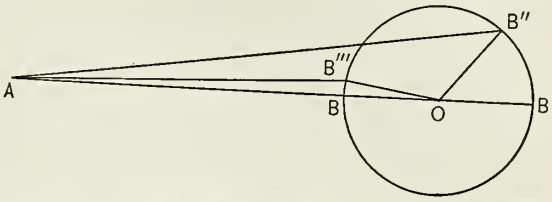


FIG. 396. — Voltage Diagram of a Polyphase Induction Regulator.

A single-phase regulator may be made in the same way, using only one set of windings respectively on the primary and secondary cores. In this case zero voltage will be produced in the secondary winding when its coils are so moved as to be parallel to the magnetic flux from the primary winding, and will increase to maximum value when the coils are rotated to the right or left from this point until they are at right angles to the flux. Compensating short-circuited coils such as are used in series motors * must evidently be used for a single-phase regulator of this kind to prevent an excessive reactance in the secondary circuit when its windings are not at right angles to the flux from the primary winding. These compensating coils are wound on the core with the primary winding but magnetically at right angles thereto. The transformer characteristics of these induction regulators makes their operation subject to the condition that the vector difference between the primary ampere-turns and the secondary ampere-turns plus the ampere-turns of any compensating coil makes the exciting ampere-turns.

Autotransformers may also be used as voltage regulators,

* Art. 206.

as has heretofore been pointed out; * and a form of regulator for single-phase circuits was commonly used in former years which consists of two coils placed at right angles on the inner barrel of a hollow cylindrical laminated core, one of the coils being connected across the circuit and the other introduced in series in the feeder concerned. A segmented rotatable core is used to direct the magnetic flux set up by the primary winding through the secondary winding or shunt it therefrom.

Such regulators are commonly made to vary the voltage of the circuit to which they are attached over a range from 10 per cent lowering to 10 per cent boosting. The primary winding must be designed for the voltage of the main circuit, and the secondary conductors must be capable of carring the full feeder current. When the regulator is raising the voltage, the outgoing current is smaller than the current in the main circuit leading to the device, and when the regulator is lowering the voltage, the outgoing current is larger than the current in the circuit leading to the device.

161. Connecting Alternators for Combined Output. — The conditions required for successfully connecting alternators so that their outputs may be combined are quite different from those obtaining in the case of direct-current machines. In order that the output of alternators may be added, it is evident that the voltage waves impressed by them upon the circuit must be in exact consonance. That is, the voltage waves must be of equal period or **In synchronism** and also of corresponding phase or **In step** with each other. If this is not the case, the machines will be in opposition during all or a portion of the current wave. The following discussions concerning combined output are applicable to both single-phase and polyphase alternators. In case of the latter, the voltage and current in the windings of a single phase should be understood to be referred to, while the power of the single phase should be multiplied by the number of phases to get the total machine power.

161 a. Alternators in Series. — The alternators will be assumed in this discussion to be constructed so as to give equal currents at equal frequency. The form of the current waves will also be assumed to approximate to sinusoids and the armatures to have negligible reactive effects upon the field magnets.

* Art. 144.

In Fig. 397 let the curves A and A' represent the voltage waves measured respectively across the terminals of two alternators with their armatures connected in series, the machines being driven independently, but so as to give practically the same frequencies and voltages. The ordinates of curve R are the algebraic sums of the corresponding ordinates of curves A and A', and hence curve R represents the resultant voltage impressed on the external circuit by the two machines. Curve C is assumed to be the curve of lagging current flowing in the circuit. Assuming the two machines to be running synchronously, but to be out of step by an angle 2ϕ, makes ϕ the phase difference between the resultant voltage wave and either component wave. Finally, the current lags behind the resultant voltage by an angle θ, on account of self-inductance in the circuit. The work put into the circuit by each machine is proportional to the algebraic summation of the products of the ordinates of the respective voltage wave with the current wave. The total work done in the circuit is equal to the sum of the products of the ordinates of the current and resultant voltage curves. Therefore, since the voltage wave of the lagging machine is

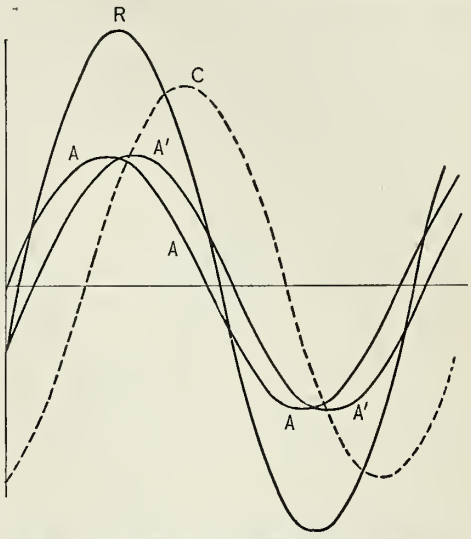


FIG. 397. — Curves of Voltages and Current showing Unstable Conditions existing when Alternators are Connected in Series.

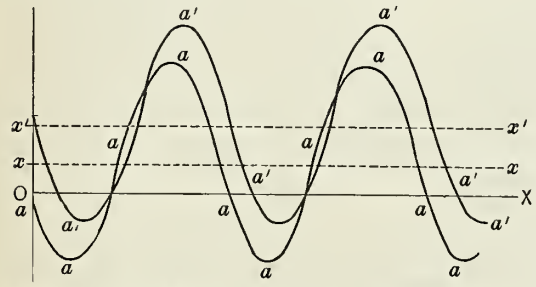


FIG. 398. — Power Loops of Alternators producing the Voltages and Current shown in Fig. 397.

nearest the current wave, that machine furnishes more work to the circuit than does the leading machine. The power loops for the two machines are shown by the curves a and a' in Fig. 398. The power delivered to the circuit by one machine is represented by the height of the line xx above the X-axis, and the power delivered to the circuit by the other machine is represented by the height of the line $x'x'$ above the axis.

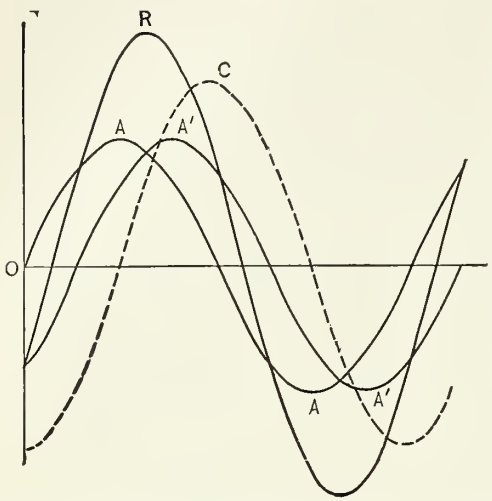


FIG. 399. — Curves of Voltages and Current showing Unstable Conditions when Alternators are connected in Series.

If the two machines were rigidly connected together, this condition would continue indefinitely. Assuming, however, that the machines are driven by separate belts, or attached to separate engines, the lagging machine being the more heavily loaded, tends to fall farther behind its more lightly loaded mate, and a still greater percentage of the load is thrown upon it. At the same time, as is illustrated by Fig. 399, this reduces the total work done in the external circuit, for the total voltage wave is now of less height than it was when the component curves were nearer coincidence of phase. The power loops for the condition of Fig. 399 are shown in Fig. 400. The height of the line xx has decreased, and that of $x'x'$ has increased, but the sum of the heights is less than before. The tendency of the lagging machine to fall farther behind may

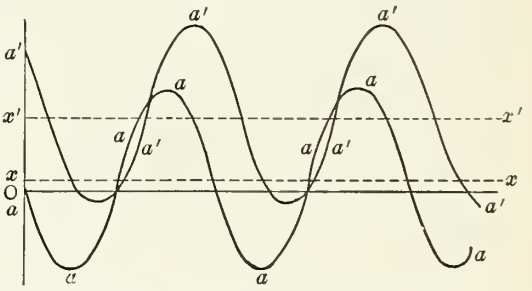


FIG. 400. — Power Loops of Alternators producing the Voltages and Current shown in Fig. 399.

continue until the voltage waves of the two machines approach exact opposition to each other in the series circuit (Fig. 401). When in opposition the machines are in stable equilibrium, but are delivering no power to the external circuit.

If the machines were started with their voltage waves in exact step, they would do equal work, but their equilibrium would be unstable, and any disturbance of their relations would cause them to tend towards opposition. It is therefore not possible to operate alternators in cumulative series on an inductive circuit unless they are rigidly united by a mechanical coupling. This result also follows when the normal voltages of the machines are different, in which case the voltage impressed on the circuit when equilibrium is attained is the difference of the machine voltages. Even if there were no reactance in the external circuit on which the machines were working, the resultant voltage and current would have the same phase, and the machines would be in equilibrium; but the equilibrium would be unstable, for after any disturbance of the operation of the machines they would have no tendency to return to their former operating state.

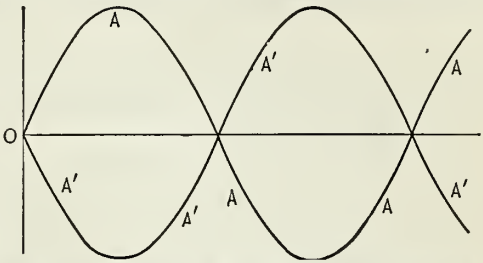


FIG. 401. — Voltages of Two Alternators connected in Series, after they have the Stable Position of Series Opposition.

The condition of operation of two alternators connected in series on an inductive circuit is also plainly indicated by means of a vector diagram (Fig. 402). In this diagram the lines represent quantities as follows:

Voltage of leading machine	$= OA.$
Voltage of lagging machine	$= OA'.$
Resultant voltage in circuit	$= OR.$
Current in circuit	$= OC.$
Power given to circuit by first machine	$= Oa \times OC.$
Power given to circuit by second machine	$= Oa' \times OC.$
Total power given to circuit	$= Or \times OC = (Oa + Oa') \times OC.$

It is evident from the construction that if the angle ROC is fixed by the conditions of the circuit, then the length of OR de-

creases as the angle 2ϕ increases, and that Oa decreases at the same time; but Oa' increases for a time and then decreases at a less rate than Oa, so that the machines tend to get farther apart in phase. When $\phi = 90° - \theta$, the length of Oa vanishes, the first machine gives no power to the circuit, and all the power is furnished by the second machine. When ϕ approaches more nearly 90°, or exceeds that angle, as when the machines are approaching opposition, one machine will run as a motor if its prime mover is disconnected. Of course, when OR decreases, if the resistance of the circuit is unaltered, the current OC also decreases, but the relative outputs and phases of the machines are not altered thereby.

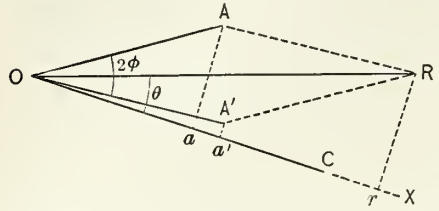


FIG. 402. — Vector Diagram of Voltages and Currents for Two Alternators in the Unstable Condition of Series Operation.

In case capacity reactance could be placed in the series circuit between the machines sufficient to cause the current to lead the resultant voltage by an angle θ, the condition is represented by Fig. 403, in which the vectors are lettered as in Fig. 402. In this case, Oa is greater than Oa'; that is, the leading machine furnishes the greater amount of power, and the machines tend to come together and run in cumulative series. Similar deductions can be drawn from diagrams like Figs. 397 and 398, but with the current wave in the lead of the resultant voltage wave. Since the alternator windings always contain self-inductance, special insertions of capacity in the series circuit would have to be made to get sufficient capacity reactance to produce this result, and it may therefore be said that, in general, alternators in series are not stable in operating.

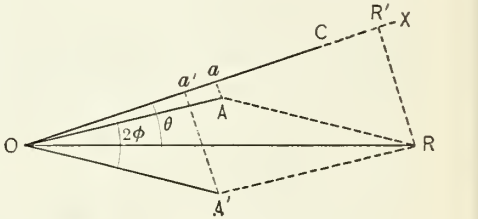


FIG. 403. — Vector Diagram of Voltages and Current of Two Alternators in Series when working on a Load of High Capacity Reactance.

161 b. Alternators in Parallel. — When the machines have reached series opposition of phases, as explained above, their voltages are in the proper relation to cause them to work in

parallel in delivering current to an external circuit attached between the lead wires connecting their terminals; for it will be seen by reference to Figs. 401 and 404 that when machines A and A' are in series opposition to each other, the points R and S must at every instant be of opposite sign, and that, therefore, the machines will deliver current through the load circuit m, m, m. This is also the proper connection for direct current generators arranged for parallel operation; that is, the voltages are opposed to each other in the series circuit through both armatures.

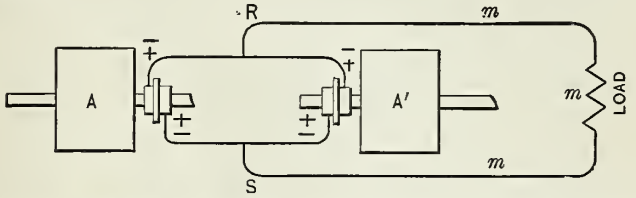


FIG. 404. — Diagram of Two Alternators connected for Parallel Operation.

The operation of alternators in parallel was first achieved by Wilde in 1868,* but this work was overlooked during the period of development of the direct-current dynamo. In 1884 Dr. John Hopkinson showed by mathematical analysis the practicability of working them in parallel. This was done without a knowledge of Wilde's earlier experiments, and it led to some experiments which were carried out by Hopkinson and Adams upon De Meritens magneto machines.† These experiments fully bore out Hopkinson's deductions, but their practical bearing was not fully appreciated until a few years later, when the transformer system of alternating-current distribution was developed.

161 c. Synchronizing Current.—Successful parallel operation of alternators depends upon their holding each other in synchronism and step (by the motor action referred to later in the article), even when the prime movers do not naturally synchronize. The effort of the machines to do this causes the flow of a cross current between them, which is commonly called the synchronizing current. This component of the total current from a machine may be considered as a series current circulating be-

* See *Philosophical Magazine*, Vol. 37, 4th series, 1869, p. 54.

† Adams, *Jour. Inst. E. E.*, Vol. 13, 1884, p. 515.

tween the machines considered as pairs connected to the station bus bars. The remaining component of the current from each machine joins in parallel with that of the other machines and flows through the load. The total synchronizing current flowing between a machine and the station bus bars may be resolved into two components, one in phase with the voltage which causes the synchronizing current to flow, and the other lagging 90° behind the voltage. The former is usually quite small. Thus, suppose in Fig. 405 that *Oa* and *Ob* represent equal vector voltages of two generators *A* and *B*, taken in series circuit through the two armatures. The machines may be considered as connected to the same bus bars of a central station. Curves *a* and *b* are sinusoids representing the voltage waves. Assume for the present that the machines are carrying no external load and that the prime mover of *B* is not supplying quite enough power to drive it unloaded while that of *A* is not only able to drive *A* but also to aid *B*. Then machine *B* will tend to fall back in phase or step and its voltage curve *b* will lag with reference to the voltage curve *a* of the machine *A*. As will be seen this will tend to cause a motoring synchronizing current to flow through the series circuit comprising the armatures of the machines. Thus, since *a* and *b* are slightly out of opposition, there will be a resultant voltage *Oq* (curve *q*), tending to set up a series synchronizing current. The relations are simply shown in the vector diagram in the figure, where the outer ends of the lines representing the vector voltages and currents corresponding to the curves are marked with the same letters as the respective curves. The series current set up by voltage *q* may be considered as made up of a component *s*, in phase with *q*, and of another component *u*, in quadrature with *q*, the former being the active and the latter the wattless component. The component *s* has the same effect upon both *a* and *b* during a complete period, *i.e.* each machine furnishes one half the power required to drive *s* through the series circuit, hence it can have no effect in tending to draw the trailing machine into the proper phase relation with its mate; but the component *u* (90° from *q*), which is dependent upon the self-inductance of the series circuit, must, from its position, cause an accelerating torque on the lagging machine (*b*), and a corresponding retarding torque on the leading machine; and it thus tends to

draw the machines into synchronism and step for proper parallel operation. It will be noticed by reference to the figure that u is in phase with m and in opposition to n, and that m and

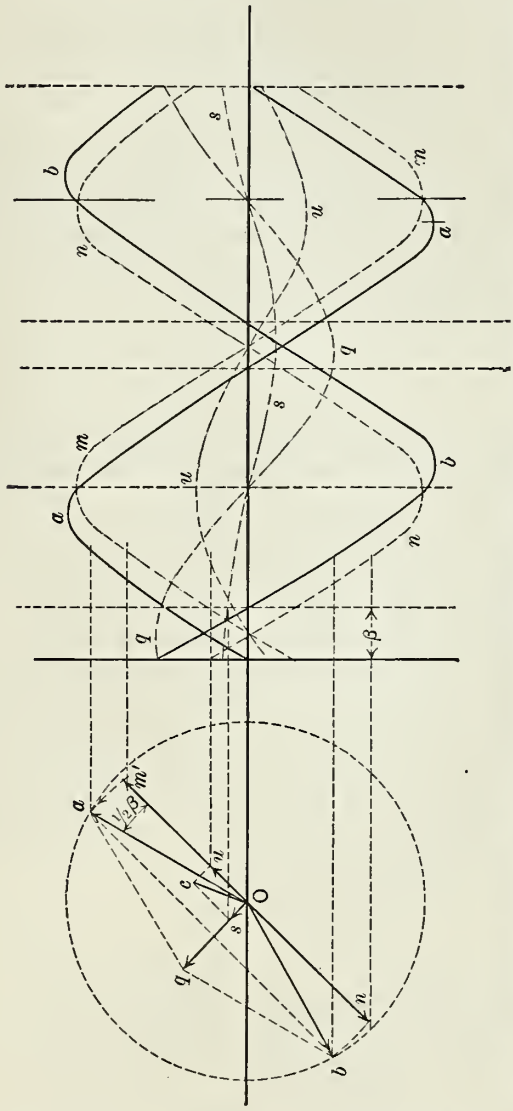


FIG. 405. — Diagram showing Curves and Vector Relations of Voltages and Synchronizing Current of Two Alternators connected in Parallel.

n are respectively the components of a and b, which are in opposition to each other in the series circuit, and are therefore impressed on the parallel circuit ; also that the motoring effect

on B is proportional to Ou times On, while the retarding effect on A is proportional to Ou times Om. If the machine B led the machine A, then the synchronizing current would retard instead of assisting the former, as the figure plainly shows.

The effect of the synchronizing current in dragging the machines into step depends upon its magnitude and relative phase; while its magnitude at each instant depends: (1) upon the algebraic sum, or, what is the same thing, the arithmetical difference between the instantaneous voltages developed by the machines; (2) upon the reciprocal of the impedance of the machine armatures. In other words, the instantaneous synchronizing current flowing in series relation through the two armatures, if the two machines are alike, is

$$i_s = \frac{e_a + e_b}{2\sqrt{R_a^2 + 4\pi^2 f^2 L_a^2}},$$

where e_a and e_b are the instantaneous voltages of the machines which are of opposite sign and equal when the machines are in exact synchronism and step, and R_a and L_a are respectively the resistance and the inductance of the armature circuit of each machine including the leads from the bus bars. It is here assumed that the effective voltages of the two machines are numerically equal, which is an essential condition for the synchronizing current to be practically wattless, and is the condition in which it is aimed to run alternators in practice.

Now suppose the voltage curve of the machine B to lag behind the phase of the voltage curve at the bus bars by an angle β. Let the effective values of these voltages be E, and the corresponding maximum voltage be e_m. At any moment the instantaneous voltages are e_a and e_b, and

$$e_a = e_m \sin \alpha = \sqrt{2}\, E \sin \alpha,$$
$$e_b = e_m \sin (\alpha + 180^\circ - \beta) = \sqrt{2}\, E \sin (\alpha + 180^\circ - \beta),$$

when considered to be acting in the series circuit. The instantaneous voltage causing a synchronizing current to flow is the algebraic sum of these, or

$$e_a + e_b = \sqrt{2}\, E [\sin \alpha + \sin (\alpha + 180^\circ - \beta)].$$

It is evident from Fig. 405 that this is a maximum at the instant when e_a and e_b are equal and of similar signs, in which

case $\alpha = \frac{1}{2}\beta$. Then the maximum value of the voltage causing a synchronizing current is found by substituting this value of α in the expression for $e_a + e_b$ and $(e_a + e_b)_m = 2\sqrt{2}\,E \sin \frac{1}{2}\beta$.

The effective value of the voltage is then $2\,E \sin \frac{1}{2}\beta$, and the synchronizing current is

$$I_s = \frac{E \sin \frac{1}{2}\beta}{\sqrt{R_a^2 + 4\pi^2 f^2 L_a^2}}.$$

For smooth and successful working in parallel I_s must become sufficiently great, in case the machines tend to get out of step, to pull the machines together before β becomes of appreciable magnitude. Hence it is desirable that the denominator in the expression for I_s be reasonably small. In other words, armature impedance must be small to give smooth parallel working. Figure 405 shows plainly that the voltage (Oq) causing I_s is behind the phase of the voltage of the trailing machine by an angle $90° - \frac{1}{2}\beta$. The synchronizing current ($I_s = Oc$) lags behind the voltage Oq by an angle θ_s (angle qOc), the tangent of which is $\frac{2\pi f L_a}{R_a}$. Consequently the synchronizing current is out of phase with the trailing machine voltage by an angle of $90° + (\theta_s - \frac{1}{2}\beta)$ and out of phase with the leading machine voltage by an angle of $90° - (\theta_s + \frac{1}{2}\beta)$. A current (wattless) which has a phase difference of 180° or 0° compared with the voltage of a machine has the strongest effect in bringing the machine into step by tending to accelerate it or to retard it. This effect is to accelerate or retard the refractory machine depending on whether the phase difference is 180° or 0°, which in turn depends upon whether the machine is trailing or leading.

The component $Ou(= I_\mu)$ of the synchronizing current just found is

$$I_\mu = I_s \sin \theta_s,$$

and therefore,

$$I_\mu = \frac{E \sin \frac{1}{2}\beta}{\sqrt{R_a^2 + 4\pi^2 f^2 L_a^2}} \cdot \frac{2\pi f L_a}{\sqrt{R_a^2 + 4\pi^2 f^2 L_a^2}},$$

or,

$$I_\mu = E \sin \tfrac{1}{2}\beta \cdot \frac{2\pi f L_a}{R_a^2 + 4\pi^2 f^2 L_a^2};$$

and, with a fixed value of R_a, this will have a maximum value when

$$R_a = 2\pi f L_a \text{ or when } \tan \theta_s = 1 \text{ and } \theta_s = 45°.$$

The maximum possible value of I_μ where the machine circuit has a given resistance is therefore

$$I_\mu \max = \frac{E \sin \frac{1}{2}\beta}{2 R_a},$$

and the corresponding value of I_s, the total synchronizing current, is

$$I_s = \frac{E \sin \frac{1}{2}\beta}{\sqrt{2}\, R_a}.$$

The limits in the value of R_a are fixed by considerations of economy in construction and of efficiency, and the frequency is fixed by conditions of operation; L_a is therefore the only independent variable in the preceding equations. In order to have the most sensitive mutual control, the self-inductance of the armature circuit, which at its least value is always many times larger than $\frac{R_a}{2\pi f}$, must be as small as commercially possible. If it were possible to reduce the reactance to the value of the resistance, the jerking of a refractory alternator into phase would probably be too severe for good working, and the stresses imposed on the machine might be injurious; but in most commercial machines such trouble is not likely to exist on account of the unavoidable magnitude of the self-inductance, which cannot be reduced beyond certain limits.

The foregoing equations are developed on the hypothesis of two equal alternators operated in parallel, but the conclusions are equally applicable to two alternators of unequal internal resistances and reactances and also to the conditions of many alternators connected in parallel to the same bus bars. In either case the impedance of the cross current circuit for any particular machine is equal to the sum of the impedances of the armature and connecting leads of that machine in series with the circuit composed of its mates in parallel and their connections. The bus bar voltage may be considered to be unaffected by the cross-current flow; then the equations for i_s and I_s show that their values are as large as the values given in the foregoing paragraphs, for the reason that the resultant series voltage Oq for one machine only acts in a circuit of impedance made up of the armature and leads of only one machine.

The vector diagrams, Figs. 406 and 407, contrast the conditions of a leading and a trailing alternator with respect to bus bars of fixed voltage OA. Figure 406 shows the position of the voltage OB of the trailing alternator and the position of the synchronizing current Oc which tends to accelerate the alternator into step; while Fig. 407 shows the corresponding conditions for an alternator with leading voltage OB and synchronizing current Oc which tends to retard the alternator into step.

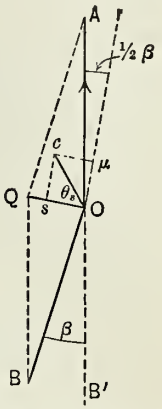


Fig. 406. — Vector Diagram of Voltages and Synchronizing Current when One Alternator in Parallel with Others lags in Step.

In each of these figures,

OA represents the bus bar voltage in phase and magnitude.

OB represents the machine voltage in phase and magnitude.

OQ represents the resultant, or synchronizing, voltage in phase and magnitude.

Oc represents the synchronizing current in phase and magnitude.

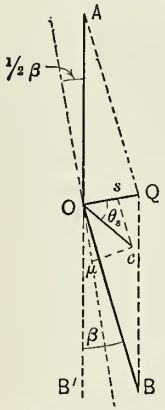


Fig. 407. — Vector Diagram of Voltages and Synchronizing Current when One Alternator in Parallel with Others leads in Step.

Ou represents its wattless, or phasing, component.

β is the angle by which the machine differs from its proper phase for parallel working, OB'.

θ_s is the angle QOc by which the synchronizing current lags behind the resultant voltage.

The figures show plainly that if OA and OB remain constant, then as β increases OQ increases and at the same time Oc and Ou increase. The product $OB \times Ou \times \cos \frac{1}{2} \beta$ is proportional to the synchronizing torque exerted on the machine. It is negative in Fig. 406 and positive in Fig. 407, so that the torque is exerted on the machine and accelerates it in one case, and it is exerted by the machine and retards it in the other case. For the conditions of $OB = OA$ and a given value of β, the torque is a maximum when $Ou \times OB$ is a maximum, which

occurs when $2\pi fL = R$, or $\theta_s = 45°$. In a 2200-volt 500-kilowatt machine giving a frequency of 60 periods per second and having a resistance in the armature circuit of .5 ohm, it is required that $L = .0013$ to give a maximum synchronizing effect, which is smaller than the smallest value which is now commercially attainable in any type of alternator of that size.

An inspection of the figures shows that if the synchronizing current Oc led the resultant voltage OQ, there would be no tendency for the machine to come into parallel step with the bus bar voltage; but this condition will not occur in practice, since it is impossible to build alternators without self-inductance in the windings.

In the foregoing discussion, the usual condition in which a prime mover tends to lag or race in speed is considered; and it is shown how a synchronizing cross current is set up through alternators operating under such conditions, which current tends to hold the generators at speeds giving a common frequency for the voltage. This current causes a transfer of power from a leading machine to its mates or from the mates to a trailing machine. The current is therefore wattless with respect to the system, except for I^2R losses.

162. Division of Load between Parallel Alternators. — Alternators may be completely synchronized but yet fail to properly divide the load of the external circuit between them, as the driving torque of the prime mover of each machine may not be adjusted to exactly overcome at the correct speed the resisting moment of the load which that machine should carry. The condition is illustrated in Fig. 408, where the conditions are the same as assumed in the preceding article except that the external circuit is receiving power. In this figure, OT is the bus bar voltage E which is assumed to be constant in length and is used as the reference phase position with respect to time. The vector of current I in the external circuit is represented by the line OH, which is shown lagging behind the line voltage by an angle θ, and which is dependent for its value upon the character and value of the impedance of the external circuit.

Assume first that two exactly equal machines are connected to the bus bars and are excited so as to give equal total voltages. Further assume that the governors of the driving engines are so adjusted as to give exactly equal torques. Then

the voltage of the generators must be in exact opposition of phase so far as the series circuit through the two armatures are concerned, and no series current will flow, but they are in exact parallel relation with reference to the external circuit. These relations, under these circumstances, are quite similar to that of two equal direct-current generators delivering equal loads to an external circuit.

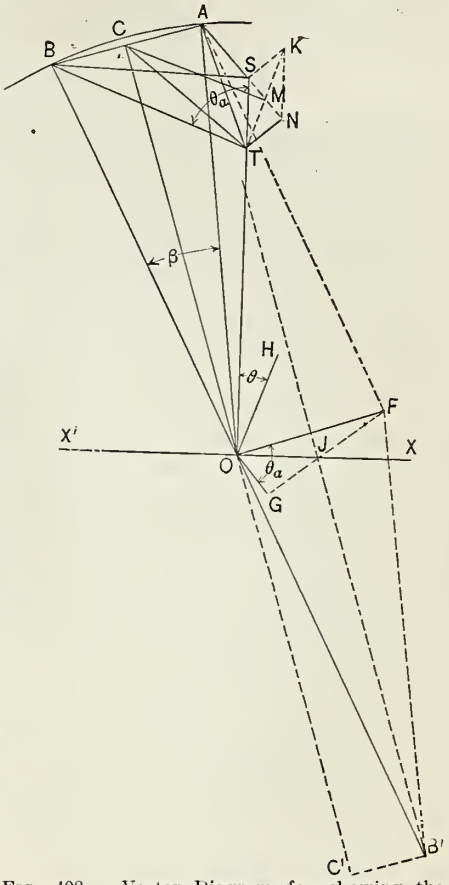


Fig. 408. — Vector Diagram for showing the Division of Current and Load between Two Equal, equally Excited Alternators connected in Parallel.

The value and phase position of the alternator voltage required to furnish the external power, $P = EI \cos \theta$, may be obtained as follows: lay out TM parallel to OH and of such a length that it equals $\frac{1}{2} R_a I$, where R_a is the resistance of one of the armature windings. At right angles to TM lay off MC of a length equal to $\frac{1}{2} X_a I$, where X_a is the synchronous reactance of one of the armature windings. Then $CT = IZ_a = E_a$ represents the voltage drop in the parallel impedances of the two armatures when current I flows. The generated armature voltages must then be vectorially equal to OT plus TC, or $\overline{E}_g = \overline{E} + \overline{E}_a$. When considered with reference to the series armature circuit the total generated voltages may be laid off as OC and OC', as was done in Figs. 405, 406, and 407. The load given out by each machine is,

$$\tfrac{1}{2} P_e = OT \times \frac{TM}{R_a} \cos \theta = \tfrac{1}{2} IE \cos \theta,$$

since $$TM = E_R = \tfrac{1}{2} IR_a \text{ and } \tfrac{1}{2} I = \frac{TM}{R_a};$$

and the total power generated by each machine is

$$\tfrac{1}{2} P_g = \tfrac{1}{2} P_e + \tfrac{1}{4} I^2 R_a.$$

Suppose, now, that the governors of the driving engines are so adjusted that one has a greater torque than the other, but so that the external load is not changed. The two machine voltages will then fall apart in phase, say by the angle β (Fig. 408). Suppose also that the total voltages of the two generators are again equal in scalar value. Construct the isosceles triangle OAB so that the base AB passes through the point C and is at right angles to OC, and the sides OA and OB make the required angle β. OA and OB are the required generator voltages and combine in parallel to send the load current $I = OH$ through the circuit composed of the line in series with the two generator armatures in parallel. The resultant AB being at right angles with the voltage OC it has no effect upon the external current OH. Now draw the lines TA and TB and erect the internal armature voltage triangles TNA and TSB. The current generated by machine A is proportional to the active side of TNA, or TN or $I_A = \frac{TN}{R_a} = \frac{E_R}{R_a}$, when E_R is the active voltage lost in the generator copper. Likewise, the total current flowing in generator B is proportional to TS or $I_B = \frac{TS}{R_a} = \frac{E_R}{R_a}$. As the generators are similar and equal, the ratio $\frac{X_a}{R_a}$ is the same for both and for the two in parallel. Scale TN and TS in terms of currents and then we have their resultant TK equal to the total line current OH or double TM, *i.e.* $\bar{I} = \bar{I}_A + \bar{I}_B$; while their vector difference equals SN and the series current is $\bar{I}_S = \frac{SN}{2} = \frac{\bar{I}_A - \bar{I}_B}{2}$. The total power generated by machine A is the product of the projection of TN on OA with OA and of machine B the product of the projection of TS on OB with OB.

For indicating the series relation, one of the generator voltage vectors can be conveniently reversed. Thus, we obtain

the parallelogram $OAFB'$ which is similar to those of Figs. 405, 406, and 407. The resultant series voltage OF equals BA, while the series current which would flow because of OF is $\frac{OG}{2}$, which is equal to $\frac{SN}{2}$.

If OA and OB are not equal, the triangle OAB is no longer isosceles, but the construction can be carried out in the same manner as above.

However, instead of going further into this special case of two alternators, consider the more general case of one alternator connected to bus bars to which are connected a large capacity of other alternators, so that the bus bar or external voltage remains practically constant in value. In order to find the current and power that will be produced by a machine having an induced voltage E_g of a given scalar value equal to the radius of circle MM with center at O, but of variable phase position, the following construction can be made; lay out OV, as in Fig. 409, equal to the bus bar voltage E, and assume that the driving engine torque is such that the generated voltage E_g makes angle β with $\bar{E}$.

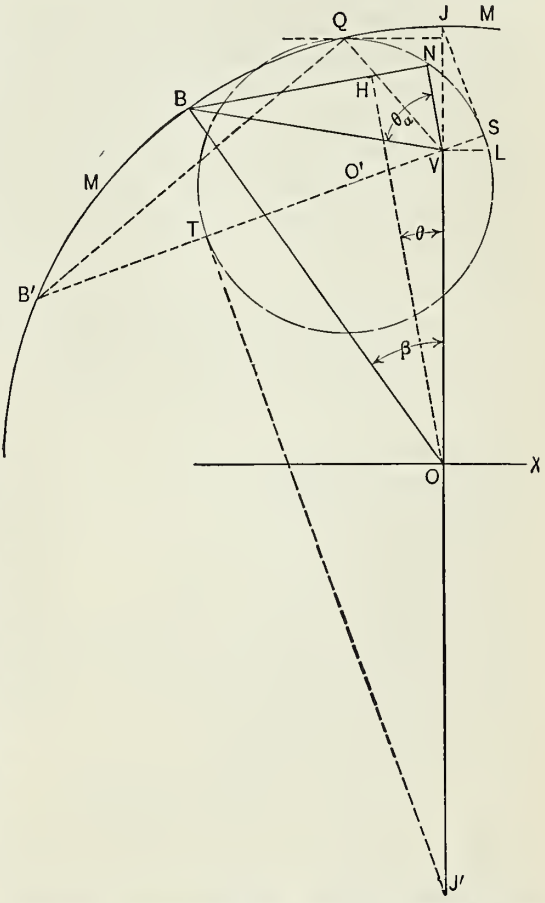


FIG. 409. — Diagram showing Locus of Current for Constant Bus Bar Voltage and Generator Voltage.

Then $OB = \bar{E}_g$. The internal voltage is then BV. Lay out on BV the internal voltage triangle BVN, in which $BV = IZ_a$ is the total loss of voltage internally, and is composed of the active and reactive voltages $VN = IR_a$ and $NB = IX_a$, respectively. In this case, as in those preceding, synchronous impedance and reactance are used. The apparent voltage or impedance triangle in which E_g or $\frac{E_g}{I}$ is the hypotenuse is OHB. The apparent impedances in this case are composed of the self-inductances, field reactions, voltage reactions, and resistances encountered throughout the system in absorbing E_g. With any change of the driving engine torque, which changes β, the form of the triangle OHB changes. In the figure with $\beta =$ angle VOB the generator current leads the bus bar voltage by the angle JVN, or θ is negative.

The side OH, however, must always be parallel to VN, the active voltage line, and the angle OHB is always a right angle.

In finding the current locus as β varies or B moves along its circular locus MM, the following symbols are used:

The generated voltage,	$OB = E_g$.
The bus bar voltage,	$OV = E$.
The angle between E_g and E,	$VOB = \beta$.
The internal voltage,	$BV = E_Z$.
The internal active voltage,	$VN = E_R$.
The internal reactive voltage,	$NB = E_X$.
The armature current scaled to equal	$VN = I$.
The internal angle of lag,	$NVB = \theta_a$.
The angle between I and E,	$NVJ = \theta$.
The armature resistance	$= R_a$.
The synchronous armature reactance	$= X_a$.
The synchronous armature impedance	$= Z_a$.

The locus of the point B is determined by the formula for the triangle OBV,

$$E_g^2 = E_Z^2 - 2\,E_Z E \cos[180^\circ - (\theta_a - \theta)] + E^2,$$

where θ is intrinsically positive or negative depending upon whether it is to the right or left of OJ; in the figure for point B it is negative. When E and E_g are of constant values and

θ is variable, this is also the formula in polar coördinates of the circular arc MM having the radius E_g and the pole at V. To find the locus of the current when θ varies, the above formula may be modified by multiplying both sides by $\cos^2\theta_a$, which gives

$$E_g^2\cos^2\theta_a = E_R^2 - 2\,E_R(E\cos\theta_a)\cos[180^\circ-(\theta_a-\theta)] + E^2\cos^2\theta_a.$$

This is evidently the formula in polar coördinates of a circle having a radius $E_g\cos\theta_a$ and in which the initial line is on a diameter; also one of the polar coördinates E_R, or taken to proper scale it is $\frac{E_R}{R}$, which equals I, the current flowing in the armature. Now, swinging E_g over to the position OJ, the triangle of internal voltages becomes VSJ, the angle JVS being equal to θ_a. When E_g is thus in phase with E, the internal drop $E_Z = VJ$ and equals $E_g - E$. Therefore, under those conditions the active voltage is

$$E_R' = VS = (E_g - E)\cos\theta_a.$$

Likewise, when E_g is swung around to OJ', the active voltage is

$$E_R'' = VT = (E_g + E)\cos\theta_a.$$

By this construction TVS is a straight line and its value is

$$TS = E_R' + E_R'' = 2\,E_g\cos\theta_a.$$

One half of this or $E_g\cos\theta_a$ is the radius of the circular locus for E_R given above. Therefore, bisect the line TS, as shown at O', and describe the side TNS by the radius $O'T = E_g\cos\theta_a$. When E_g is at the position OJ the second coördinate is $O'V$, and

$$O'V = O'S - VS = E_g\cos\theta_a - [E_g - E]\cos\theta_a,$$

or

$$O'V = E\cos\theta_a,$$

which is the value given in the formula. V is evidently the origin and TS the initial line for the circle TNS as given by the formula. For the pair of coördinates VS and VO' the angle included is 180°, which is correct as $180^\circ-(\theta_a-\theta)$ is 180° when $E_g = OJ$. The circle TNS is then the locus of the active internal voltage E_R, which has its pole at V and its initial

line TS; or, if the scale is changed by a ratio equal to $\frac{E_R}{R}$, it is the locus of the armature current.

The power given out is

$$P = EI \cos \theta,$$

where θ is the phase angle between E and I or is NVJ when E_g is at the position OB. The maximum power is reached when E_g has swung forward so that I takes the position VQ, where Q is the point of tangency of the current locus to a perpendicular to OV. At this point E_g has swung to OB', where $\beta = \theta_a$. If the driving torque is great enough to drive the armature farther ahead, the machine will break from synchronism. The minimum power is found when E_g has fallen back so that I lies at the position VL, which is perpendicular to $OV = E$. If the torque of the driving engine is reduced so that E_g falls still farther back, current is driven through the armature by the voltage of the bus bars and the machine runs as a synchronous motor.

Heretofore we have dealt with a machine in which the induced voltage was kept constant. An equally important consideration is the effect of the variation of this voltage. Assume that the load, including the internal copper losses, and the bus bar voltage are kept constant, but that the generator excitation is varied. In order that the load may be kept constant, the torque of the driving engine, as determined by the setting of its governor, is kept constant. Evidently, so long as the driving engine gives constant torque and the machine is held to its synchronous speed, the total power must be constant irrespective of variation of the excitation of the alternator field magnets or other variables. In Fig. 410 the triangle OHB is similar to the triangle OHB in Fig. 409. In this case the power given to the bus bars is as before

$$P = EI \cos \theta.$$

The total power generated which is to be kept constant is

$$P_t = E_g I \cos (\beta + \theta),$$

or from the figure,

$$P_t = E_g I \sin (\delta + 90^\circ - \theta_a).$$

Therefore,

$$P_t = E_g I \{\sin \delta \cos (90^\circ - \theta_a) + \cos \delta \sin (90^\circ - \theta_a)\},$$

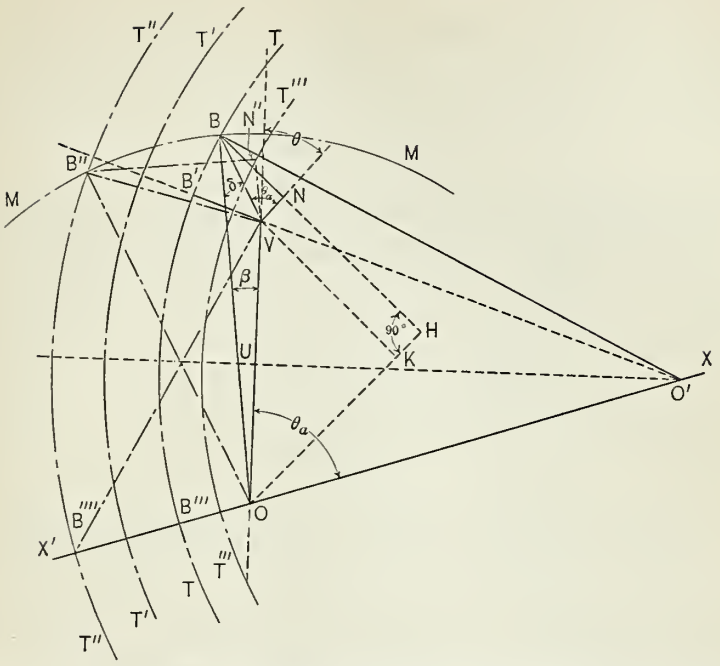


FIG. 410. — Voltage Locus and Diagram of a Generator running in Parallel when its Excitation is varied.

or

$$P_t = E_g I (\sin \delta \sin \theta_a + \cos \delta \cos \theta_a).$$

But from the triangle OVB it is seen that

$$E \sin \beta = E_Z \sin \delta,$$

or

$$\sin \delta = \frac{E \sin \beta}{E_Z} = \frac{E \sin \beta}{IZ_a},$$

and

$$E_g = E_Z \cos \delta + E \cos \beta,$$

or

$$\cos \delta = \frac{E_g - E \cos \beta}{IZ_a}.$$

The symbols used here are the same as in the discussion of Fig. 409. Substituting these values in the final formula given for P_t, there results

$$P_t = E_g I \frac{E \sin \beta}{IZ_a} \sin \theta_a + E_g I \frac{E_g - E \cos \beta}{IZ_a} \cos \theta_a,$$

or $$P_t = \frac{E_g}{Z_a} E \sin \beta \sin \theta_a + \frac{E_g^2}{Z_a} \cos \theta_a - \frac{E_g}{Z_a} E \cos \beta \cos \theta_a,$$

and $$P_t = -\frac{E_g}{Z_a} E \cos (\beta + \theta_a) + \frac{E_g^2}{Z_a} \cos \theta_a.$$

The variables in this expression are E_g and β. Multiplying through by $\frac{Z_a}{\cos \theta_a}$ to give a term containing only E_g^2, there results

$$\frac{P_t Z_a}{\cos \theta_a} = -\frac{E}{\cos \theta_a} E_g \cos (\beta + \theta_a) + E_g^2.$$

If we add to each side of this $\frac{E^2}{4 \cos^2 \theta_a}$, we have

$$\frac{P_t Z_a}{\cos \theta_a} + \frac{E^2}{4 \cos^2 \theta_a} = \frac{E^2}{4 \cos^2 \theta_a} - \frac{E}{\cos \theta_a} E_g \cos (\beta + \theta_a) + E_g^2,$$

which is the formula for a circle expressed in polar coördinates, where the left-hand member is the square of the radius and where the initial line passes through the center, $\frac{E}{2 \cos \theta_a}$ is the coördinate between the initial point and the center of the circle, E_g is the variable coördinate, and $(\beta + \theta_a)$ is the angle included between the two. Then, to construct the locus of E_g, when β is varied by changing the excitation but the total generated electrical power P_t remains constant, take some point O (Fig. 410) as the pole. Lay off $E = OV$, vertically for convenience. At the angle θ_a to the right of OV lay off the initial line XX'. Measure off the distance $\frac{E}{2 \cos \theta_a}$ from O, giving OO'. With O' as a center, describe the arc TT having a radius equal to the square root of the sum of the terms on either side of the polar equation. This is the required locus of E_g, which for the field excitation that gives it the value OB, shown in the figure, leads the terminal voltage OV by the angle β. From the relation given, it is seen that the perpendicular $UO' = \frac{E}{2} \tan \theta_a$, and $UO = \frac{E}{2}$; therefore O' may be at once found by laying off the

distance $\frac{E}{2}\tan\theta_a$ on the perpendicular bisector of VO. The required locus is then found as before by striking a circle having the radius

$$\frac{P_t Z_a}{\cos\theta_a} + \frac{E^2}{4\cos^2\theta_a}.$$

Since the total power generated in the windings is kept constant, the output will be a maximum when $\bar{E}_Z (= VB)$, which varies with the current and therefore with the copper loss, is at a minimum. This occurs when $\bar{E}_Z$ lies on a radius of the locus TT and therefore when BV extended passes through O', that is, when the field excitation has a value which makes the induced voltage equal to OB' and the drop of voltage in the armature is therefore $B'V$. From the geometrical construction of the figure it can also be seen that at this point $\bar{E}_R$ takes a position on OV extended and hence that the external power factor is unity. If the excitation be made less than when $\bar{E}_g = OB'$, the current is larger and leads the bus bar voltage E; and when the excitation is increased, as when $\bar{E}_g = OB$, the current lags behind E. The lagging and leading components are so related to the series circuits, comprised of the generator under consideration and the other generators (which maintain the constant terminal voltage E), that their effects tend to be respectively those of leading and lagging motoring currents. Thus, when the generator voltage is low, it tends to cause the voltage of other machines connected to the line to fall off and when high to increase. This is explained more fully in the article dealing with the synchronous motor.* Substituting w for $\frac{E^2}{4\cos^2\theta_a} + \frac{E}{\cos\theta_a} E_g \cos(\beta + \theta_a) + E_g^2$ in the last equation given, an expression for P_t may be found in which w stands for the radius of the circle locus of E_g with fixed power generated (*i.e.* circle TT of Fig. 410, which may be called a **Power circle**), thus :

$$P_t = \frac{\cos\theta_a}{Z_a}\left(w - \frac{E^2}{4\cos^2\theta_a}\right).$$

Draw a number of power circles concentric with TT with any convenient radii, such as $T'T'$, $T''T''$, and $T'''T'''$. Then, for any given or fixed voltage E_g, the current and voltage relations

* Art. 167.

are immediately determinable for any power. Thus, take E_g equal to OB, Fig. 410, and draw a voltage locus MM as in Fig. 409; then if the power increases from that corresponding with radius $O'B$ to that corresponding with radius $O'B''$, E_g swings over from OB to OB'' and the current swings forward from the lagging position VN to the leading position VN'' with respect to the terminal voltage.

The curves of current for constant power and varying excitation, when plotted with generator voltages as abscissas and armature currents as ordinates, are sometimes called V-curves from their shape.* They can readily be obtained from the figure by giving E_g a series of values.

Prob. 1. Given an alternator having an armature resistance of $\frac{1}{2}$ an ohm and synchronous reactance of 4 ohms. Construct the locus diagram (Fig. 410) of the generator induced voltage when the bus bar voltage is 1000 volts and the generated power is 100 kilowatts. When the power is 50 kilowatts. When 25 kilowatts.

Prob. 2. Determine from the diagram of Prob. 1 the power output of the generator when the total powers are 100, 50, and 25 kilowatts.

163. Maximum Possible Load and the Regulation of Prime Movers in Parallel Operation. — An engine torque greater than the resisting torque of the maximum possible electrical load, will, of course, cause the machine to break from synchronism. With very large machines, the excessive currents that result from such a condition may cause injury to the armature conductors by the excessive magnetic forces set up between the conductors by the large currents which may occur before the circuit breakers get open; for at the ends of the armature core where the windings extend beyond the core the conductors are sustained in position by insulating materials of comparatively small tensile strength. This is similar to the case of transformers.†

The maximum possible load which a prime mover can deliver to its alternator, which is operating in parallel with others under the conditions explained in the last article where dealing with Fig. 410, is readily determined from the construction of that figure. Thus, with fixed field excitation, the generated voltage OB swings to the left when increased power is supplied from the

* Art. 170. † Art. 131.

prime mover, because the increased driving torque tends to make the machine race. The vector OB therefore cuts larger and larger power circles (TT, $T'T'$, etc.) as the engine power increases until, if the power becomes sufficiently great, it lies upon the initial line or radius XX'. At this point it reaches the largest power circle possible for the particular field excitation of the alternator. Then the angle $\beta = 180° - \theta_a$. Any greater power from the prime mover causes the point B to move toward smaller power circles; the speed increases and the generator falls out of synchronism. Reducing the excitation may likewise then throw the generator out of synchronism. Thus, suppose the machine is running under constant power represented by the power circle $T''T''$. Then suppose that the voltage is reduced to OB'''' (Fig. 410), making the triangle OVB'''' of the bus bar voltage $\bar{E} = OV$, $\bar{E}_g = OB''''$, and the internal voltage drop $\bar{E}_z = B''''V$. Any smaller excitation, giving a smaller value of OB'''', will cause the point B'''' to touch a smaller power circle and the engine will speed up, causing the generator to break from synchronism. The excessive currents caused to flow in either of these limiting cases, as indicated by the large internal drop $B''''V$ in the latter and an even larger drop in the former when the larger constant generated voltage OB was used, are far beyond the capacity of the windings of modern commercial alternators. Further, when the voltage drop line approximates the position $B''''V$, the reactive effect upon the field magnets of the generators connected to the bus bars would be very great and cause the break from synchronism to occur before the voltage vector had been rotated to a point where it reached the line XX'. It is evident, therefore, that destructive results may follow an excessive lead imposed by the prime mover, such as might occur through accident to the governor, or great weakening of the field current. In studying Figs. 408, 409, and 410, it is well to note that the armature angle of lag θ_a has been made smaller than is likely to be found in commercial machines.

When alternators not rigidly connected together by mechanical means are operated in parallel, it has been shown above that the load division is dependent on the relative torques which the prime movers tend to exert upon the rotors of the generators to which they are respectively attached, and consequently the division of load between alternators operated in parallel is de-

pendent on the speed regulation of the prime movers. If one prime mover tends to lag in speed behind the other, it produces less power than is the case if it tends to lead. As explained* the lagging machine under these circumstances is relieved of load by the leading machine. This may be considered to be due to the accelerating effect on the lagging machine caused by the synchronizing current, or to the resultant reduction of load on the lagging machine and its consequent increase on the leading machine. The instantaneous torque of the leading machine may be considered equal to that caused by its load sent to the external circuit plus the torque exerted upon the lagging machine to keep it in step, while the torque of the lagging machine (if of equal internal impedance and equally excited) may be considered equal to that of an equal load to the external circuit minus the torque expended upon it by the leading machine. Evidently, when one of two equal steam engines driving alternators in parallel has its governor set for a higher speed than the other, it acts automatically and takes more steam when held below its normal speed by the more sluggish second engine, while the second engine, being urged forward above its normal speed by the first engine, automatically takes less steam. Thus, the torque of the first engine is greater than that of the second. As the two engines are running at the same or synchronous speed, the first must then take a larger proportion of the load than the second.† It is then seen that the division of load on alternators in parallel must be accomplished by regulating the governors of the prime mover. This is the ordinary practice. The governors of several engines are not apt to regulate exactly alike, and the parallel operating alternators which are connected to them are apt to divide their loads unequally, that is, one machine "robs" the others, unless the governors are adjusted with great care. It is sometimes desirable to fasten the governors of all the engines except one at the point of full load and permit the governor of the one engine to maintain the speed regulation for all and carry the variation in load. Then when the speed tends to rise, the governed machine tends to retard the others by absorbing a cross current, that is, it shifts its load on to its mates and thereby prevents their racing. When the speed tends to fall, the governed machine tends to take on the excess load.

* Art. 161. † Art. 162.

164. Effect of Form of Voltage Curve and Variation of Angular Velocity on Parallel Operation and Division of Loads. — The able designer, C. E. L. Brown, early operated his own alternators with smooth iron armature cores in parallel with Ganz alternators which have highly reactive pole-type armatures.* The former machine gave a voltage curve which approximated a sinusoid, while the curve given by the latter was quite irregular and peaked. Dr. Steinmetz also early operated machines with smooth iron armature cores in parallel with others having toothed cores.† These experiments showed that machines with different voltage curves can be run together satisfactorily, as is now very frequently done, but they require a considerably

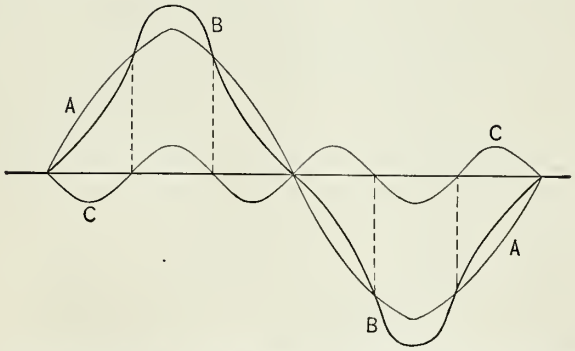


FIG. 411. — Curves Voltage to Cause Flow of Harmonic Cross Currents between Paralleled Alternators having Voltage Curves like *A* and *B*.

increased exchange of current as compared with the synchronizing current of machines with voltage curves which are exactly alike. For, since the unlike curves cannot coincide even when the machines are in exact step, a current of more or less irregular form must be exchanged by the machines, and this is superposed upon the true synchronizing current. This superposed current may have a very different frequency from that of the machines and is not necessarily wattless. For instance, in Fig. 411 curves *A* and *B* represent the voltage curves of two alternators operating in parallel, in exact step, and with the same effective voltage. The instantaneous differences between the two curves leave residual voltages forming the curve *C* which acts to set up a cross current in the series circuit comprising the two armatures.

* *Jour. Inst. E. E.*, Vol. 22, p. 600. † *Electrical World*, Vol. 23, p. 285.

Also, higher harmonic cross currents which may flow when polyphase armatures which are operating in parallel act much as they do in polyphase induction motors.* The third harmonic sets up a rotary armature magneto-motive force opposed to that of the field magnet. It has already been pointed out that the armature currents in a polyphase alternator tend to set up a rotary field which reacts upon the alternator field magnet.† When the cross currents have sufficient magnetizing force, they may cause the machines to work unsatisfactorily in parallel.

The angular velocity of reciprocating engines used for driving dynamos is by no means uniform throughout the revolutions, although the revolutions may be isochronous. Curves showing the instantaneous crank velocities taken from engines of well-known types are often quite irregular. If alternators are driven from separate engines, this irregularity of angular velocity may be the cause of more or less difficulty in parallel working, and some machines may work quite satisfactorily in parallel when driven from the same countershaft or from separate hydraulic or steam turbines (which give a uniform angular velocity), when they will not do so if driven by separate reciprocating engines. It is evident that when engines of different types are used to drive alternators that it is desired to parallel, the angular velocities may vary so differently for the different engines that the synchronizing current may be unable to hold the alternators together. It is therefore desirable to specify, in obtaining reciprocating engines for parallel working, a minimum variation in angular velocity. This variation should not cause a displacement of the rotor from its normal position of more than two electrical degrees during the time of one cycle.‡

Sometimes it is necessary to cause reciprocating engines to move with their cranks in the same angular space relation, though it is difficult to do this, and it results in the variation in frequency due to variation in angular speed being transmitted to other synchronous apparatus connected to the system. These other machines are then apt to hunt.§

165. Synchronizers and Synchronizing. — Mechanical imperfections in engine governors and machine pulleys cause slight

* Art. 198. † Art. 152. § Art. 173.

‡ Standardization Rules of Amer. Inst. of Elect. Eng., Rules 61–64.

differences in the speeds of machines intended to run at equal velocities. Consequently it is desirable to arrange some device for determining the moment a machine is in synchronism with one with which it is to be thrown in parallel. When the alternators are to be connected in parallel, the terminals of each may be connected directly to appropriate bus or main conductors through convenient indicating instruments, switches, and safety devices. Before switching a new machine upon the bus conductors it should be brought to normal speed and to the voltage of the other machines. Then at a moment when it is in synchronism and in step with the voltage wave of the bus bars, it may be switched into circuit without causing a disturbance among the other alternators.

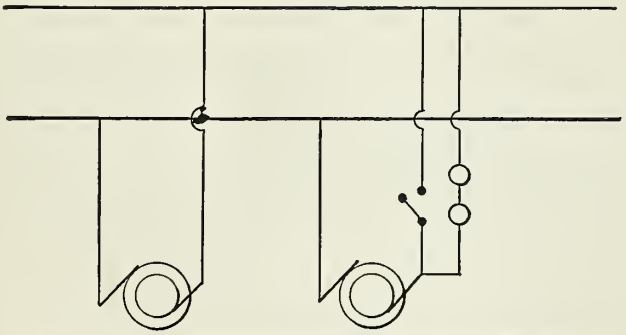


FIG. 412. — Simple Lamp Synchronizer, One Phase only shown.

Any device for indicating the synchronous relation is called a **Synchronizer.** Its simplest form for low voltage machines consists of one or more incandescent lamps in series, which are connected as in Fig. 412. One terminal of a phase of the alternator is connected directly to a bus conductor, while the other is connected through the lamps to another bus. When the voltage waves are equal but not in opposition, the lamps are illuminated, and at the moment of opposition the illumination dies out. If the frequencies of the alternator and the bus bars differ materially, the flashes of illumination or "beats" are quite rapid. As the frequencies approach synchronism, the beats lengthen out, exactly as do the beats of two tones which are approaching unison. The alternator should be connected to the bus bars at an instant of no illumination during a period

when the beats are fairly long. This indicates that the voltage of the alternator is substantially in synchronism with that of the circuit and that it is in proper step or angular phase position. Continued darkness of the lamps under these circumstances can only occur when the machines produce the same voltage and run at absolutely the same frequencies, which is not a practical occurrence unless the machines are rigidly connected together.

For polyphase machines the lamps may be connected to a single phase, as shown in Fig. 412, since if one phase is in step with the bus bars the others must be, if the machines are properly connected to the bus bar leads. When connecting a polyphase machine for the first time to the bus bars, much care must be taken to give assurance that the various phases are in proper relation, or disastrous results may follow from the flow of excessive currents caused by the reversed relation of one or more phases of the machine with reference to their respective bus bar phases. The correct connections can be obtained by placing synchronizing lamps, or equivalent devices, in each lead from the alternator to the bus bars. When the connections have been properly made, the synchronizers will show opposition of voltage phase between each alternator lead and its respective bus bar. Synchronizing lamps connected as in Fig. 412, placed in each lead, should all be dark together.

Since alternators are commonly built for high voltages, it is usual to use a transformer with the synchronizer lamps. The primary winding of this transformer may be composed of either one or two circuits. When it is composed of one winding, one terminal of the alternator is connected to a bus bar through it, the other terminal of the alternator being connected directly to the other bus bar (Fig. 413). In this case the lamp on the secondary circuit acts exactly as the synchronizer lamps already described. When the primary winding is composed of two circuits, one is connected between the bus conductors and the other between the terminals of the alternator (Fig. 414). In this case the proper phase relation for switching the alternator into circuit may be indicated either by darkness or by illumination of the lamps, depending upon the connections of the primary windings of the synchronizer transformer. This arrangement is advantageous, since it allows the use of a double-pole

switch at the dynamo, while the previously described arrangements require the use of single-pole switches. The latter arrangement may be modified by using two separate transformers,

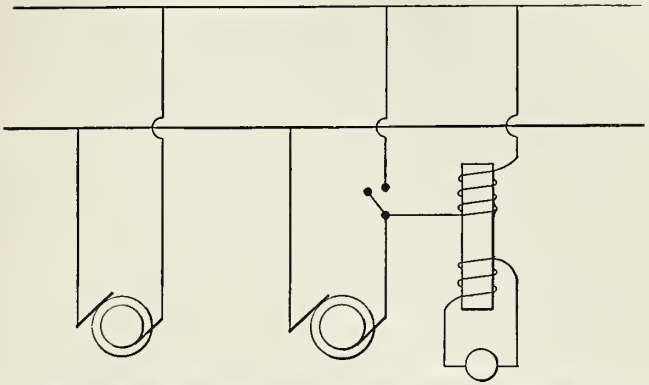


FIG. 413. — Synchronizer Lamp connected to the Secondary Winding of a Simple Voltage Transformer.

the primary circuit of one being connected to the alternator and that of the other to the bus bar circuit. The secondary windings are connected together in series with one or two lamps.

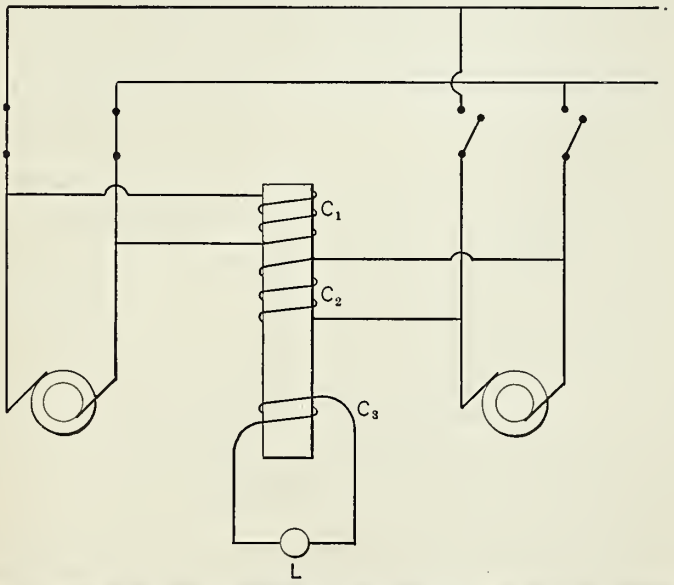


FIG. 414. — Sychronizer Lamp connected to the Secondary Winding of a Transformer having Two Primary Coils.

If the secondary windings are connected directly in series, as in Fig. 415, darkness of the lamps indicates the instant for connecting the alternator to the bus conductors. If the secondaries are cross-connected as in Fig. 416, maximum illumination of the lamps indicates the moment when the machines are in proper step. In this country the general practice has been to connect lamp synchronizers so that darkness indicates when the machines are in step. This has the evident advantage that darkness is a condition which is more readily distinguished than the condition of maximum brightness. This practice, however, is not always followed.

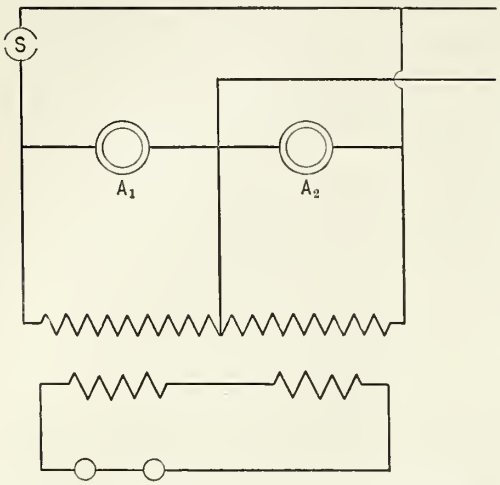


FIG. 415. — Synchronizer Lamps connected in the Secondary Circuit of Two Transformers having their Secondary Windings in Series.

In any of these methods, the lamps may be replaced by a sensitive high resistance alternating-current amperemeter or galvanometer which is dead-beat.

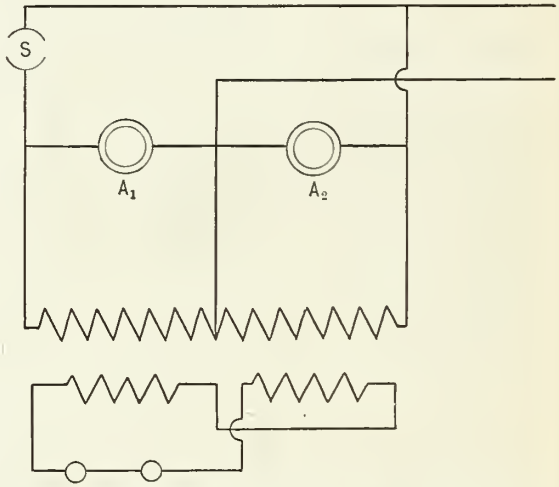


FIG. 416. — Synchronizer Lamps arranged with Two Transformers, where One of the Secondary Windings is reversed with Respect to the Other.

An ingenious device for use as a synchronizer, which consists of two electromagnets made with

iron wire cores, has been used sometimes. The windings of these are connected respectively to the alternator and the bus bar circuit. Each magnet has placed in front of it an iron diaphragm which emits a tone which has a pitch due to the frequency of the current flowing in the winding. In front of the magnets are placed resonators which magnify the sound emitted by the diaphragms. When the two tones are not in exact harmony, interference causes beats, and the synchronizer emits an intermittent sound. If the speeds of the machines are brought nearer and nearer to synchronism, the beats become less rapid. At exact synchronism, the beats die out and a clear tone results. The alternator may or may not be in step when the clear tone is given, but a small machine may be safely thrown into circuit, for the interaction of the current waves will bring it into proper phase relations. This synchronizer was designed for use with synchronous motors. It cannot give satisfaction with large alternators or with alternators that are furnishing current at constant voltage for incandescent lighting, since placing an alternator in the circuit while it is out of step is likely to momentarily disturb the voltage. A visual synchronizer, having much the same principle, consisting of a set of vibrating reeds, can also be constructed.

It is possible to do without a synchronizer when connecting small alternators in parallel, provided they are driven at approximately equal speeds. In this case if the alternator is brought to the proper voltage and is then connected to the bus bars, the machine reactions will bring it into step. This plan was practiced to some extent in one or two earlier American plants, but it is vicious in its working unless inductance coils of considerable magnitude are introduced in circuit with the incoming alternator. Throwing machines on to the bus bars without synchronizing usually causes a disturbance in the voltage, and it also strains the armature even of a small alternator on account of the sudden torque impressed upon it to bring it into step, and in large machines is apt to cause destructive strains in the ends of the armature coils. In order to prevent an undue rush of current when alternators are thrown together while slightly out of step, external inductance coils are sometimes inserted temporarily in series between the machines and the bus bars.

A transformer having three legs and windings like the three-phase transformer of Fig. 294 may be used for operating the synchronizing indicator. The windings of the two outer legs are attached respectively to the bus bars and the machine, and a voltmeter or lamp is connected across the middle winding. At the point of synchronism the magnetic fluxes produced by the two exciting coils are in parallel in the middle core and the voltmeter shows a maximum reading.

An instrument which indicates whether the machine to be synchronized is running fast or slow, as well as the condition of step, is sometimes called a **Synchroscope** or **Phase indicator.** One of the simplest arrangements of this kind consists of a disk attached to the alternator shaft or that of a small synchronously running motor. This disk is painted in alternately white and black sectors, each sector having an angular width equal to the polar pitch. In front of the disk is placed an arc lamp attached to the bus bars through the necessary transforming device. The arrangement is the same as that sometimes used for measuring the slip in an induction motor.* Then when the illuminated sectors of the disk apparently stand still, the frequency of the machine is the same as that of the bus bars. When the illuminated sectors of the disk apparently move in the direction opposite to the direction of the rotation of the rotating part of the machine, the speed of the latter is too slow; and when the illuminated sectors apparently move in the direction of the rotating part, the speed of the machine is too fast. The step phase when the bus bars and machine are in synchronism is indicated by the position of the stationary sectors, but the additional use of a lamp synchronizer is desirable.

In a three-phase machine the use of three lamps, as shown in Fig. 417, may be adopted. In this case the three lamps, through suitable transforming devices, are placed at the corners of a triangle and a terminal of each connected to one of the three bus bars, while the other three terminals are connected to one phase wire of the machine. When the machine is in synchronism, the lower right-hand lamp in the figure is in darkness while the other two are equally bright. When the speed of the machine is slightly too high, the lamps glow one after the other, giving the appearance of a rotation in one direction; and when

* Art. 189.

the speed is too low, the rotation of the lighting is in the other direction. Various modified arrangements to accomplish this result may be made.

Synchroscopes using the magnetic relations of a movable coil to which a pointer is attached are now generally used in commercial plants and are eminently convenient and satisfactory. A common form has an armature which is somewhat similar to the field magnet of a two-coil, split-phase induction motor without its iron core. A two-pole field magnet, *C*, embracing the armature may have its coil connected to the bus bars (Fig. 418), and the split-phase armature winding, its two coils *A* and *B* wound at right angles and its shaft pivoted between the pole pieces of the field, may be connected to the machine to be synchronized, or *vice versa*. A pointer is attached to the armature for the purpose of indicating its motion. When the currents in the field winding and armature windings are in synchronism, the armature takes an angular position with reference to the field magnet which is determined by the phase difference between the currents entering the field and armature windings, the zero position being assumed when the currents are in step with each other. When the frequency of the machine is greater than that of the bus bars, the armature of the syn-

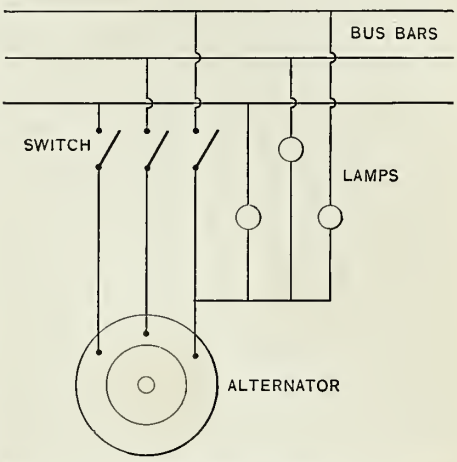


FIG. 417. — A Three-lamp Synchroscope for a Three-phase System.

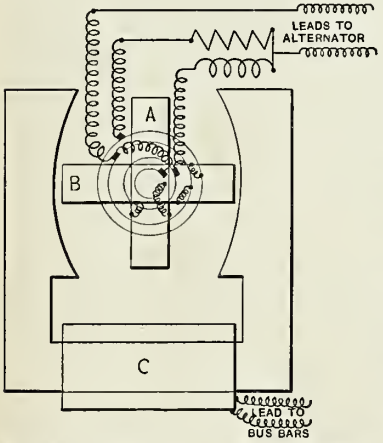


FIG. 418. — Diagram of a Synchroscope of Split-phase Type.

chronizer rotates in one direction; and when the frequency of the bus bar voltage is the greater, the rotation is in the opposite direction. Thus, when the machine is in exact step and synchronism with the bus bar voltage, coil A takes a position with its phase at right angles to the flux of the field. B is then inert, its time phase being 90° from that of the field. When the phases change, say 45°, A and B then equally tend to place themselves at right angles to the field flux, and the armature must take up a position with A 45° from its former position. At 90° difference between the machine and bus bar phases, A is inert and B takes a position at right angles to the field flux. At 180° difference of phase A takes a position 180° from its first position, and so on. Then if the machine voltage is in synchronism but out of step with the bus bar voltage, the pointer takes a definite angular position; but if the machine frequency is different from that of the bus bars, the pointer must make one complete revolution in one direction for each cycle gained and in the other direction for each cycle lost.

Considering the two coils A and B wound on the armature at right angles to each other and traversed by currents in quadrature, then the effect of the windings is to set up a rotating magnetic field which makes its complete rotation with a frequency equal to the frequency f' of the currents in the coils.* The field magnet directs a flux through the armature which has the frequency f of the current through the field winding. The armature tends to place itself in such a position that its flux shall be at all times in the same direction as the flux of the field magnet, but during one period of the field current the rotating field of the armature will rotate through $\frac{f'}{f}360°$, where f is the frequency of the current in the field coil and f' the frequency of the currents in the armature coils. Therefore the armature core will turn through an angle of $\left(1-\frac{f'}{f}\right)360° = \frac{f-f'}{f}360°$ during each period of the field flux, which is equivalent to saying that the armature will rotate $f-f'$ revolutions per second since the field flux makes f complete cycles per second. If f' is smaller than f, the armature core will rotate in the same direction as the rotating magnetic field set up by its

* Art. 186.

windings, but at a speed which is slower in the ratio of $\frac{f-f'}{f'}$; and if f' is the larger, the armature will rotate in the opposite direction, but also with the speed ratio $\frac{f-f'}{f'}$. The bus bars being permanently connected to one of the synchroscope circuits, the machine to be synchronized must be connected to the other circuit, and the direction of rotation of the needle then always indicates whether the incoming machine is running too fast or too slow. When the pointer of the instrument stands still, the two frequencies are equal. If the currents in A and B stand respectively in phase with and 90° behind the voltage impressed on their joint circuit and the field flux is 45° behind the voltage impressed on the field magnet circuit, the pointer will stand at zero when the voltages of bus bars and machine are in both synchronism and step. The two differing phases can be obtained for the armature coils by the use of inductance and resistance as shown in the figure, or where the system is polyphase, directly from the machine or bus bars.

Various other arrangements for accomplishing the result described in the preceding paragraph are also possible.

Automatic synchronizers in which the main switch is closed by the action of an electro-magnet with its coil in the synchronizing circuit are sometimes used, though they have not fully shown their desirability. The arrangement usually comprises an electro-magnet in series with the synchronizer lamp circuit, the armature of which either directly trips the main switch, which closes by a spring, or closes a switch which in turn causes a strong electro-magnet attached to the main switch to become excited, and through the attraction of it the armature closes the main switch.

166. Methods of Connecting Alternators in Parallel to Feeder Circuits. — Switchboards for connecting alternators in parallel are almost as various in details of construction as are the number of switchboards built, that is, each board is designed to meet the conditions it must fill, although it is usual to follow general standards. They are usually of much the same character so far as relates to the underlying duty they are to perform. Figure 419 is the diagram of a suitable board designed by one of the large manufacturing companies for controlling

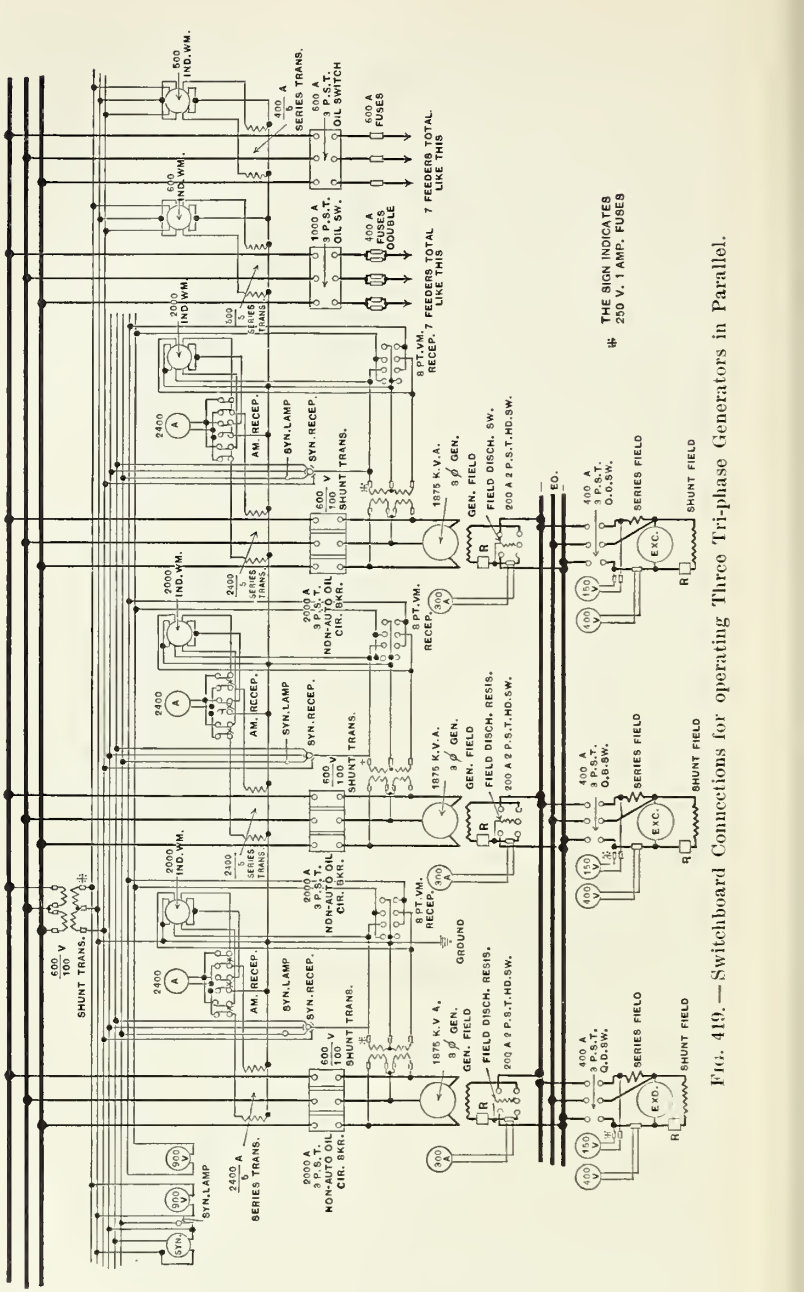


FIG. 419. — Switchboard Connections for operating Three Tri-phase Generators in Parallel.

tri-phase alternators. The lower heavy lines represent the exciter bus bars and connections, the upper heavy lines represent the alternating current bus bars (which are often duplicated for safety but are shown here in only one set) and connections, and the light lines show the measuring instrument connections. The feeders to the load are shown at the right, and the synchronizing apparatus and voltmeters to the left. The details of the connections may be readily obtained from a study of the figure. Additional generators and feeders can be added to the system shown in the figure by extending the bus bars. Ordinarily all high voltage is kept away from the face of the switchboard, and the highest voltage switches are usually situated at a distance in fireproof cells and operated by suitable mechanical or electrical transmission, the high voltage bus bars being located in fireproof ducts. Thus Fig. 420 shows a cross section of a switchboard in which the high voltage oil switches and the current and voltage transformers for operating the switchboard instruments are placed on a framework at the rear of the board. The instruments and controlling levers are in front of the control board. The switches may be made to act as automatic circuit breakers by adding suitable automatic tripping devices. A switch or circuit breaker with

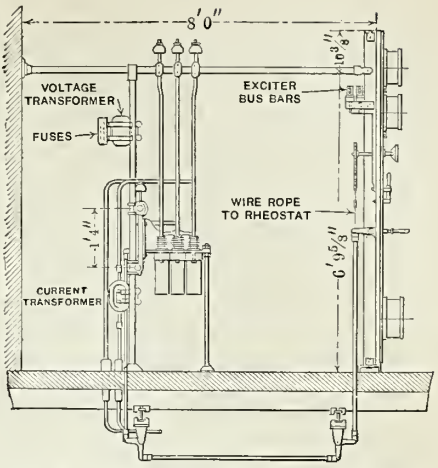


FIG. 420. — Cross Section of an Alternating-current Switchboard.

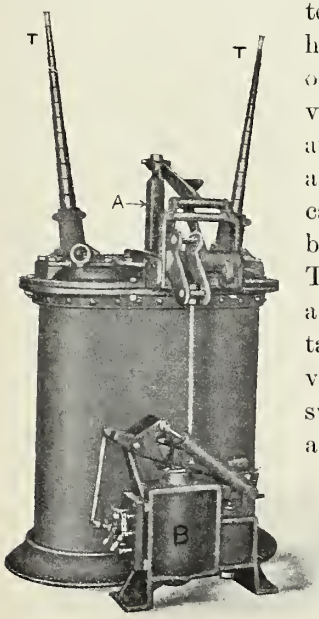


FIG. 421. — View of High Voltage Automatic Switch.

its parts immersed in oil, called an oil switch or circuit breaker, suitable for voltages of several tens of thousands of volts is shown in Fig. 421, and in Fig. 422 is a cross section of the same apparatus. The contact parts are immersed in oil in the large tank and the contact is opened and closed through the intervention of the wooden connecting rod *A*. Movement for operating the switch is transmitted to the rod *A* by means of the solenoid *B* and associated links. The current for the solenoid is controlled from the control board. A small electric motor may be substituted for the solenoid. The terminals for leading current into and out of the switch are shown at *TT*, Fig. 421.

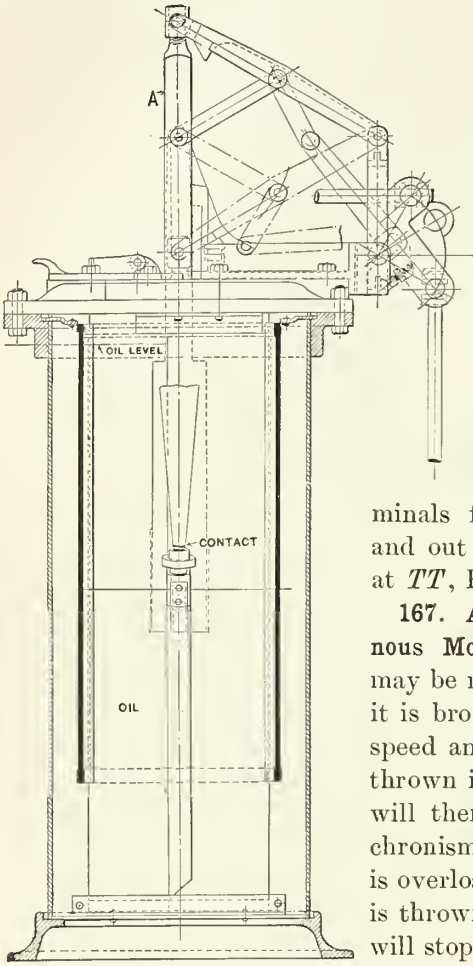


FIG. 422. — Section of High Voltage Switch.

167. Alternators as Synchronous Motors. — Any alternator may be run as a motor, provided it is brought up to synchronous speed and into step before it is thrown into circuit. The motor will then run in complete synchronism if left to itself. If it is overloaded, or by other means is thrown out of synchronism, it will stop. In general, the action of an alternator used as a synchronous motor is quite similar to that of an alternator operated in parallel with another. A great disadvantage of single-phase synchronous motors is the fact that they are not self-starting: but must be brought up to speed before they will operate. The starting of a single-phaser may be done by a small series-wound auxiliary

motor made with laminated fields.* A small two-phase motor with a device for splitting the current into two phases may be used, or the exciter of the alternator may be run as a motor from a direct current source and used to bring the alternator into synchronism.

A polyphase synchronous motor may be started by an ordinary polyphase induction motor coupled to its shaft; or it may be self-started as an induction motor. In the latter case the armature of the synchronous machine becomes the primary portion, and the field magnet the secondary portion, of the rotating field induction motor. For starting, the field circuit is usually opened. Current is permitted to flow from the main circuit at reduced voltage through the armature windings, an auto-transformer (Fig. 423) or a transformer having several voltage taps from the secondary winding being ordinarily used for the purpose of giving the reduced voltage. When the machine has been brought up to speed by the torque exerted between the eddy currents induced in the pole faces and the rotating magnetic field set up by the currents in the armature windings, the field switch may then be thrown on to the direct current exciter bus bars, and the machine will continue to rotate as a synchronous motor. When the field coils are large, the winding must be opened in a number of places to prevent excessive voltages being generated in them by the rotating armature flux. When the rotor of the machine has been brought to as high a speed as possible as an induction motor, it may jump into synchronism before the field switch is closed to send direct current through the field windings. The action is similar to that in large unloaded induction motors † with appreciable polar projections, that is, the magnetic flux from the armature rotating but very slightly faster than the projecting poles of the field magnet, magnetizes the poles as it passes over them and they take up synchronous speed, continuing to obtain magnetization from the armature magneto-motive force. When the field switch is closed, the magnetic flux already set up in the fields by the armature reaction may be in the wrong direction, in which case the armature must fall back the distance of the polar pitch before the regular field flux can produce forward operating torque. The processes for starting rotary converters are given in greater

* Chap. XII. † *Ibid.*

detail later,* and as they are essentially the same on the alternating current side as for synchronous motors, many details concerning starting the latter are deferred for the present.

Figure 423 shows the elementary connections for starting a quarter-phase synchronous motor when an autotransformer controlled by a cylinder switch is used. The rheostat in the motor field circuit, the main switch, and measuring instruments are not shown. The exciter may be driven mechanically by the synchronous motor and should build up its voltage while the

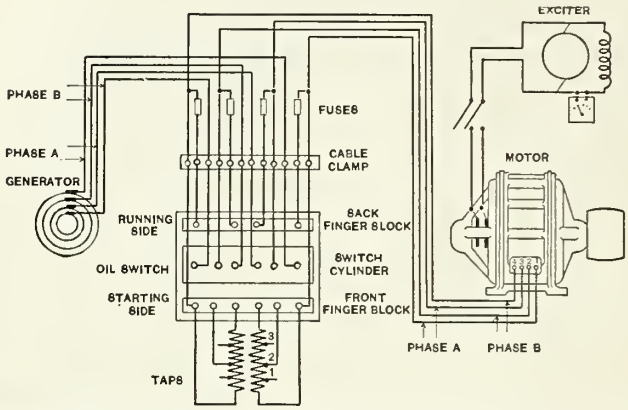


Fig. 423. — Elementary Diagram of Connection for a Quarter-phase Synchronous Motor.

synchronous motor is coming up to speed. In order to make the machine field frame assume the function of the secondary conductors of a short-circuited induction motor, it is sometimes desirable to place copper conductors about the field poles in such a manner as to make short-circuited paths (Fig. 442). When the motors are very large, or are not designed to start as induction motors without an excessive flow of current, they are commonly brought into synchronism by starting motors,† as are single-phase machines.

168. Relation of Voltages, Currents, and Power in Synchronous Motors. Synchronous Impedance. — When a synchronous motor is put in the circuit, a peculiar relation exists between the strength of the field of the motor and the current in its armature. In direct-current motors, if the strength of the field is

* Art. 181. † Chap. XII.

slightly changed without altering any of the other conditions, the speed of the motor changes inversely, and the current in the armature remains practically unchanged; but the speed of a synchronous motor cannot change permanently, and, consequently, upon first consideration, it would appear that the strength of the field magnet of a synchronous motor must be exactly adjusted, in order that the machine may operate satisfactorily. This is not the case, however, as was seen in the discussion of the parallel operation of alternators,* on account of the effect which may be gained through variations of the relative phases of the current and of the impressed and counter voltages. The active voltage, which at any instant causes current to flow through the armature of a motor, is equal to the difference of the corresponding instantaneous values of the impressed and counter voltages. If the field strength of a motor is so adjusted that the value of the impressed and counter voltages are equal, and the motor armature is brought by external means into exact step with the impressed voltage curve, then, when the motor is switched on the supply main, it will fall back in phase with respect to the impressed voltage sufficiently to permit the proper driving current to pass through the armature. Now suppose that at some instant the load is increased, the difference of instantaneous voltage at that instant will be insufficient to cause the larger current to flow through the armature which is necessary to produce the torque for the new load. The motor, therefore, falls back in its phase or armature step without losing synchronism, if the load is not too great, and then continues operating in synchronism, but with a greater lag in step. When a motor lags in step behind the phase of impressed voltage, its counter-voltage lags to an equal extent. The armature current ordinarily takes an intermediate phase, so that it is behind the resultant voltage but in advance of opposition to the counter-voltage.

Were it not for the just described automatic adjustment of the phases of the impressed and counter-voltages, it would be necessary to adjust the field excitation of a synchronous motor so that its counter-voltage would be less than the impressed voltage and the range of load carried with a given excitation would be limited to one value. The effects due to the auto-

* Arts. 162, 163.

matic adjustment of phase, however, make it possible to fix the field excitation once for all, and yet have the motor operate on widely varying loads. It is even possible, on account of the automatic adjustment of the voltage phases, to operate a motor when its excitation is much greater or much less than its normal value.

The adjustment is assisted by the effect of armature reactions on the motor, in which a current lagging with respect to the impressed voltage tends to strengthen the field magnet and a current leading with respect to the impressed voltage tends to weaken the magnet.* When a single motor is operated from an alternator of about its own size, the automatic adjustment of the machines is still more marked, since the current which strengthens the field magnet of the motor tends to weaken that of the alternator as the load is varied, and *vice versa*. A balance for each load is reached when the two voltages have attained such values that their vector difference or resultant is just sufficient to drive a current through the circuit which gives a torque sufficient to keep the motor running synchronously.

It is evident from the preceding paragraphs that the armature current of a synchronous motor must have a quadrature component which depends upon the phase difference of the impressed and counter voltages and the internal angle of lag, and it may readily be seen that a particular excitation of a synchronous motor field magnet reduces the armature current to a minimum (or makes the power factor a maximum) when the motor is carrying a particular load.

The current which flows through the armature is not only dependent upon the ordinary impedance, consisting of the resistance of the windings and their self-inductances, but also upon the field reactions caused by the quadrature component of current. For convenience the effect of the field reactions is frequently included and treated as part of the armature impedance instead of as a factor directly affecting the value of the counter-voltage of the motor. The impedance then obtained is called synchronous impedance. As explained earlier † it may be defined as the ratio of the voltage which would be developed at the given excitation, if the machine was run as a generator at synchronous speed without current in the armature, to the am-

* Art. 152. † Art. 156.

peres that would flow though the armature at the same excitation when it is short-circuited. In the diagrams for the parallel operation of alternators and those to follow for the synchronous motor the machine voltages used are those that would be generated were field reactions absent, while to compensate for this assumption, the synchronous impedance of the armature, which includes the effect of reactions, has been used instead of the ordinary impedance caused only by the self-inductance and resistance of the armature windings. The synchronous impedance has also, for simplicity, been considered constant when the excitation has been varied. As a matter of fact the field reactions and the true reactance of the armature vary with the space position of the armature coils with reference to the field magnet and the time phase of the current, as well as upon the reluctance of the leakage magnetic circuits, so that some error is involved in assuming the synchronous impedance to be constant. Thus, it is evident that the synchronous impedance is less for very high excitations than for those of low values, since in the first case the magnetic circuits are more highly saturated and a given armature magneto-motive force has less effect upon the field magnetic flux than in the second case.

In order to bring out more clearly the relation of voltages and currents in the motor, recourse may be had to a diagram in which relations of current and voltage are shown much as in parallel working. It was shown* that in parallel working the machines were held in step by a motor action, caused by a wattless current flowing, and that if one machine was cut off from its prime mover it would continue to run in synchronism as a motor. In Fig. 424 let OI be the current flowing from a circuit through the armature of an alternator acting as a motor, let SR be the drop of voltage in the synchronous reactance, equal and opposite to the reactive voltage OL of the armature, and let OS be the drop of voltage in the armature resistance; then OR is the resultant voltage IZ, required to act on the synchronous impedance of the armature to pass the current OI through the circuit. Suppose the voltage OE_1 is impressed at the armature terminals, and the motor is excited to give a numerically equal voltage OE_2; then OE_1 and OE_2 must be in such phases with respect to each other as to give

*Art. 161.

the resultant OR, while the components of voltage resolved upon the current vector OI must be such that the component of OE_1 has the same direction as the current, and the component of OE_2 is in opposition. To obtain this relation the armature voltage must fall back from direct opposition to the impressed voltage by a space angle equal to $\theta_2 + \theta_1$. In other words, the armature conductors must take up a step relation

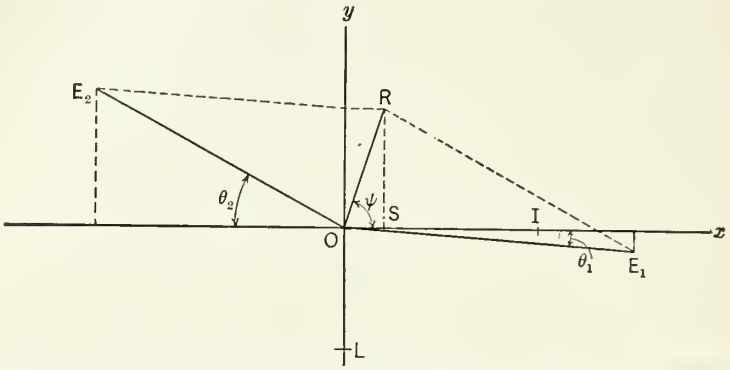


FIG. 424. — Diagram showing Relation of Voltages and Current in a Synchronous Motor for a given Field Excitation and Armature Current. Impressed Voltage Equal to Counter-voltage.

with respect to the field magnetism that is determined by the vector values of E_1, E_2, and IZ. The work delivered to the motor by the circuit is $IE_1 \cos \theta_1$, and that utilized by the motor armature in furnishing power and overcoming the magnetic and frictional losses is $IE_2 \cos \theta_2$; while that lost, due to armature resistance, is their difference, and is equal to

$$I^2Z \cos \psi = OS \times OI.$$

It will be seen that for small loads on the motor the current with the assumed voltage may lead the impressed voltage, as shown in Fig. 424, that is, θ_1 is negative; but that as the load increases (and the length of OR therefore increases), the motor counter-voltage is caused to swing farther back in step compared with the impressed voltage, so that the current takes a lagging position, as shown in Fig. 425. The construction indicates that the current is always more than 180° in the lead of the counter-voltage, when the impressed and counter voltages are equal, and that the value of $(\theta_2 + \theta_1)$ increases when

the load on the motor is increased. The value of the current I is one half greater in Fig. 425 than in Fig. 424, the input of the motor $IE_1 \cos \theta_1$ is 50 per cent greater, and the output $IE_2 \cos \theta_2$ about 45 per cent greater. The latter is increased in a smaller proportion, because the I^2R loss increases directly as I^2. The construction of these figures makes it manifest that the parallelogram of voltages OE_1, OE_2 and resultant OR determines the values of θ_1 and θ_2 so that each has a particular value for each given value for the impressed and counter-voltages and the magnitude of the motor load.

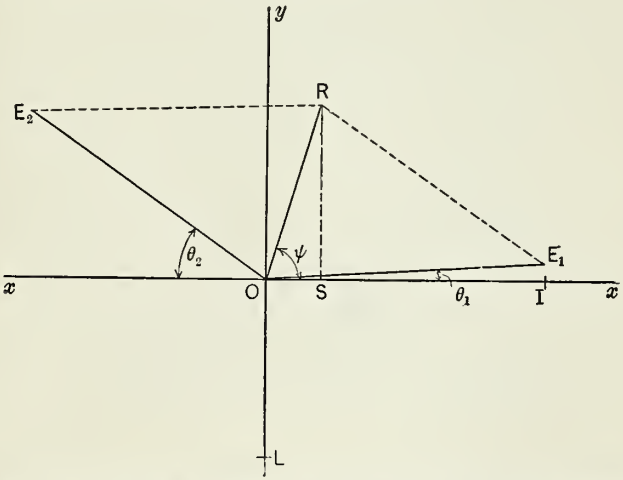


FIG. 425.—Vector Diagram of Voltages and Current when the Load is increased, but the Conditions are Otherwise as in Fig. 424.

The motor will continue to operate, as the load is increased, until θ_2 has attained such a value that $E_2 \cos \theta_2 \times I$ becomes a maximum ; then, if an additional load is put on the motor, the corresponding increase of θ_2 will cause $IE_2 \cos \theta_2$ to decrease, and the motor will fall out of synchronism and stop, because the maximum value of its torque is not sufficient to pull the load. In the case under consideration (when the impressed and counter voltages are equal) this will not occur with well-designed alternators until a load much above that for which the machine is rated is reached.

As was stated, the counter-voltage at which the motor is run, and therefore its excitation, has an important bearing upon the

stability of operation and the efficiency of transmission. If the motor voltage is made larger than that of the generator, the current and voltage relations may be shown by a construction similar to that used in Fig. 424. If OR in Fig. 426 represents the magnitude and direction of the resultant voltage, and the impressed and counter voltages have magnitude OE_1 and OE_2, then the parallelogram can be completed with the values of the angles θ_1 and θ_2 shown in the figure. In this illustration the counter-voltage is greater than the impressed voltage. Now

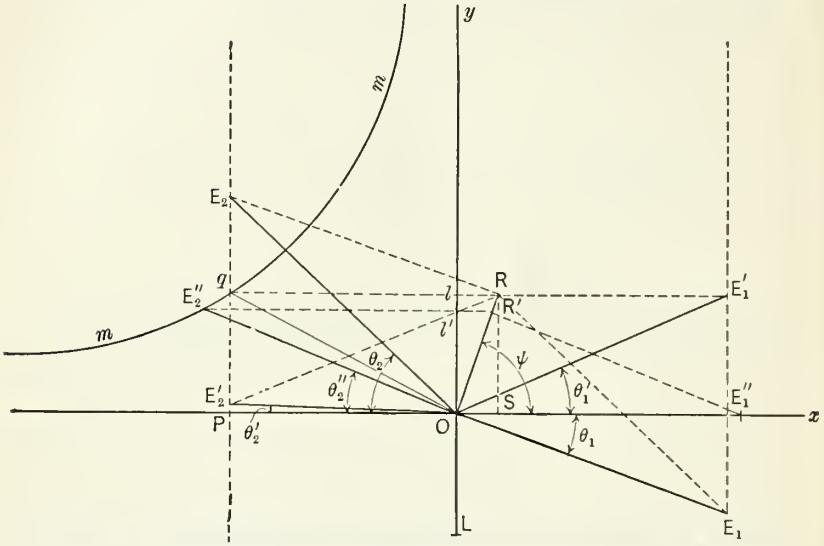


FIG. 426. — Diagram showing the Effect of Variation of Excitation in a Synchronous Motor.

suppose the motor excitation is changed so that the counter-voltage is OE_2', having the same horizontal projection as OE_2, while OR and the impressed voltage have the same values as before; then the phase relations are as shown by the lines OE_1', OR, and OE_2'. The values of $OE_2 \cos \theta_2$ and $OE_2' \cos \theta_2'$ are equal by the construction, since the points E_2 and E_2' are in the same vertical line; and since OR has the same magnitude and position in the two cases the current is the same. Also, $OE_1 \cos \theta_1$ and $OE_1' \cos \theta_1'$ are equal; but in the first instance the counter-voltage is greater than the impressed voltage and the current leads the impressed voltage, and in the second instance the impressed voltage is the greater and the current

lags. This construction shows (with a fixed value of impressed voltage) that for every load on the motor, except that corresponding to a zero angle of lag, there are two values of the excitation which cause the same armature current to flow, the current leading the impressed voltage with the higher excitation and lagging by an equal angle with the lower excitation; hence an over-excited motor acts upon the line current very much like a condenser, and an under-excited motor acts like an inductance coil.

Figure 433 shows the armature current of a synchronous motor operating under fixed load on a circuit of fixed voltage, plotted as a function of the field-exciting current. This shows plainly that the armature current assumes the same numerical value for each of two excitations. The vector diagrams show that the lesser excitation corresponds to a current lagging and the greater excitation to a current leading with respect to the impressed voltage.

The excitation at which the motor will do the most work with a given current flowing, or will carry a given load with the least current (and hence do it with the best power factor), is, as is proved later (Art. 169), that which causes the current to come into phase with the impressed voltage. In this case the current for a given load is a minimum, and $E_2 \cos \theta_2$ is a maximum for the conditions concerned. If the value of E_2 in Fig. 426 is increased (by increasing the excitation of the motor), either the value of I and the length of OR which is proportional to it, or the impressed voltage, must be increased, provided $IE_2 \cos \theta_2$, which is equal to the motor load, is constant. On the other hand, taking the vector rectangle OE_2RE_1, if E_2 decreases numerically while E_1 remains numerically constant, the values of the angle θ_1 and the current decrease until the current and impressed voltage come into phase, after which a further decrease of E_2 causes the current to increase again and the angle of lag to increase in the positive direction. All this is shown by the relations between E_1, I, and E_2 in the figure.

The value of $IE_2 \cos \theta_2$ (that is, the load carried when the current is $\frac{OR}{Z}$, when Z is the synchronous impedance, and the counter-voltage is either OE_2 or OE_2') is proportional to the area of the rectangle $OlqP$, since Ol is proportional to I. Now if the motor load is kept constant, but its field excitation is changed, the corner of the corresponding rectangle must move

from q, but the locus of the positions of the corner is a hyperbola with its origin at O and the rectangular axes x and y, since the rectangles included between the axes and the ordinate and abscissa for each point must be all of equal area. Consequently, the point of the vector representing E_2 at the least current for the given load must be in the rectangular hyperbola mm which passes through q. This point, which is E_2'' for the given load and the impressed voltage E_1, is found by laying off the horizontal line equal in length to OE_1 which will just reach between the hyperbola and the line OR. This cuts OR at R', and the sought for minimum current is $I = \frac{OI'}{X} = \frac{OR'}{Z}$, where X is the synchronous reactance and Z the synchronous impedance. The impressed voltage and current are, under these conditions, in phase with each other. If a larger or smaller counter-voltage than OE_2'' is used, such as OE_2 or OE_2', the impressed voltage and current are thrown out of phase, and the current in the circuit is, therefore, increased, if the impressed voltage remains of the same numerical value, which causes increased I^2R losses and armature reactions. The excitation of the motor which brings the current and impressed voltage into step for a given load depends upon the relation of the synchronous impedance of the motor circuit to its resistance. When the angle ψ is 60°, that is, when $Z = 2\,R$, the counter-voltage is equal to the impressed voltage at zero lag of the current, since $\frac{E_1}{E_2} = \frac{2\,R}{Z}$ under conditions of maximum output (Art. 169); and when Z is greater or less than $2\,R$ the counter-voltage is respectively greater or less than the impressed voltage at zero lag. Putting X and Z for synchronous reactance and impedance, then $X = \sqrt{3}\,R$ when $Z = 2\,R$, since $Z = \sqrt{R^2 + X^2}$. In the machines which are now commonly built, the synchronous impedance of the armature circuit is commonly larger than twice the resistance, and the maximum power factor and output are obtained in such synchronous motors by an excitation which gives a counter-voltage that is greater than the impressed voltage.

Figure 427 shows polar diagrams for various excitations at which a small motor under constant load was provided with power by a generator of exactly the same size and type as the motor. Lines OE_1, OE_2, and OR are the generator, motor, and

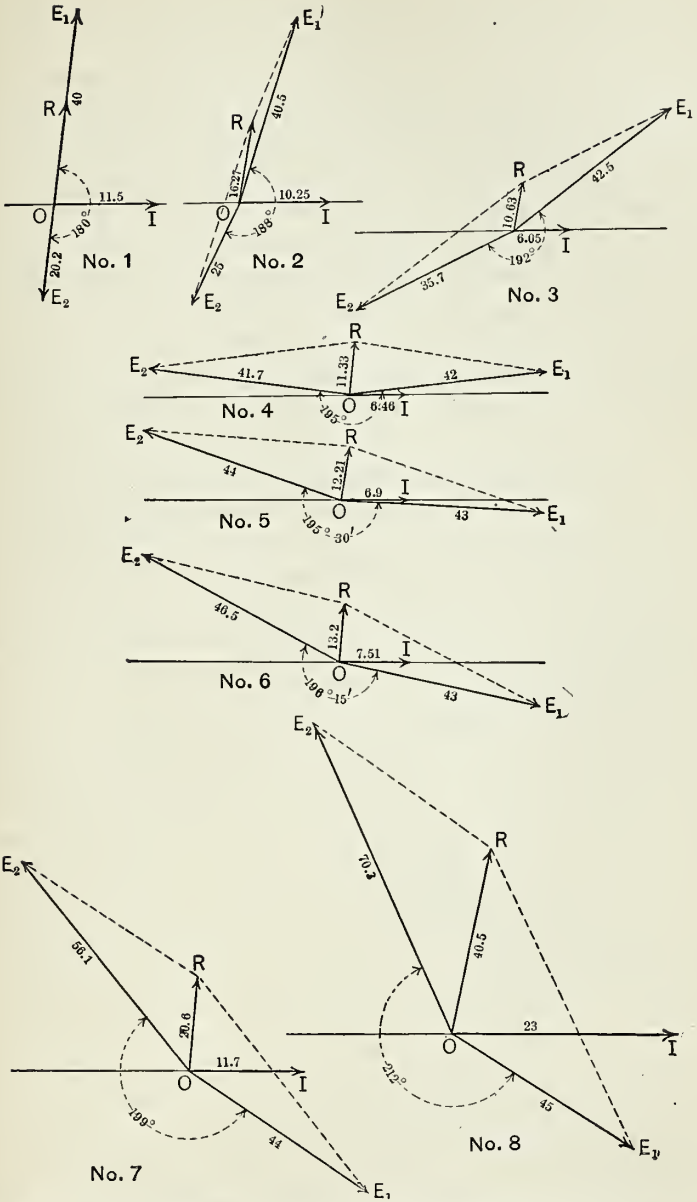


FIG. 427. — Vector Diagrams obtained from Experiments made with Two Small Similar Machines, One Running as a Motor, the Other as a Generator.

resultant voltages respectively acting in the circuit comprising the two machines, and OI is the current. These were obtained from a diminutive pair of single-phase alternators connected together and operated as motor and generator in a classic experiment of Professors Ryan and Bedell. They illustrate the vector relations of current and voltages for motors of all sizes. The angle by which the current lags behind the resultant voltage was obtained from the synchronous impedance of the armature circuit. It may be clearly seen from the diagrams that the current swings from a position of large lag with reference to the generator voltage at a small excitation of the motor as in diagrams Nos. 1, 2, and 3; into phase with it, as between diagrams Nos. 4 and 5, the point of minimum current for the given load on the motor; and finally into a position of large lead when the motor is greatly over-excited, as in diagrams Nos. 7 and 8. The numerical values of E_1, E_2, I, and E_Z and the angular relation of E_1 and E_2, as observed, are shown in the figure. The ratio of E_Z to I is the internal impedance of the motor.

Figure 428 shows the armature current for different motor field excitations when the motor was under the same small constant load. Curves D, F, and G in the same figure show plainly the tendency of the armature reactions and leakage reactance to reduce differences between numerical values of motor and generator voltages. Curve D represents the generator voltage, and, although the generator excitation was constant, the generator voltage rises as the motor voltage is increased. This is due to the reaction caused by the current swinging from a lag to a lead with reference to the generator voltage. At the same time the motor voltage, which is represented by curve F, at first is larger and then grows smaller than would be the case

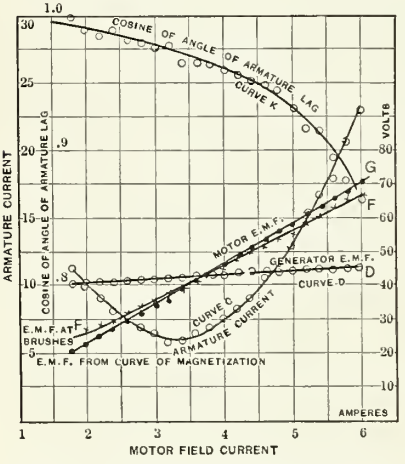


FIG. 428. — Curves showing Relation of Voltages and Armature Current for Constant Load and Variable Excitation in a Synchronous Motor.

were no reactions present. The curve G represents the motor voltage considering reactions absent. The effect in the motor is caused, as in the generator, by the current changing its lead with reference to the motor voltage.

This series of experiments and the preceding discussions show that there was some foundation for the statement of earlier experimenters, that alternators must have self-inductance in their armature circuits if they are designed to be run in parallel. The application of that statement to the case, however, is fallacious, since alternators operating in parallel should require much less than the torque of normal load to hold them in step, so that the synchronizing tendency of armatures with small inductance is ample to make them run in parallel, and for either parallel working or for operation as synchronous motors a small armature impedance is of the greatest importance.

169. Locus Diagrams of Currents, Voltages, and Loads of the Synchronous Motor. — In Fig. 426 a locus is given whereby the motor excitation can be obtained when the load is constant and the current varies. Suppose now that the current and impressed voltage are maintained of constant values, then Fig. 429 results. The locus of the impressed voltage is a circle $E_1E_1'E_1''$, with its center at O. The current represented by OI being constant, OR, the resultant voltage, is constant, if it is assumed that the synchronous impedance is constant. The latter assumption is not exact for commercial machines on account of the effect of saturation in the magnetic circuit with different excitations of the field magnet as well as the effect of changes of the power factor of the motor, but the assumption may prevail for the purpose of the following exposition. Then in order that the impressed and counter-voltages may always combine to equal OR the locus of the counter-voltage E_2 must be the circle $E_2E_2'E_2''$ with its center at R and its radius equal to E_1. The geometrical construction of the figure shows that these relations are necessary to fulfill the condition of constant current and impressed voltage. When the excitation gives the counter-voltage OE_2, the construction shows that the mechanical load is $OT \times IO (= IE_2 \cos \theta_2$, where θ_2 is the angle between OE_2 and IO), which will be assumed to be the amount of power required to drive the armature at synchronism without external load. The current OI leads OE_1 by a large angle so that there is a large

leading reactive component of current equal to VI, while the active component of OE_1 ($E_1 \cos \theta_1$, where θ_1 is the angle between E_1 and I_1) is equal to OW, and $OW \times OI$ is in this instance only sufficient power to furnish the I^2R losses and the power required to overcome magnetic and friction losses. The output of the motor is a maximum when E_2 equals OE_2' (and is equal

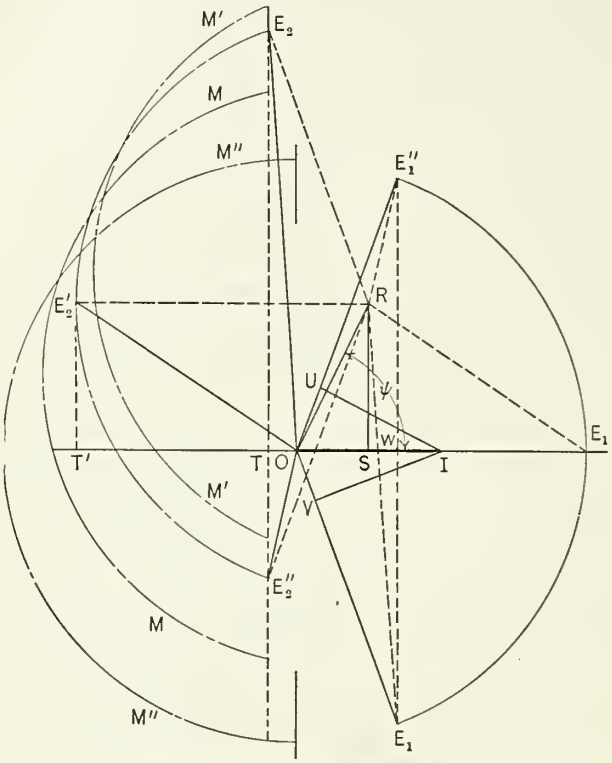


FIG. 429. — Locus of Counter-voltage in a Synchronous Motor when the Current and Impressed Voltage are Constant. The Current Vector is made the Initial Line.

to $OT'' \times IO$) for at that value of E_2 the active component of the counter-voltage is a maximum, the vertical line $E_2'T''$ being tangent to the arc $E_2E_2'E_2''$. The construction shows that with this excitation of the motor, the power is taken from the circuit at unity power factor. This obviously must be true under the conditions, since a maximum output with fixed losses predicates a maximum input. As a similar construction may be made with the current I of any fixed value, the motor ex-

citation that gives unity power factor at any current gives maximum output for that current. The minimum excitation which will keep the motor running is when the counter-voltage equals OE_2''. At this point the power converted by the armature is again equal to $OT \times IO$ and is just sufficient to drive the machine without external load, while the impressed voltage OE_1'' again supplies an active component OW, and $OW \times OI$ equals the power input required to supply all losses. The current now has a heavy lagging reactive component UI. The motor cannot operate with the current OI, even without an external load, if the excitation is either increased or decreased beyond the limits of OE_2 and OE_2'' and the impressed voltage is maintained constant; but it must be remembered that the armature reactions may contribute to the excitation, and a motor, without external load, may, therefore, run synchronously, even when not provided with external direct current excitation. If, however, the motor is coupled to a prime mover such as a steam engine, the counter-voltage may take any position on the complete circle of which $E_2E_2'E_2''$ is an arc. In this case when the motor voltage is near the top or bottom of the circle locus so that its active component is less than is needed to provide the load torque, it will draw power partly from the electric mains and partly from the engine. When the active component is on the right of O and equal to or greater than $-OT$ the machine becomes a generator working in parallel with the mains. The voltage locus of which $E_1E_1'E_1''$ is an arc may also be drawn as a full circle; and when a voltage vector is to the right of O power is delivered, but when the vector is to the left of O power is absorbed. The arc MM in the figure is the locus of E_2 when I equals $\frac{1}{2}\ OI$; $M'M'$ is the locus when I equals $\frac{5}{4}\ OI$; and $M''M''$ is the locus in the limiting case which might occur if the armature reactions could be negligible and the motor losses could be exactly supplied mechanically through the motor pulley, in which case E_1 and E_2 would have to always be equal and opposite.

A more generally useful locus is that of the current when the impressed and counter voltages are maintained constant and the load is varied. The construction is shown in Fig. 430, in which OE_1 is the fixed vector of impressed voltage E_1. Consider first the case where the counter-voltage E_2 equals E_1. Then when the counter-voltage is in the position OJ

opposite to OE_1, no current can flow as there is no resultant voltage. This position is imaginary, as the machine will not operate unless supplied with at least enough power to overcome the losses. Assume now that the motor armature, and hence E_2, falls back in step to the position OE_2, because of the application of load. The resultant voltage is then OR, and the current is OI lagging behind OR by an angle $\psi = \cos^{-1}\frac{R}{Z}$, where R is the armature resistance and Z the synchronous impedance. The current is $I = \frac{OR}{Z}$. When the point E_2 moves along the

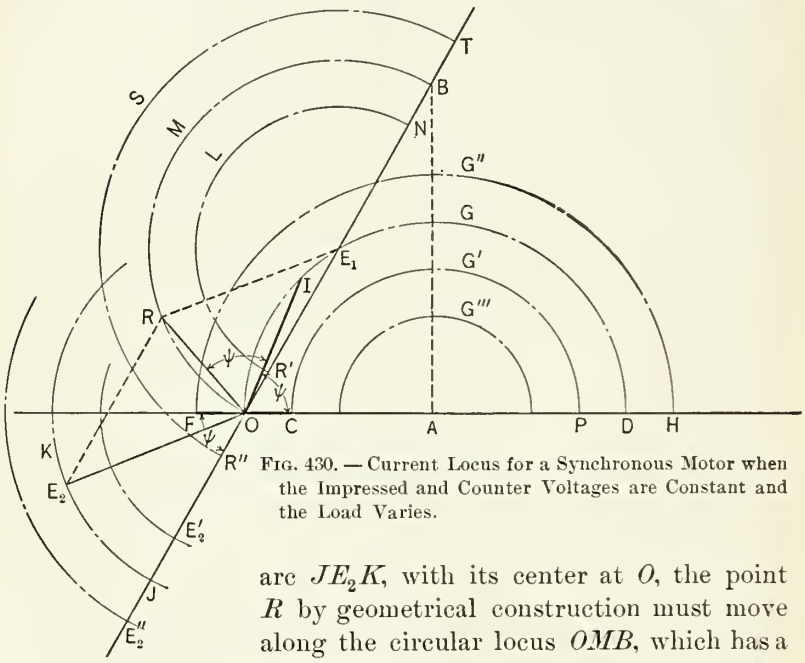


FIG. 430. — Current Locus for a Synchronous Motor when the Impressed and Counter Voltages are Constant and the Load Varies.

arc JE_2K, with its center at O, the point R by geometrical construction must move along the circular locus OMB, which has a radius OE_1 and its center at the point E_1. Therefore, since the current maintains a constant angle ψ with OR and varies in length in proportion to OR, the point I describes a circular locus OE_1GD drawn on a diameter OAD, which makes an angle equal to ψ with the line OE_1. The diameter of this circle is equal to $\frac{OB}{Z}$, that is, the diameter of the locus of R divided by the synchronous impedance. The diameter OB is the theoretical value of OR that would exist under the

hypothetical condition of OE_2 in consonance with OE_1 and the motor on the circuit with its induced voltage in series relation with the impressed voltage, that is $OB = E_1 + E_2$. The angle DOB equals ψ, as Z is considered constant. The diameter of the circle OGD equals

$$\frac{E_1 + E_2}{Z} = \frac{2\,E_2}{Z} = \frac{2\,E_1}{Z},$$

since $E_2 = E_1$ by the premises; and it makes an angle with OE_1 which is

$$\psi = \cos^{-1}\frac{R}{Z}.$$

Now suppose that E_2 is made OE_2', which is less than E_1 by such an amount that, when the two are in direct opposition, the resultant voltage OR' in phase with OE_1 drives the current OC through the armature. By the geometrical construction it is seen that, when OE_2' lags behind opposition to OE_1, OR' describes the circular locus $R'LN$, and therefore, the current which, as before, must maintain a constant angular position with respect to the resultant voltage and be proportional to its length, must describe the circular arc $CG'P$. The diameter of this circle is

$$\frac{R'N}{Z} = \frac{2\,E_2}{Z}.$$

Again, suppose E_2 is made greater than E_1 and is given the value shown by the line OE_2''. If E_2'' takes the position OE_2'', shown in the figure, the resultant OR'' drives the current OF through the synchronous impedance. For this position of OE_2'' the machine must evidently be driven as a generator, as it is driving current against the impressed voltage E_1. It will not run as a motor until OE_2'' has fallen back sufficiently to cause the current vector to take an angle of less then 90° from the impressed voltage and have an active component sufficient to supply not less than the losses of the machine, other than field losses. The locus of the resultant voltage OR'' must by geometrical construction, as before, be on the semicircle $R''ST$, and the current locus in accord with the reasoning for the previous cases is $FG''H$. By continuing this method the current loci for any desirable number of counter-voltages may be laid off with the respective loci of counter and resultant voltages.

The input can be obtained from the product of current, impressed voltage, and the cosine of the angle of lag between the two. The maximum current input for the machine when running as a motor with minimum excitation under which the motor will run is slightly less than current OA which equals $\frac{E_1}{Z}$. The maximum possible current input for the machine when running as a motor is when there is maximum excitation under which the motor will run, and this can then be more than current OD, and therefore more than $\frac{2\,E_1}{Z}$. The mechanical load including core losses equals the input minus the armature copper losses, or the total power transferred from the mains to cause rotation equals the impressed voltage times the active component of current minus the I^2R losses, which can be readily calculated. Therefore, we have the input,

$$P_i = E_1 I \cos \theta_1;$$

and the total transfer of electrical energy by the armature into mechanical energy and core losses,

$$P_o = E_2 I \cos \theta_2 = E_1 I \cos \theta_1 - I^2 R,$$

where θ_1 and θ_2 are respectively the angles between impressed and counter-voltages and the current. The mechanical power given out is, as intimated, less than P_o by an amount equal to the mechanical and magnetic power losses caused by rotation.

In order to determine the current that is required at a given power factor to supply a specific armature power P_o, a locus of currents for constant power may be constructed in accordance with the foregoing principles. Thus, in Fig. 431 let OE_1 be the direction of the impressed voltage, which is also made the initial line. Assume now that a certain output P_o, including rotative magnetic and frictional losses, is being consumed by the rotor or delivered from its shaft. Then the input is

$$E_1 I \cos \theta_1 = P_o + I^2 R,$$

where E_1 is the impressed voltage, I the current flowing, θ_1 the angle of lag between the current and impressed voltage, and R the resistance of the armature. From this:

$$I^2 - \frac{E_1 I \cos \theta_1}{R} = -\frac{P_o}{R},$$

and $$I^2 - \frac{E_1 \cos \theta_1}{R} I + \frac{1}{4} \frac{E_1^2 \cos^2 \theta_1}{R^2} = \frac{1}{4} \frac{E_1^2 \cos^2 \theta_1}{R^2} - \frac{P_o}{R},$$

hence, $$I = \frac{1}{2} \frac{E_1 \cos \theta_1}{R} \pm \sqrt{\frac{1}{4} \frac{E_1^2 \cos^2 \theta_1}{R^2} - \frac{P_o}{R}}. \qquad (1)$$

From this expression for I when the armature power is P_o it is seen that there are two values of armature current for each

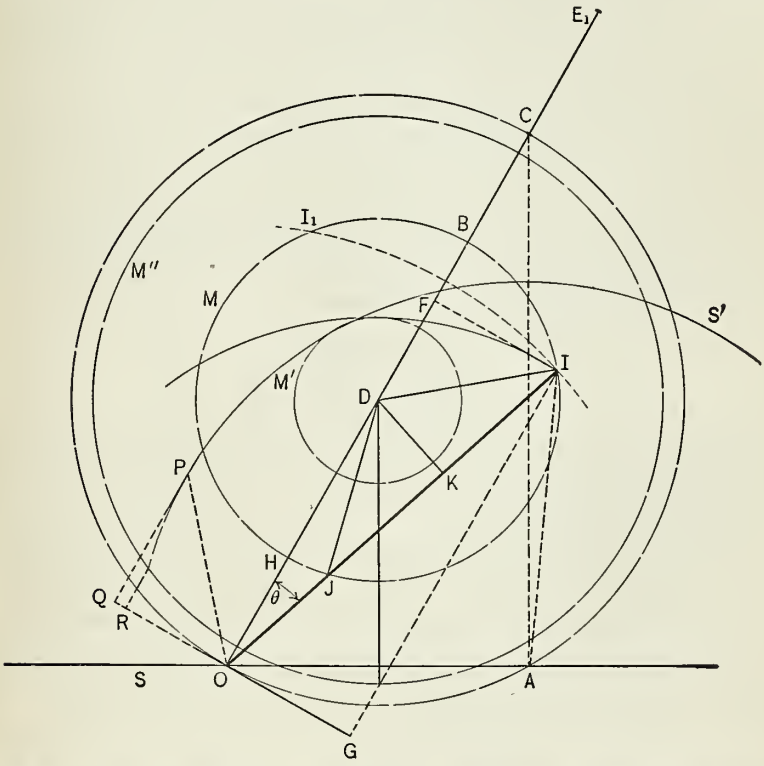


FIG. 431. — Diagram showing Current Loci in a Synchronous Motor for Constant Power when the Impressed Voltage is Constant.

value of $\cos \theta_1$ except one, and also that the currents are numerically equal but of opposite algebraic signs for equal positive and negative values of $\cos \theta_1$. The values of the currents contain imaginary terms when $\frac{P_o}{R} > \frac{1}{4} \frac{E_1^2 \cos^2 \theta_1}{R^2}$. To more clearly observe these conditions the locus of constant power, P_o,

HMB is drawn. The locus is a circle having a radius equal to $\sqrt{\frac{1}{4}\frac{E_1^2}{R^2} - \frac{P_o}{R}}$, as may be seen from the following considerations. From the last formula it is seen that the current I equals $\frac{E_1}{R}$ when P_o and θ_1 equal zero. Then the current is in phase with the voltage and flows through the windings of the armature in accordance with Ohm's law and all the energy is utilized in I^2R loss. Such a condition could only exist, of course, when the rotor has its rotative losses supplied from external mechanical sources, and has swung to such a position that the counter-voltage forms a resultant with the impressed voltage of such phase angle that the current is in consonance with the latter. These conditions are only fulfilled when the counter-voltage is at right angles with the current and of definite length. This current $\frac{E_1}{R}$ is laid off at OC in the Fig. 431. When θ_1 only is zero, the formula for current becomes

$$I = \frac{1}{2}\frac{E_1}{R} \pm \sqrt{\frac{1}{4}\frac{E_1^2}{R^2} - \frac{P_o}{R}}$$

The larger value obtained by this formula is laid off as OB and the smaller value as OH in the figure. The distances OH and BC are evidently equal.

Now take any value of current I obtained for a given P_o and for any angle. Assume that OI at lag angle θ_1 is such a current. Then draw the perpendicular DK from a point D, midway between O and C or H and B, and connect the points D and I. The length $OD = \frac{1}{2}\frac{E_1}{R} = \frac{1}{2}\,OC$, and by construction

$$OK = \frac{1}{2}\frac{E_1}{R}\cos\theta_1,$$

and

$$DK = \frac{1}{2}\frac{E_1}{R}\sin\theta_1 = \frac{1}{2}\frac{E_1}{R}\sqrt{1-\cos^2\theta_1}.$$

Also

$$KI = OI - OK = I - \frac{1}{2}\frac{E_1}{R}\cos\theta_1.$$

By construction $(DI)^2 = (DK)^2 + (KI)^2$.

Substituting in this the values of DK and KI and using the value for current given in the formula for current in Equation (1), then

$$[DI]^2 = \left[\frac{1}{2}\frac{E_1}{R}\sqrt{1-\cos^2\theta_1}\right]^2 + \left[\left(\frac{1}{2}\frac{E_1\cos\theta_1}{R}\right)\right.$$

$$\left. \pm\sqrt{\frac{1}{4}\frac{E_1^2\cos^2\theta_1}{R^2} - \frac{P_o}{R}} - \left(\frac{1}{2}\frac{E_1}{R}\cos\theta_1\right)\right]^2, \qquad (2)$$

which reduces to

$$(DI)^2 = \frac{1}{4}\frac{E_1^2}{R^2} - \frac{P_o}{R}$$

or

$$DI = HD = \pm\sqrt{\frac{1}{4}\frac{E_1^2}{R^2} - \frac{P_o}{R}}. \qquad (3)$$

Since I is assumed in the premises to be the current which can supply the power P_o at a given value for the angle θ_1, the point I so long as P_o is constant must travel the locus $HMBI$, which must be a circle having a radius equal to $\sqrt{\frac{1}{4}\frac{E_1^2}{R^2} - \frac{P_o}{R}}$, with its center D at a distance equal to $\frac{1}{2}\frac{E_1}{R}$ from the origin O. Any power P_o was assumed, hence by assuming a series of outputs, the circular loci for the outputs are concentric circles such as those marked M, M', and M''. The largest locus possible has a radius about equal to DO, P_o then being only sufficient to supply the no-load rotative losses. The smallest is the point D, where the radius is zero, the power factor is unity and the greatest output occurs. For this locus there is only one current OD, half the input power is used up in I^2R loss and the other half in turning the armature since $\frac{P_o}{R} = \frac{1}{4}\frac{E_1^2}{R^2}$ and $I = \frac{1}{2}\frac{E_1}{R}$. The efficiency is thus less than 50 per cent. Commercial synchronous motors could not approach this amount of load without causing excessive and disastrous heating. The maximum efficiency occurs when the copper losses equal the rotation losses, which may usually be assumed without serious error to be of fixed value. By projecting OI upon the voltage line OE_1 and at right angles, the active and quadrature components of the current, OF and OG, are obtained. By suitably changing the scale of

the figure, OI, OF, and OG, may be read off directly as total kilovolt-amperes, kilowatts, and quadrature volt-amperes respectively. The output may be determined for any locus by marking the locus with the power assumed in its construction. Since the radii of the current loci are scaled in amperes so that

$$r = \sqrt{\frac{1}{4}\frac{E_1^2}{R^2} - \frac{P_o}{R}},$$

the radius of the locus circle depends upon the output, as E_1 and R are constant. Therefore, loci for various outputs can be readily laid out by making the radius

$$r = \sqrt{A - \frac{P_o}{R}}$$

where A is a constant equal to $\frac{1}{4}\frac{E_1^2}{R^2}$. By combining Figs. 430 and 431 all of the information desired concerning the motor voltages and phase angles is obtained for any output when the voltage and power factor of the input are known. A figure or construction similar to that of Fig. 410 may be made for a synchronous motor having constant power output, variable induced voltage, and constant impressed voltage. By a process of reasoning similar to that used in discussing the division of load in alternators,* it may be shown that the induced voltage locus (constant power circle) is a circle having a radius $\left[\frac{E_1^2}{4\cos^2\theta_a} - \frac{P_o Z_a}{\cos\theta_a}\right]$ and the polar coördinates E_2 and $\frac{E_1}{2\cos\theta_a}$, where P_o is the motor output, E_1 the impressed voltage, E_2 the motor counter voltage, and θ_a the internal angle of lag. The angle included between the coördinates is $(\theta_a - \beta)$. The construction is like that of Fig. 410, except that the angle $O'OB$ is $(\theta_a - \beta)$ instead of $(\theta_a + \beta)$ and the locus radius is shorter than OO', which throws the motor counter-voltage to the right of the terminal voltage OV and causes the loci TT, etc., to be at the right of OV.

170. Curve showing the Relation of Armature Current to Excitation. — The relation of armature current to field excitation may be plotted for a motor operating under the conditions considered above, namely, when driving a constant load, by

* Art. 162.

taking the corresponding values of E_1 and I from a chart made like Fig. 426, or like Fig. 431. This gives a curve like Fig. 432, which has two values of its abscissas for every value of the ordinate except the lowest and highest, points A' and C' in the figure; or for each value of the armature current there may be two values of the excitation, one giving a counter-voltage

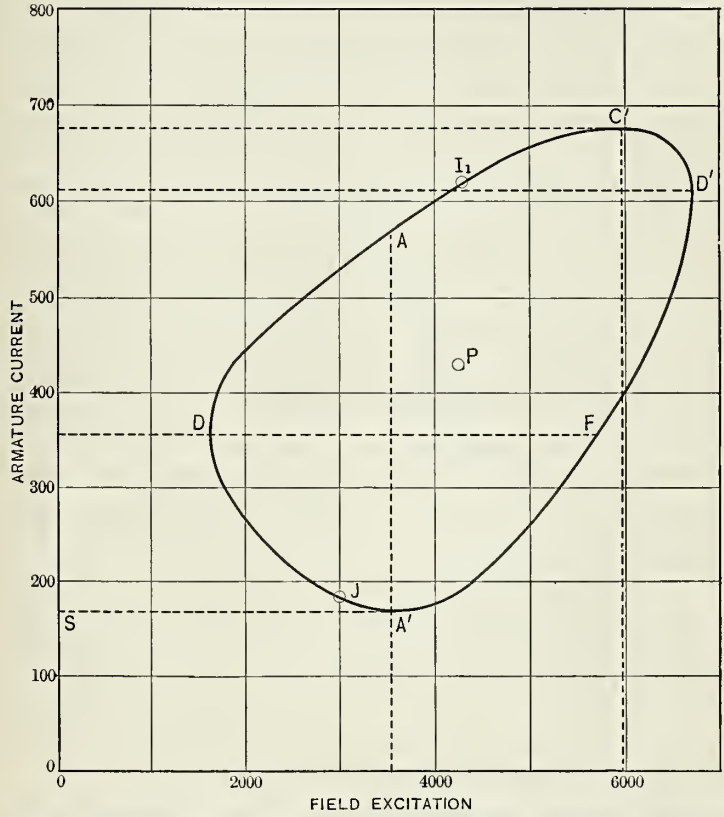


FIG. 432. — Calculated V-Curve for a Synchronous Motor, showing the Relation between Field Excitation and Armature Current for a Constant Load.

greater and the other a counter-voltage less than the impressed voltage, except at the points of minimum and maximum armature current, which correspond to but one excitation, as has already been explained. Likewise there are two values of the armature current for each value of the field excitation except for the minimum and maximum excitations at which the given

load can be carried, namely at the points D and D' in the figure. The way in which Figs. 426 and 431 are constructed shows that the smaller the angle ψ, or the smaller the armature self-inductance, the less will be the difference in the two excitations corresponding to any armature current; and hence the curve showing the relation of excitation to current in a machine having a large reactance compared with the resistance is broad and rounded, but the curve for an armature having a small reactance compared with the resistance is sharp and narrow. In actual working, only the lower part of the closed curve of Fig. 432 is available for use. This corresponds to curve C of Fig. 428 and the curve shown in Fig. 433. These curves are ordinarily called V-curves on account of their shape. Curves for loads smaller than the one for which the curve is shown in the figure expand and fall outside of that curve, while for larger loads they contract and fall within it. At maximum load, represented by point D, Fig. 431, the curve reduces to the point P (Fig. 432).

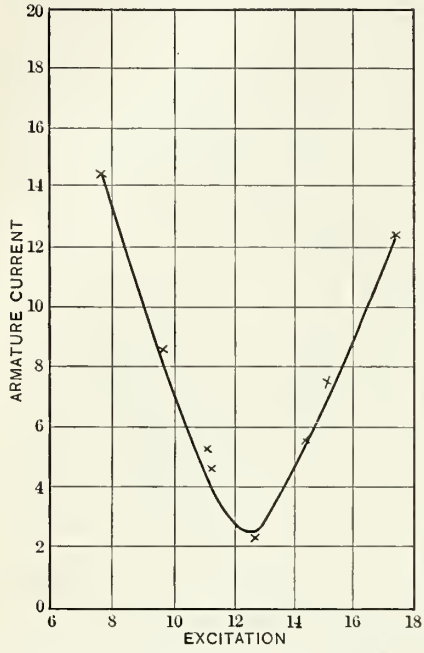


FIG. 433. — V-Curve of a Synchronous Motor Plotted from the Observed Value of Armature Current and Excitation when operating under Constant Load.

In Fig. 430 it was seen that the counter-voltage, which is proportional to the field flux, is also proportional to the radius of the current locus. The latter is numerically equal to $\frac{E_1}{Z} = OA$ when $E_1 = E_2$. Hence if Z, the synchronous impedance, has been determined, the counter-voltage for any current locus is $E_2 \propto r\ Z$, where r is the radius of the current locus under consideration. By drawing the line OA in Fig. 431 so that it makes the angle ψ with the line OE_1, it is possible to

determine a plot for the V-curve very readily for any output P_o. Considering the power locus $HMBI$ to be for the load P_o (Fig. 431), the current OI laid out as an ordinate in Fig. 432 has an abscissa proportional to AI, the radius of the current locus which passes through I in Fig. 430. If the field reluctance is constant and field reactions are included as part of the synchronous impedance, the armature current locus may be obtained by drawing an arc with the radius OI in Fig. 431. Where it cuts the constant power output locus $HMBI$ a second time at I_1 the excitation is proportional to the distance between A and I_1. In Fig. 431 it is seen that as $OC = \frac{E_1}{R}$ and, in Fig. 430, $OA = \frac{E_1}{Z}$, therefore since DOA, Fig. 431, equals ψ, $OA = \frac{E_1}{Z}$ in Fig. 431 is the horizontal side of the right-angled triangle OAC. The curve of Fig. 432 is constructed from Fig. 431 as follows: the points on the curve whose ordinates are equal to currents OI and OI_1, Fig. 431, are I_1 and D' in Fig. 432, with abscissas equal to AI and AI' of Fig. 431, while the current OJ is represented by the ordinate of point J with abscissa equal to AJ of Fig. 431. The current OI (Fig. 431) is so large compared with OD, which represents the current when the armature IR drop is equal to $\frac{1}{2} E_1$, that the efficiency would be very low. The useful part of the curve of Fig. 432 is, therefore, only a short distance on either side of A', the ordinate and abscissa of which are equal to OH and AH of Fig. 431. The maximum load for any excitation may readily be determined from Fig. 431; thus, if the excitation is proportional to AS, the largest load which can be carried is represented by the power locus M', as this is the smallest power locus that is cut by the current locus SS' having AS as a radius.

It must not be supposed that the armature voltages actually rise in proportion to the amperes of field excitation, even for constant reluctances, as indicated by the abscissas of Fig. 432, because the quadrature current when the excitation is less than AH (Fig. 432) strengthens the fields, and when greater than AH weakens the fields; but this quadrature current calls for a proportional decrease or increase of field current in the same amount that would be called for if the reactions were zero and the counter-voltages were proportional to the abscissas in the figure.*

* McAllister's *Alternating Current Motors*, p. 195.

171. Application of the Parallel Operation and Synchronous Motor Diagrams to Machines Wound with Any Number of Phases. — The synchronous motor diagrams given in the previous articles are suitable for use for any single or polyphase machines. In the case of single-phase machines the power input, voltage, and current used in the diagrams are those measured in the supply mains by wattmeter, voltmeter, and amperemeter. For quarter-phase machines one full phase may be conveniently considered using its voltage, current, and power as in the single-phase machine. The powers must then be doubled to obtain the values of motor input and output. Tri-phase machines may be dealt with in the same manner, using the voltage, current, and power in one branch for obtaining the diagrams and then multiplying the inputs and outputs by three. When the tri-phase motor is wye connected, the branch voltage is $\frac{1}{\sqrt{3}}$ the line voltage, and when the motor is delta connected the branch current is $\frac{1}{\sqrt{3}}$ the line current. Six-phase mesh-connected motors may be treated in the same way as similarly connected tri-phase machines, except that the power shown by the single-phase diagrams must be multiplied by six to get the entire power. For six-phase machines with a star connection, the voltage, current, and power per phase are the same as in the mesh connection.

Prob. 1. Given a three-phase wye-connected synchronous motor operating under 2200 volts impressed voltage at its terminals with a frequency of 60 periods per second. The synchronous impedance per phase, when the frequency is normal, is considered constant at a value of 5 ohms and is composed of such proportions of resistance and reactance that the current lags 75° behind the resultant voltage. Giving the vector of impressed voltage a fixed position, draw the current loci for 100, 200, and 300 amperes; the constant total armature mechanical power loci for 100, 200, and 400 horse power, assuming that the mechanical and magnetic losses are constant at 12 horse power; and the V-curves for the latter powers. Also, giving the current vector a fixed position, draw the impressed voltage and counter-voltage loci for currents of 100, 200, and 300 amperes. Complete the locus circles in the latter diagrams

and mark the positions where the machine is a generator working in parallel with the supply mains.

Prob. 2. Draw the V-curves for the generator and conditions of Prob. 1, Art. 162, by the method given in that Article, when the total powers are 100, 50, and 25 kilowatts.

172. Methods of testing Alternators. — The general definitions of efficiencies, which have already been explained with regard to transformers,* are applicable to alternators, although in the case of alternators the power received by the field coils is usually supplied from a direct current exciter circuit. The principal causes of loss are the same as those in direct-current machines; but on account of increased frequencies of the magnetic cycles in the armature cores, the effects of eddy currents and hysteresis are intensified. On this account particular care is required in selecting and annealing the iron for the armature cores, and in insulating the armature disks from each other. Advantage is also taken of every opportunity for ventilation.

The conditions under which a test of alternator efficiency is to be executed are usually specified with great care, including the speed, the load, the voltage, the current, the rise of temperature of windings, etc., corresponding to the circumstances under which it is desired to have the machine work. Field exciting current, rise of temperature of the bearings and the collector rings, and other phenomena of the operation of the machine are also ordinarily observed and recorded. The conditions for making standard tests are given in the Standardization Rules of the American Institute of Electrical Engineers.

1. *Efficiency by Rated Motor and Stray Power Measurements.* — In case a transmission dynamometer is used to measure the power absorbed by an alternator, the commercial efficiency is equal to the electrical output shown by a wattmeter divided by the input shown by the dynamometer readings plus the power used for separately exciting the field magnet. When the machine is separately excited, as is the case in standard, modern, commercial alternators, the energy supplied to the field magnet may be measured by amperemeter and voltmeter or by wattmeter. The machine friction may be determined from the dynamometer readings when the machine is run with its field magnet unexcited and the external circuit open. The

* Art. 137.

loss due to hysteresis and eddy currents in the armature core, armature conductors, and pole faces may be approximately determined from the dynamometer readings when the machine is operated with its field magnet separately excited and the external circuit open, and the total loss may be approximately separated into its component parts by substituting in the formulas * previously given, the power being indicated from the dynamometer readings for two values of field excitation. The I^2R losses in field and armature conductors may be readily computed from the respective hot resistances. In the special case where the machine is self-excited by a rectified current, the field current will be less than that calculated from the voltage and resistance. In that case it is advisable to use an alternating-current amperemeter, such as an electrodynamometer, to determine the effective current. From this the I^2R loss may be computed if the hot field resistance is known; or, the field loss may be directly determined with more assurance of accuracy by a wattmeter in the field circuit.

Instead of a mechanical transmission dynamometer, which is generally of a low degree of accuracy even if available, it is convenient to mechanically connect the alternator under test to an electric motor, of which the no-load losses and curve of efficiency as a function of the load are known. The input of the alternator under test can then be calculated from the electrical input of the machine that is used as a driving motor by deducting its losses. A machine used in this way as the driving motor is commonly said to be "rated" when the relations between its outputs and inputs are known for all loads and a number of speeds. It is desirable to connect the rated motor by a flexible coupling to the alternator under test. When the motor is connected to the test machine by belt, the loss of energy caused by the belt is included in the power delivered by the motor. This loss may be approximately determined by running the test alternator as a synchronous motor with and without the belt and deducting from the difference of the input readings the friction and windage losses of the rated driving machine. During this test the rated machine should be disconnected from the supply mains and be unexcited.

* Art. 110.

One of the methods of obtaining frictional or fixed losses is sometimes called the retardation test. The machine under test may be brought to full speed and then have the driving force removed. The speed is then observed at certain intervals of time. The friction loss is determined from the formula,

$$P = \frac{2\pi^2 K[V_1^2 - V_2^2]}{T},$$

where P is the power in watts dissipated in friction, V_1 and V_2 are the speeds in revolutions per second at the beginning and end of the time interval of T seconds, and K is the moment of inertia in kilogram-(meters)^2 of the rotating part. By exciting the field magnet, core losses plus friction may be approximately obtained in this way. This method requires that the weight and dimensions of the rotating part be known in order that K may be determined, and even at the best is not very accurate for ordinary dynamo tests.

It is usually best to measure the losses under conditions of no load operation and compute the efficiency by taking the ratio of the output under consideration to the output plus the losses, rather than to measure the input and output and take the ratio of their measured values. Windage, friction, and core losses at no load, the field magnet being excited with the normal excitation for the load concerned minus the armature back turns for that load, may be measured by means of the rated motor, and the I^2R losses may be computed after the hot resistance of the windings has been measured. A correction must then be made for the effect of armature reactions on the distribution and density of magnetism in the armature core which alter the core losses and also the eddy current and hysteresis losses in the pole faces of the field magnet. The magnitude of this correction depends upon the type of machine, but it ought not to be large in well designed and constructed machines. To obtain the armature resistance per phase of a wye-connected tri-phaser, the measurement may be made from each line terminal to the neutral point. Twice the resistance is obtained by making the measurement from one line terminal to another. For a delta-connected tri-phaser, two thirds of the resistance per phase is obtained by measuring from one line terminal to another.

The load given out by the alternator under test can sometimes be utilized, at least in part, by being delivered to the service mains of the plant in which the test is being made. However, frequently all and usually part of the load must be absorbed in metal or water rheostats.

A test of only poor accuracy can be made by using a rated steam engine or other heat engine as the driving machine in place of a rated electric motor, in which case the input to the engine is obtained by means of indicator cards.

The efficiency of direct-connected steam engines and alternators, especially of steam turbo-generator sets, is often given in terms of pounds of dry steam per kilowatt of output, in which case the input is determined from the condensed steam at the engine exhaust. Or when the efficiency is given in terms of the ratio of power output to input, the input is measured by means of a steam engine indicator, when this is possible. The output is always to be measured in such cases by carefully calibrated wattmeters. It is of course understood in the case of a steam turbine set that the mechanical power delivered to the dynamo is difficult to determine, but it may be approximately computed by determining the no-load losses of the alternator and turbine rotor by driving the alternator as a synchronous motor and using the data thus measured as a basis for computing the dynamo input from its output plus its losses.

In the case of water turbines the input to the turbine can be determined by the head and rate of flow of the water used.

The following methods are especially adapted to shop testing of the heating and efficiency of an alternator.

2. *Feeding Back Method for measuring Efficiency.* — It is sometimes desirable to make a test with actual full load, while avoiding the use of a power dynamometer or the necessity for supplying the full rated input from the shop circuits. With this in view a modification of Hopkinson's method of testing * may be made (Fig. 434). Two equal alternators are rigidly coupled together in proper step for parallel working. Their armatures are electrically connected together with a wattmeter and an amperemeter in the circuit. The field magnets being properly excited by a separate exciter, so that one machine will act as a generator and the other as motor, the system may be

* Jackson's *Electromagnetism and the Construction of Dynamos*, p. 256.

driven by supplying only sufficient power to make up the losses of the two machines. Assuming the armature and stray losses of the two machines to be equal, and representing the wattmeter

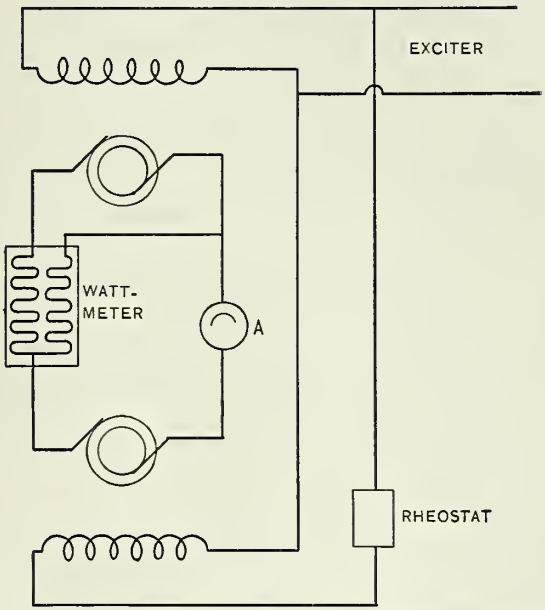


FIG. 434. — Two Alternators connected up for an Efficiency or Heat Test by a Feeding Back Method.

reading by P_e, the power supplied by P_m, and the resistance of the connections by R_1; then the efficiency of the generator is

$$\eta = \frac{P_e}{P_e + \frac{1}{2}(P_m - I^2R_1) + I_f^2R_f},$$

if $I_f^2R_f$ is the field loss of the generator. If the machines are self-exciting, the power in the field circuits must be measured and proper allowance made. The power supplied to make up the losses may be measured by a rated direct-current or alternating-current motor (as explained earlier in the article).

3. *Efficiency by Rated Motor.* — Where approximate determinations of the various losses of conversion and of the commercial efficiency are sufficient, the alternator tested may be driven directly from a "rated" direct or alternating current motor as explained earlier in the article. By means of amperemeter

and voltmeter the power supplied to the motor may be determined with the alternator operated under such various conditions as may be necessary to determine the losses. In each case the power supplied to the alternator is equal to the power absorbed by the rated motor multiplied by the efficiency of the motor given in per cent for its load in the particular test. The motor may be rated by determining its efficiency by the stray power method. If the motor efficiency for various loads is determined by some exact method, the power transmitted to the alternator may be determined with considerable exactness. When it is desired to carry the accuracy of this method of testing beyond a fairly good approximation, the efficiency at various loads of the rated motor should be plotted, so that the efficiency at any load may be readily read off.

4. *Mordey's Method.* — A neat arrangement for testing a single alternator with a stationary armature by a method akin

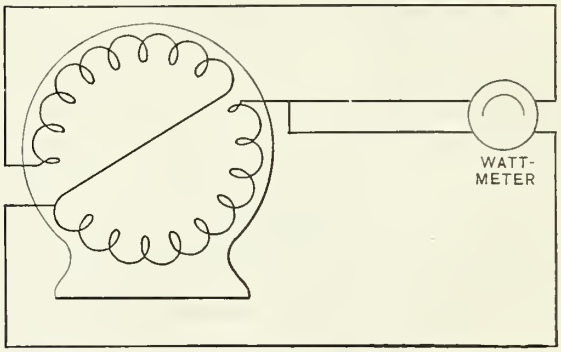


Fig. 435. — Unbalanced, Divided Armature Connection for making Efficiency or Heat Test of Alternator with Revolving Field Magnet. Applicable to Single or Polyphase Machine.

to Hopkinson's two-machine method was devised by Mordey.* It is applicable to alternators which have stationary armatures, and where the individual coils may be connected in the desired combination. Such an armature may be divded into two parts which are connected in such a way as to oppose each other (Fig. 435). If one part gives a somewhat higher voltage than the other, the first part will operate as a generator and the second as a motor if the alternator is driven in the usual

* Testing and Working Alternators, *Jour. Inst. E. E.*, Vol. 22, p. 116.

manner. By properly adjusting the difference between the voltages of the two parts, the current flowing in the machine may be caused to have any desired value. The efficiency of the machine is gained by measuring the power absorbed by the machine operating as a self-contained motor-generator and measuring its output by a wattmeter, the voltage coil of the wattmeter being connected across the terminals of the motor or generator coils and the current coil being connected directly in the circuit. The power thus measured by the wattmeter is evidently approximately one half of the total energy of the machine; consequently the corrected efficiency is

$$\eta = \frac{2\,P_e}{2\,P_e + P_m + I^2 R_f},$$

where P_m is the power supplied to the driving machine and P_e is the power read on the wattmeter.

The difference between the voltages developed in the parts of the machine which is necessary to cause the desired current to circulate may be caused by an unsymmetrical division of the armature (Fig. 435), or by supplying a little additional voltage to the generator by means of a transformer. The latter may be supplied from another alternator operating in synchronism with the machine under test or it may be supplied directly from the test machine (Fig. 436). The transformer is likely, however, to introduce uncertain elements of loss. Figures 435 and 436 may be considered to represent the windings of a single-phase alternator or one phase of a polyphase alternator. In the case of polyphase machines each phase is connected independently as in the case of the one shown in the figures. In order that the alternator may be

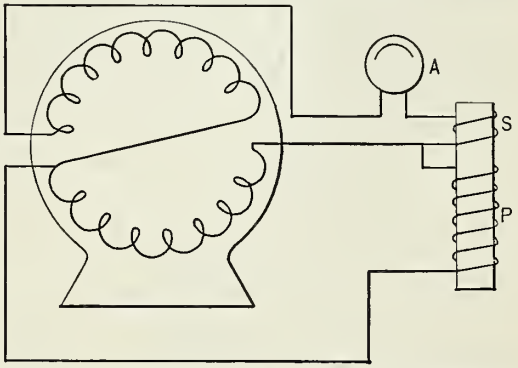


Fig. 436. — Divided Armature Connection for Efficiency and Heat Tests. Extra Voltage supplied by External Transformer. Applicable to Single or Polyphase Alternators with Revolving Field Magnet.

tested for various loads, the secondary winding of the transformer shown in Fig. 436 should contain a number of voltage taps, or a regulating impedance may be placed in series with its primary winding. For unity power factor this impedance should consist of resistance and for current lags of resistance and reactance. Instead of a transformer an autotransformer or compensator may be used. It will be noted that such methods as this one in which the divided armature is used are especially desirable for heat runs (that is, for operating the machine for a period under full excitation and full load current to determine the rise of temperature), as only a small fraction of the full load power is required to obtain full heating effects, but they are not so desirable for efficiency measurements, since dividing the armature alters the armature reactions and changes the core losses.

5. *Divided Field Method of measuring Alternator Losses or Efficiency.* — As pointed out by Professor Ayrton,* the arrangement preceding may be modified so as to apply to alternators with either rotating armatures or rotating field magnets. In this case opposite halves of the field magnet are magnetized in opposite directions, and the armature is short-circuited through an amperemeter (Fig. 437). By adjusting the relative excitation of the halves of the field magnet by means of rheostats or otherwise, a current of any desired value may be caused to circulate in the armature. If the excitation is practically normal, the measured losses of the machine when any current is flowing will be approximately equal to those when the machine is operating normally on the same current, provided it may be assumed that the losses in an alternator are appreciably equal when driven as a generator and as a motor. This assumption is reasonable for most instances in which the effects of armature reactions and leakage reactance are not large. The arrangement here described is not applicable to machines with armatures with the halves wound in parallel, since under the test conditions an amperemeter connected between the terminals of such an armature would not indicate the current circulating in the armature coils. The plan does not put full voltage on the armature circuit and therefore does not afford a complete test, besides altering the effect of armature reactions.

* *Jour. Inst. E. E.*, Vol. 22, p. 136.

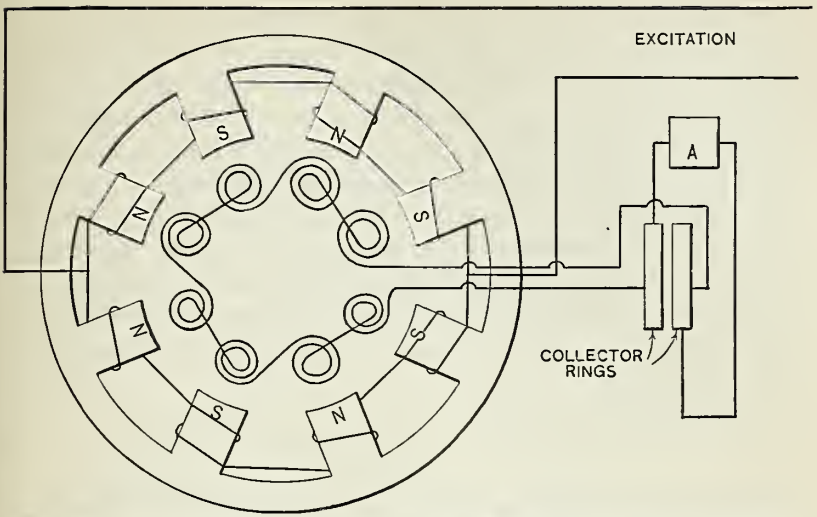


FIG. 437. — Divided Field Connection for making Efficiency and Heat Tests. Applicable to Single and Polyphase Alternators.

6. *Motor-generator and Similar Methods for Measuring Efficiency, Losses, and Heating of an Alternator.* — Mordey also suggested the following purely electrical method of testing alternators which have stationary armatures.* The machine being properly excited, one half of the armature winding of each phase is connected as a generator to an external load R (Fig. 438). The other half of the armature is connected to another

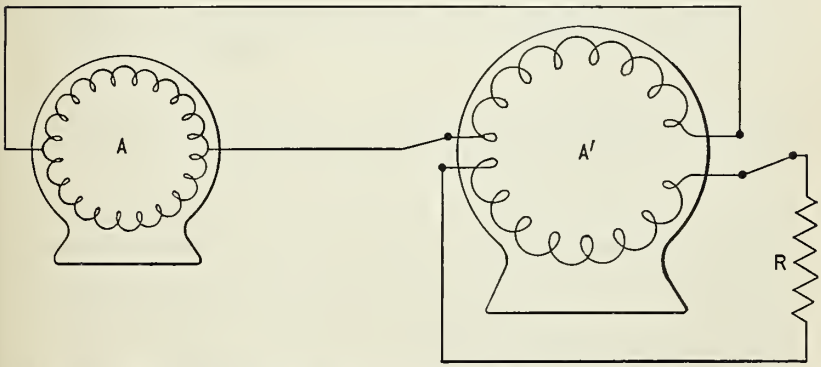


FIG. 438. — Motor-generator Method of obtaining Efficiency, Losses, and Heating of Single or Polyphase Alternator having a Rotating Field Magnet.

* *Jour. Inst. E.E.*, Vol. 22, p. 122.

alternator and driven as a synchronous motor. The total losses under these conditions are evidently equal to the power absorbed by the motor half of the armature, minus the output of the generator half of the armature ; that is, $P_m - P_g$, where P_m and P_g are respectively the power absorbed by the motor and the output of the generator. Since the output P_g is the output of only half the armature, but the losses thus determined are those of the whole machine, the losses will be approximately the same when the machine is run as a generator with an output of $2\ P_g$, provided generator losses and motor losses are the same. The efficiency of the machine as a generator is therefore approximately

$$\eta = \frac{2\ P_g}{2\ P_g + P_m - P_g + I_f^2 R_f} = \frac{2\ P_g}{P_g + P_m + I_f^2 R_f}.$$

The input and output of the machine under test may be measured by wattmeters.

Another somewhat similar plan is to divide the stationary armature into three divisions which are connected in series.

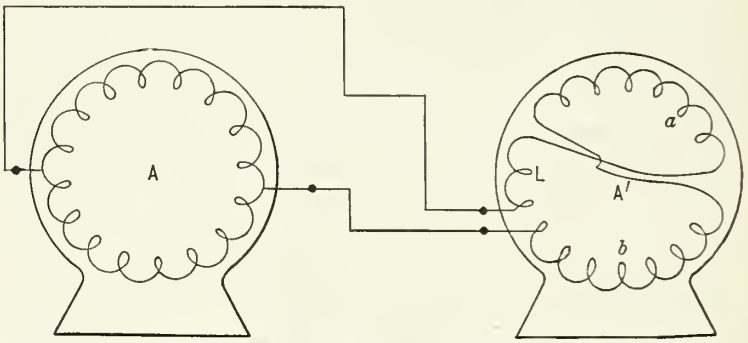


FIG. 439. — Synchronous Motor Method of testing Rotating Field Single-phase Alternator. Armature divided into Three Parts.

Two of these are made equal and are connected in opposition. The third consists of only a small portion of the armature. The machine is electrically connected to another alternator and operated as a synchronous motor under the influence of the small armature division (Fig. 439). By a proper choice of the number of coils composing the small division any desired current may be sent through the machine to be tested. The energy given to the machine represents the losses in the machine

when operated as a generator and producing the same current with the same excitation. This method is also applicable to determining the losses in machines with revolving armatures the coils of which are all connected in series. In this case the fields are excited with the poles on one half reversed (Fig. 440), one half of the field being slightly stronger than the other. If the armature is connected to that of another alternator of proper voltage, it will run as a motor. The instantaneous counter-voltage of the machine under test depends upon the relative strength of the two halves of the field magnet, and by adjusting

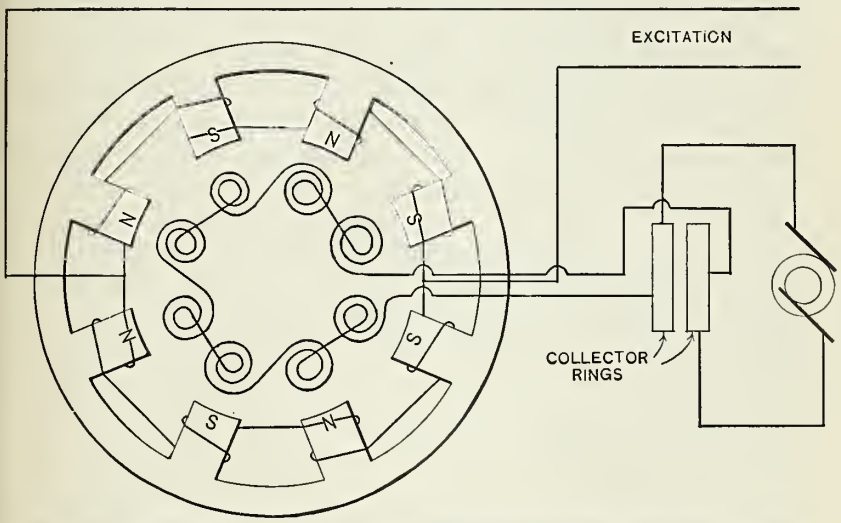


FIG. 440. — Synchronous Motor Method of testing Single-phase Alternator with Rotating Armature. Field divided into Two Parts.

this, with due reference to the impressed voltage which should be of a relatively small value, the current flowing in the armature circuit may be given any desired value. Under these conditions the losses in the test machine are equal to the power absorbed, which may be measured by a wattmeter. The core losses may be very much altered from their normal value by the practice of this method.

When, in either of the cases mentioned heretofore, the operation of an alternator as a motor is predicated, it is assumed either that the test machine is brought to synchronism with the alternating source, or that the primary generator is started from

a state of rest after the circuit connections are made with the machine to be tested, in which case the duly excited test machine will start and run with the generator.

These methods of testing are not only sometimes convenient in determining the losses and the efficiency of an alternator, but the tests, according to several of the methods, are made with the consumption of comparatively little power. This makes the methods satisfactory for use in shop tests for determining the reliability in operation and the heating limits of machines. Mordey suggested that the efficiency of an alternator with stationary armature may be determined from a test of one armature coil, but this cannot serve as a satisfactory shop test, which requires a test of the complete machine.

7. *Heating Tests.* — The heating of an alternator should be determined much as is done in the case of a transformer.* The machine should be run at specified load under normal conditions until the temperature has become constant. The temperatures of field and armature windings should be determined by both resistance measurements and thermometers. The temperatures of the cores, collector rings, bearings, and other parts must be measured by thermometers. After a machine has run until thermometers on the stationary parts have reached constant readings and is then stopped, the thermometers on ventilated stationary parts will again begin to rise. This is due to the heat from interior masses of the constructive material tending by conduction to equalize the temperatures at the surface and interior. The temperatures recorded for the test should be the highest obtained. The heating test can sometimes be made conveniently by alternately running the machine with the armature short-circuited, with the field flux of such a value that something over full load current flows, and then running it with open circuit armature and a field flux which will give something over normal no-load core losses (this can be determined by the power input). The alternate periods of each kind of operation should be short, *i.e.* only a few minutes. The short circuit current and its period of flow should be so adjusted that the product of the average I^2R loss times the time of the runs is equal to the kilowatt hours which would be expended by the rated full load current flowing constantly during the total time of the test.

* Art. 147.

Likewise the iron loss and its period of activity should be so adjusted that its average value times the time of the runs is equal to the kilowatt hours which would be expended by the normal full load excitation during the period of the test.

8. *Insulation Tests.* — Dielectric strength tests should be made as in the case of transformers.* The voltage applied should follow the accompanying specification, which is in general applicable to all electrical machinery. The voltage wave should have an approximately sinusoidal form. In the case of field windings, the voltage of the exciter circuit should be used in obtaining the test voltage from the succeeding table, except where the alternator is to be used as a synchronous motor and is brought to speed as an induction motor, when the test voltage should be 5000 volts, since high voltages are likely to be induced in the field windings under such conditions. In testing the dielectric strength of high voltage machines care should be exercised, as in the case of transformers, to bring the testing voltage up very gradually.

It may be noted that high voltages in a large machine may cause appreciable heating due to the dielectric loss. This amounts to from 3 to 5 or more kilowatts in some of the largest machines now in service while generating voltages of 10,000 volts and over.

TABLE OF DIELECTRIC STRENGTH. TEST VOLTAGES†

Rated Terminal Voltage of Armature	Rated Output	Testing Voltages
Under 400 volts	Under 10 Kw.	1000
Under 400 volts	10 Kw. and over	1500
Under 800 and over 400 volts	Under 10 Kw.	1500
Under 800 and over 400 volts	10 Kw. and over	2000
Under 1200 and over 800 volts	All	3500
Under 2500 and over 1200 volts	All	5000
Over 2500 volts	All	Double the normal rated voltage

9. *Regulation.* — The regulation of a constant voltage alternator may be obtained by loading the machine to full load at normal speed, voltage, and other normal full load conditions,

* Art. 147.

† Standardization Rules of American Institute of Electrical Engineers.

and then removing the load without change of speed. The regulation is the ratio which the difference between the voltage at full load and the voltage at no load bears to the voltage at full load. Knowledge of the regulation and the efficiency are frequently desired for fractions of full load and at various power factors. In the latter case full load is rated in kilovolt-amperes, and the normal full load conditions of the test must include the power factor concerned. Except where otherwise denominated, however, testing is done with loads of unity power factor. When the power factor is less than unity with current lagging, the excitation must be larger than is required to obtain equal voltage at unity power factor and equal armature amperes. The losses are then far greater at the lower power factor, and both the efficiency and regulation are depreciated.

In the combined regulation of direct-connected units composed of an engine and constant voltage alternator, the speed is made the normal value for full load and is supposed to be the same for no load. However, when the load is suddenly removed, there is a short fluctuation of speed. The speed regulation is then commonly taken as the ratio of the greatest instantaneous change of the speed at the instant the load is removed to the normal full speed. The rapidity with which the load is removed is apt to have an influence upon this ratio. The instantaneous values of the speed fluctuation can be obtained by a sensitive speed-recording instrument, or the necessary data can be obtained by a recording voltmeter or oscillograph connected to a small magneto-generator driven from the engine shaft, in which case the speed variation can be calculated from the variation in instantaneous voltage.

10. *Wave Form.* — The wave form of the voltage produced by an alternator can be obtained by means of the oscillograph or otherwise as already explained at some length.*

As said earlier, the conditions surrounding the measurements of dielectric strength, regulation, and other quantities during a test should, in America, be in accord with the Standardization Rules of the American Institute of Electrical Engineers, to which reference has already been given earlier in this article, unless other specifications are specially provided.

* Art. 157.

A **Power-factor indicator** is often of service in testing and also for a permanent place on a switchboard. This may be built on the same principle as the synchroscope explained earlier (Fig. 418), or it may be built like a two-phase induction motor. In the latter case a current having the phase of the line current is passed through the windings of one phase, and another having the phase of the voltage is passed through the other, and the torque of the armature is then a function of the phase angle. Revolution of the armature being restrained by a spring, the degrees of rotation of the armature are proportional to the torque and therefore indicate the phase angle. A pointer attached to the armature and arranged to play over a scale completes the essential parts of the instrument.

173. Hunting of Synchronous Motors and Other Synchronous Machines. — All machines, such as alternating current generators, synchronous motors, and rotary converters, which depend upon a

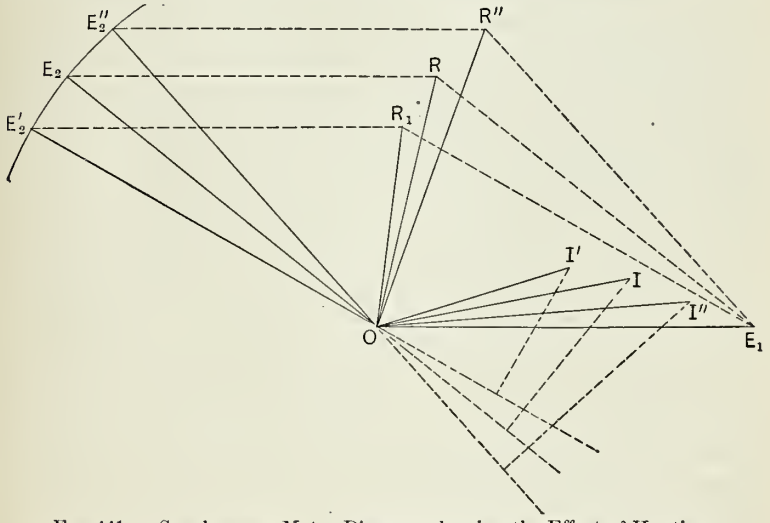


Fig. 441. — Synchronous Motor Diagram showing the Effect of Hunting.

series or synchronizing current to hold them in step, are subject to vibratory irregularities in angular velocity which are given the name of **Hunting.** In explanation of this effect consider a synchronous motor of which the voltages and current relations are illustrated in the diagram of Fig. 441. In this diagram the impressed voltage E_1 is kept in a fixed position and is assumed

to be of constant scalar value. The counter-voltage E_2 is also assumed to be of constant scalar value. Now suppose that the motor load has just been reduced from a higher value to one where the normal relations of the voltages and current are represented by the parallelogram OE_1RE_2 and the line OI, OR being the resultant voltage and OI the current. These relations will not, however, immediately prevail, because in moving forward to its new space-phase position the rotating part of the machine is required to take a momentary speed somewhat greater than that of synchronism. The added momentum gained by the rotor then tends to drive it farther ahead in space phase than the normal position, so that the counter-voltage takes a position such as OE_2', which is too far advanced for steady balance, and the resultant voltage and current vectors at the same time move forward in phase and fall off in length as illustrated in OR_1 and OI'. The component of the current in opposition to the counter-voltage also falls off in length, and the resisting moment of the load being then greater than the electro-magnetic torque between rotor and stator, the extra momentum is absorbed in the mechanical load and the speed is reduced to normal. Now the resisting moment of the load still being greater than the electro-magnetic torque, the rotor must fall back. It then must for an instant slow its speed to below that of synchronism and thus it gives up to the load some of its momentum. When it has reached the space-phase position where the counter-voltage for the load is normal, its speed is, therefore, less than normal, and it drops back still farther, as, for instance, so that the counter-voltage lies in OE_2''. But as it falls back the current increases and the electro-magnetic torque becomes greater than the resisting moment of the load. In the figure the limit to which the rotor has fallen behind proper step is supposed to have been reached when the counter-voltage is OE_2'' and the current OI''. The electro-magnetic torque while the rotor is at the retarded point is greater than the load torque, OI'' being above normal for the load, and the armature tends again to gain speed in excess of synchronism and step ahead to normal position. The excess momentum due to excess speed may again carry the rotor beyond the normal point for the load and the whole process be repeated over and over again. If there were no losses caused by this vibration, it

would continue indefinitely at full amplitude. There are, however, frictional, magnetic, and electric losses. The frictional losses of the rotor bearings and of windage absorb some of the excess energy stored in the rotor which produces the vibratory movement, just as frictional losses tend to absorb the energy of a pendulum which has been set swinging. The hysteresis loss and eddy, or short-circuit, currents in the iron core and field windings caused by the changes of the current value act in the same way. In connection with the latter, loss is also occasioned by the inductive action of the armature magnetic flux as it sweeps through the iron of the field poles because of the superimposed vibratory motion of the rotor. If no new disturbance occurs, this energy loss ordinarily causes the vibration to rapidly reduce to zero, and the voltages and currents assume and maintain their normal relation. In this case the rotating part of the machine has the characteristics of a torsional pendulum, of which the normal position gives counter-voltage in the position OE_2 (Fig. 441), and which is drawn toward that position by the electro-magnetic torque, but which possesses the moment of inertia of the rotating part of the machine with respect to the shaft. The normal position of rest of this torsional pendulum travels in synchronism with the uniform theoretical rotative speed of the rotor, and the pendulum vibration is superposed on that rotative speed.

A mechanical illustration of the conditions can easily be devised. Thus, suppose a mass of given weight is attached to the lower end of a spiral spring and is lifted at a uniform rate by a rope attached to the upper end of the spring. Suppose now that some of the mass breaks away and thus causes the remaining mass to vibrate up and down. The spring is first shortened due to the lessened weight, but the momentum given to the mass causes it to pass beyond the normal position that it should assume. The tension on the spring in pounds is then less than the supported weight, and the mass falls back. When it reaches normal position again, it is rising with less than the speed of the rope and, because of its inertia or force of momentum, it falls below the normal position. The spring is now stretched so that its tension in pounds is greater than the weight of the mass, which is therefore pulled back, and the process may be repeated over and over. The molecular friction

of the spring and the extra wind friction of the mass due to the vibratory motions dampen down the amplitude of the vibrations until they become zero. The product of the speed of the rope by the weight of the mass is equivalent to the output of the motor, or its rotor speed times the load torque. The weight is equivalent to the rotor load torque ; the momentum of the mass to the rotor momentum ; and the pull of the spring on the mass to the electro-magnetic motor torque.

The time of vibration of the mass may be expressed

$$T = a\sqrt{MK},$$

where M is its weight and K the compressibility of the spring. Likewise the same formula is approximately applicable to the motor, where M is the moment of inertia of the rotor about the rotating axis, and K represents the reciprocal of the rate of change of the electro-magnetic torque with reference to the distance of the rotor from its normal instantaneous position. It is seen, therefore, that the time of vibration is increased with increase in weight and diameter of the rotor and with decrease of the rate of change of electro-magnetic torque per unit of distance that the rotor is away from its normal instantaneous position. Now this latter factor depends upon the rate of change of the component of the current vector which is in opposition to the counter-voltage for the various positions of the rotor with reference to its normal instantaneous position. A study of the diagrams given in Figs. 424, 425, and 441 will show that the angle ψ between the resultant voltage and current should be small to give the greatest proportional change of electro-magnetic torque with a swing of the rotor through a small given angle. The armature impedance should evidently be small, but also to make ψ small, $2\pi fL$ should be small compared with R. If R is very small compared to $2\pi fL$, the conditions may permit the hunting to increase, eventually throwing the armature out of step. This result may also occur through periodic changes in the angular speed of the prime generators or of the motor load, provided the period of such changes chances to approach equality with the natural period of vibration of the rotor about its axis as shown in the preceding formula.

If the field windings are of low resistance, they act in effect like the short-circuited secondary of a transformer to prevent

the sweeping back and forth of the armature flux; that is, they reduce the apparent reactance of the armature and increase its stability. Instead of lowering the resistance of the field windings, short-circuited squirrel cage conductors are sometimes laid in the faces of the field poles, as shown in Fig. 442. These are sometimes called **Amortisseurs** or **Damping grids**. They form excellent short-circuited secondary paths for the production of currents by the vibrating armature flux and the dissipation of the energy of vibration by its conversion into heat.

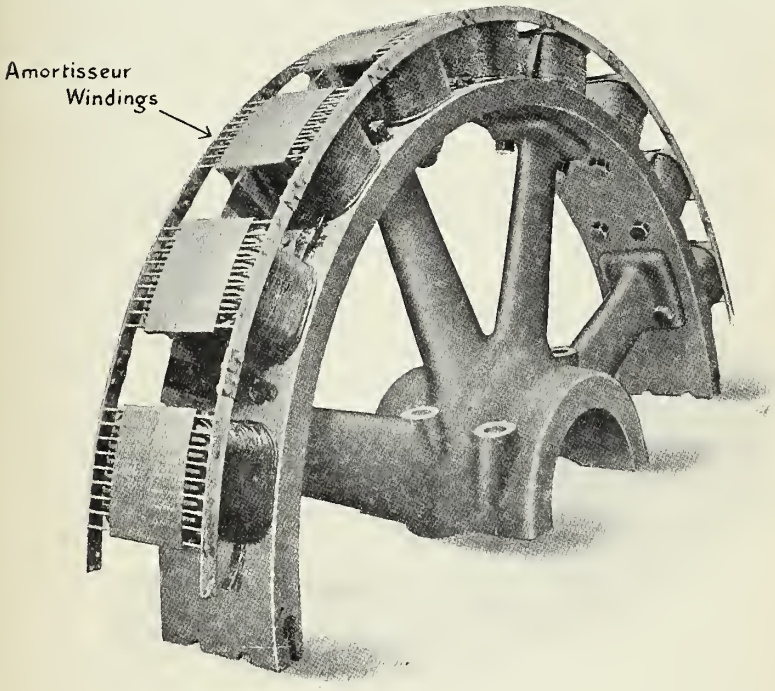


Fig. 442. — Amortisseurs for preventing Hunting.

The damping effect of these windings due to the currents caused by the rotor vibrations may be understood when it is remembered that the armature magnetic flux for a polyphase synchronous motor is normally stationary with reference to the poles of the field magnet in case the machine and line voltages are in exact isochronism. Then when relative movements or vibrations of the armature occur with respect to its normal

instantaneous positions with reference to the field magnet, the magnetic flux of the armature currents acts upon the amortisseur windings and other metal parts of the field magnet exactly as though their vibratory motion was the only one that existed. In accord with the principles of electro-magnetic induction, it is to be observed that the currents induced by this motion of the armature flux with respect to the damping grids produce electro-magnetic torques tending to destroy the motion. This is similar to considering that the mass and spring used in the earlier illustration are vibrated while the rope is stationary, which is entirely justifiable, as the vibrations have no effect on the upward movement of the rope. The effect of the amortisseur windings may also be compared to the damping effect which will be produced upon the weight by a medium having greater frictional resistance than air, such as would be the case if the weight were immersed in water or oil.

As noted earlier, amortisseur windings are of service in starting polyphase synchronous motors by means of rotating field induction motor torque.

174. The Synchronous Condenser. — It has been shown that a synchronous motor causes a quadrature lagging current to flow in the line when its field magnet is under-excited, and that it causes a leading current to flow when it is over-excited. In the above statement normal excitation is taken as that which gives unity power factor to the input of the motor, for the load under consideration.

This characteristic of the synchronous motor makes it valuable for use as a power factor regulator, and when so used it is called a **Synchronous condenser.**

As an illustration of this service, consider a long transmission line loaded at the receiving end with induction motors which cause, during a certain period when their combined mechanical load is equal to 500 kilowatts, a lagging power factor of 80 per cent. Then the total kilovolt-amperes are 625 and the quadrature kilovolt-amperes are 375. The kilovolt-ampere triangle for these conditions is marked ABC in Fig. 443. As the sides of this triangle, AB, BC, and AC, respectively, equal the impressed voltage multiplied by the active current, by the lagging quadrature current, and by the total current, they may be scaled to represent either currents or kilovolt-amperes. If it is

desired to increase the power factor of this circuit to unity, a synchronous motor supposed for the present to be without losses may be employed at the receiver end of the line, which is excited so as to demand from the circuit 375 leading quadrature kilovolt-amperes. Since the motor's input is limited by the heating of the armature conductors, its rated capacity must be at least 375 kilovolt-amperes. Mechanically it need not have this rating, for if no load is placed upon its shaft, the only torque required is that sufficient to overcome losses of rotation. If the voltage of supply is assumed to be 10,000 volts and three phase, the quadrature current of the synchronous condenser is $12.5\sqrt{3}$ amperes. Since such a motor in fact has some losses, a little true power must be supplied to it, which is represented in the figure by $BE = CD$, so that the actual kilovolt-amperes of the motor armature are represented by CE.

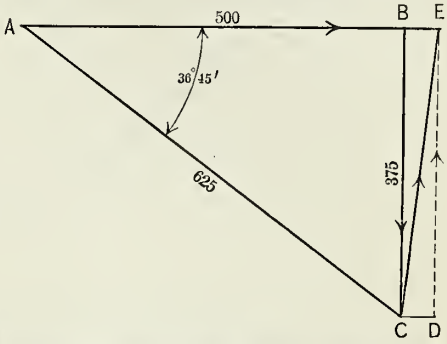


FIG. 443. — Diagram showing the Components of Apparent, True, and Quadrature Power in a Line supplying an Inductive Load; and the Rating of the Synchronous Condenser required to increase the Power Factor to Unity.

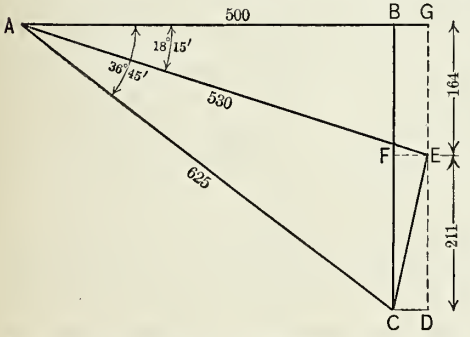


FIG. 444. — Diagram for determining the Rating of an Unloaded Synchronous Condenser required to increase the Power Factor of a Load.

If it is desired to increase the power factor from .80 to .95, the relations are as shown in Fig. 444. In this case the angle of lag must be reduced from 36° 45′ to 18° 15′. To bring this about, the synchronous condenser must provide approximately 211 kilovolt-amperes in leading quadrature, which comes to $7\sqrt{3}$ amperes in a three-phase circuit at 10,000 volts. The power losses of the condenser are assumed to equal CD, the quadrature kilovolt-amperes necessary to be provided

by the condenser are equal to DE, and the lagging component of the kilovolt-amperes still in the line is equal to BF. The rated loading of the condenser must be CE kilovolt-amperes.

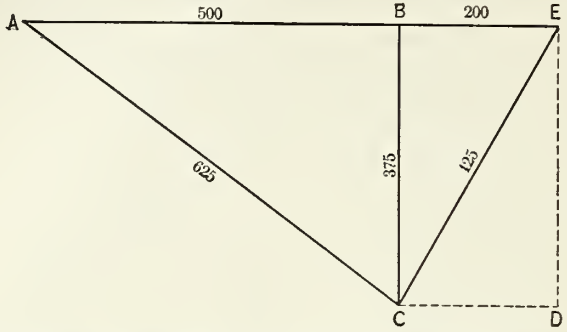


FIG. 445. — Diagram showing Power or Current Relations in a Circuit containing a Loaded Synchronous Condenser, which raises the Line Power Factor to Unity.

When it is desired to increase the power factor and at the same time obtain mechanical power from the motor, the diagrammatic constructions are as in Figs. 445 and 446. In Fig. 445 the synchronous condenser is arranged to deliver 185 kilowatts of mechanical power and is then supposed to have losses equal

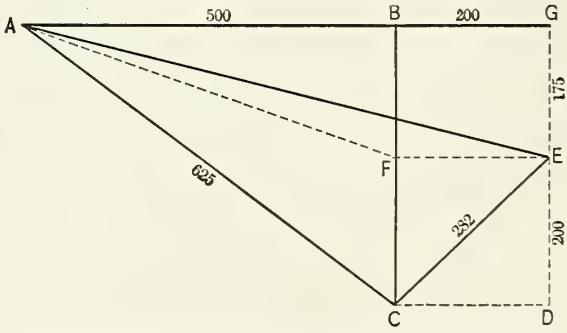


FIG. 446. — Diagram showing Power or Current Relations in a Circuit containing a Loaded Synchronous Condenser. The Line Power Factor is increased by the Condenser.

to 15 kilowatts. Therefore, to bring the circuit to unity power factor, it must receive from the mains $200 = BE$ kilowatts of power, $375 = DE = -BC$ of quadrature kilovolt-amperes, and a total of $425 = CE$ total kilovolt-amperes. In Fig. 446 the synchronous condenser is adjusted to only reduce the quadrature power by $200 = DE$ kilovolt-amperes, although it receives

the same true power as in Fig. 445. It will be noted that the true power load of the condenser itself has an influence on the angle of lag of the line current, for in Fig. 446 the reduction of the lag angle would have been less by the angle FAE had the motor run without mechanical load but had still provided 200 $= DE$ quadrature kilovolt-amperes. It will also be noticed that the total loading of the synchronous condenser of Fig. 446 is only 282 kilovolt-amperes when receiving 200 $= CD$ kilovolt-amperes true power and 200 $= DE$ kilovolt-amperes quadrature power. It is evident from the construction that the smallest motor capacity is required for a given arithmetical sum of true and quadrature volt-amperes when the two components are equal.

In general we have the loading required for a synchronous condenser,

$$P_t = \sqrt{P_a^2 + P_q^2}, \text{ or } I_t = \sqrt{I_a^2 + I_q^2},$$

where P_t, P_a, and P_q are the total, active, and quadrature volt-amperes received by the machine, and I_t, I_a, I_q, are the total, active, and quadrature currents.

From the current locus of the synchronous motor, it may be seen that the quadrature kilovolt-amperes for no mechanical load on such machines make quite a large figure when the fields are over-excited, and that these increase up to a certain point as the load is increased (see current locus $FG''H$, Fig. 430). This may also be seen by drawing a number of constant power current loci for a synchronous motor as in Fig. 431, then for a given over-excitation which gives a current locus such as SPS'' in that figure, the quadrature leading current increases with the load, thus, the quadrature leading current increases from OR at no load to OQ as the load increases from that represented by the locus M'' to the constant load locus that would pass through the point P. For further increase of mechanical load the quadrature current decreases. The line QP is parallel to OE_1, and the point P is the tangent point between QP and SPS'. It is therefore impossible to exactly neutralize the lagging current of induction motors, when the loads vary, by the use of loaded synchronous motors with fixed excitation. Thus, if the synchronous motor is excited so as to supply the right quadrature current when driving its average mechanical load and when the load of the induction motors on the line is average, it

will supply more or less than enough if its mechanical load increases or the load of the induction motors decreases, and *vice versa* if the synchronous motor load falls off or that of the induction motors increases. However, by setting the synchronous motor excitation at the average value referred to, the actual variation from the correct value, for commercial circuits, is not likely to vary more than a few per cent. When necessary, the excitation can be changed to more closely adjust the balance. This is usually done by hand, but if the need is sufficiently great, an automatic regulator can be constructed, using a phase-indicating device in the main circuit to operate relays for controlling the field rheostat or exciter rheostat of the condenser.

Synchronous condensers are desirably placed on a part of a transmission line near a group of apparatus, such as induction motors, which cause the flow of lagging current. The condenser then performs a twofold service; that is, it improves the line regulation, and at the same time reduces the I^2R loss in the line for a given number of kilowatts transmitted. Thus, suppose the line supplying the load diagrammed in Fig. 443 is compensated to unity power factor. The line loss which was proportional to $(625)^2$ becomes proportional to $(500)^2$ or it has been reduced to close to $\frac{5}{8}$ of its original value. This saving in power in itself would in many cases warrant the installation of a synchronous motor to float without mechanical load on the line. If it is not desired to reduce the line loss, the saving in copper would be nearly $\frac{5}{8}$ of that which would be demanded without the condenser, for the instance mentioned. If the total copper is large, the saving may be very material, especially if the uncorrected power factor is very low. If the synchronous condenser replaces other motors by carrying their mechanical load and has a sufficient rating so that it can carry an equal quadrature load (Fig. 446), the total kilovolt-amperes rating required over that of the motors replaced is in the ratio of 1 to $\sqrt{2}$. If the quadrature power to be neutralized is not great, the increase may be but a small percentage of the power of the total apparatus installed.

The line regulation may be improved by the use of the synchronous condenser from two causes, first, by changing the current flowing and bringing it into phase with the voltage, and thus changing the IZ drop and bringing the IX component

to a more favorable vector position; and second, by the automatic tendency in the synchronous motor to maintain constant voltage. Thus, referring to the second cause, if the line voltage drops, the motor will have a relatively higher over-excitation and thus draw additional leading current from the line, which will tend to strengthen the fields of the generating apparatus and neutralize the effect of line inductive reactance. If the line exceeds normal voltage, the counter-voltage of the motor will be less in proportion thereto, and the motor will draw less leading current from the circuit, thus leaving unneutralized inductive reactance in the line; or if the rise of line voltage is sufficiently great, the motor may draw lagging current from the line, and thus directly add to the inductively reactive element of the circuit impedance.

The first of these two effects is illustrated in Fig. 447, in which OE represents the voltage vector at the receiver end of the line, OI the current vector delivered by the line at the receiver end, and OE_0 the voltage vector at the generator

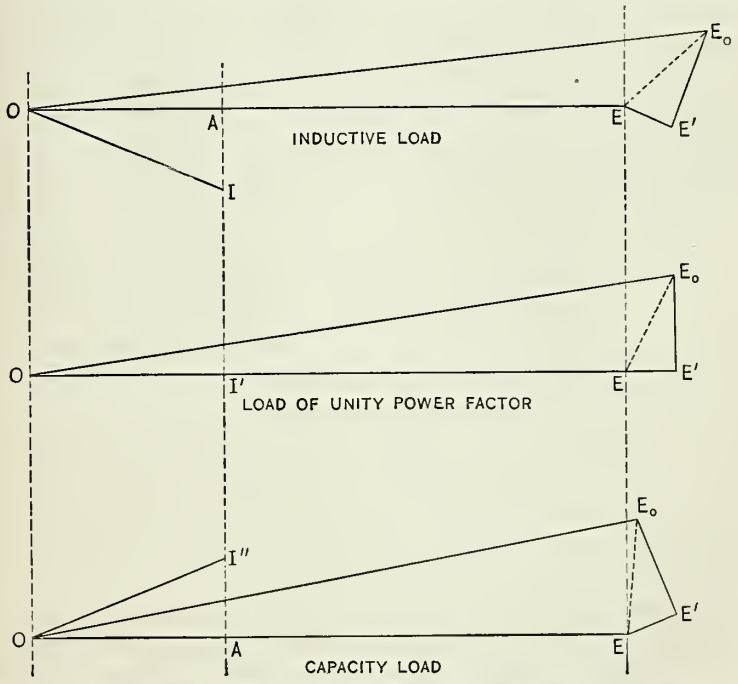


FIG. 447. — Illustration of Effect of Power Factor at Receiver End on Regulation of Transmission Line.

end of the line. In the upper diagram of this figure, the current lags behind the voltage, but gives the active component OA which is proportional to the delivered power. The vector EE' parallel to OI represents the IR drop of the line, and the vector $E'E_0$ in leading quadrature with OI represents the IX drop of the line. Hence EE_0 is the IZ drop and $\overline{OE_0} = \overline{OE} + \overline{EE_0}$ is the generator voltage. When the same power is delivered by the line at the same voltage, but the power factor is brought to unity by means of a synchronous condenser, the current takes the value OI' shown in the middle diagram of the figure, which is equal to OA of the first diagram. In this case EE' and $E'E_0$ are shorter than in the first diagram, in the ratio of the lengths $\frac{OA}{OI}$, and are respectively in parallel and quadrature with OE. Under these circumstances the generator voltage OE_0 does not differ from the receiving voltage OE by as many volts as in the first diagram. Now, if the quadrature current of a synchronous condenser causes the current to lead the voltage at the receiving end sufficiently to bring the power factor to the same numerical value as in the first diagram, the active component remaining equal to OA, the current is illustrated by OI'' in the bottom diagram in the figure which has the same length as OI in the first diagram. The IZ drop is EE_0, which has the same length as EE_0 in the first diagram, but its position now results in still less difference between the numerical values of OE_0 and OE than in the second diagram. By still further increasing the current lead, the generator voltage required to transmit the power over the line may become smaller than the voltage maintained at the receiver end.

Synchronous condensers are sometimes useful in generating stations, where they care for the lagging current coming in on the line. Synchronous motors of the ordinary type may be used as synchronous condensers; or rotary converters* may be so used, in which case the true power output of the machine, if any, is electrical instead of mechanical.

175. Converters. — Reference has already been made to the possibility of converting a direct-current dynamo into a single-phase alternator, and the fact that a machine so constructed, with

* Art. 175.

commutator and alternating-current collector rings, may be used as a rotary converter to convert alternating currents into direct currents, or *vice versa*, or as a double-current machine to generate or absorb both alternating and direct currents at the same time.* Similar combinations may also be applied to polyphase machines. When such a machine receives alternating currents through its collector rings, running as a synchronous motor, and gives out direct currents through its commutator, it is called a **Converter** or **Rotary Converter.** When it receives direct current through its commutator, running as a shunt-wound motor, and gives off alternating currents through its collector rings, it is called an **Inverted Converter** or **Inverted Rotary.**

It is possible in this manner to make a quarter-phaser to be used with separate circuits out of any direct-current machine with a closed circuit armature winding, such as the Gramme or Siemens winding, by arranging four collector rings on the shaft and connecting them to the armature windings at points which are 90 electrical degrees apart. It is also possible to make a three-phaser out of a direct-current machine by arranging three collector rings on the shaft, and connecting them to the armature windings, at points 120 electrical degrees apart. Such machines may be used as converters to transform direct currents into polyphase currents or *vice versa*. Likewise, a six-phaser can be constructed by using six collector rings connected to the armature winding at points 60 electrical degrees apart.

In Fig. 448 is shown a diagram of a multipolar machine with the connections of the armature winding indicated from A to B. The distance from A to B is 360 electrical degrees, so that the taps to be used for a single-phase machine are those numbered 2 and 5. The left-hand tap marked 2 is the first connection for the succeeding 360 electrical degrees of armature circumference, and it must also be connected to ring 2 to complete the circuit through the windings of the second 180°. For a quarter-phase machine rings and taps 2, 1, 5, and 8 are used; for a tri-phase machine the taps and rings to be used are marked 2, 4, and 6; while for a six-phase machine they are 2, 3, 4, 5, 6, and 7.

Suppose now a commutator is attached to the windings of the machine as shown in Fig. 448, with the brushes for connecting

* Art. 27.

to direct-current circuits placed upon the commutation points as indicated, then the machine so constructed may be used as a converter. It will run in synchronism when fed with alternating currents through the collector rings, and its speed therefore depends upon the number of poles in the field magnet and the

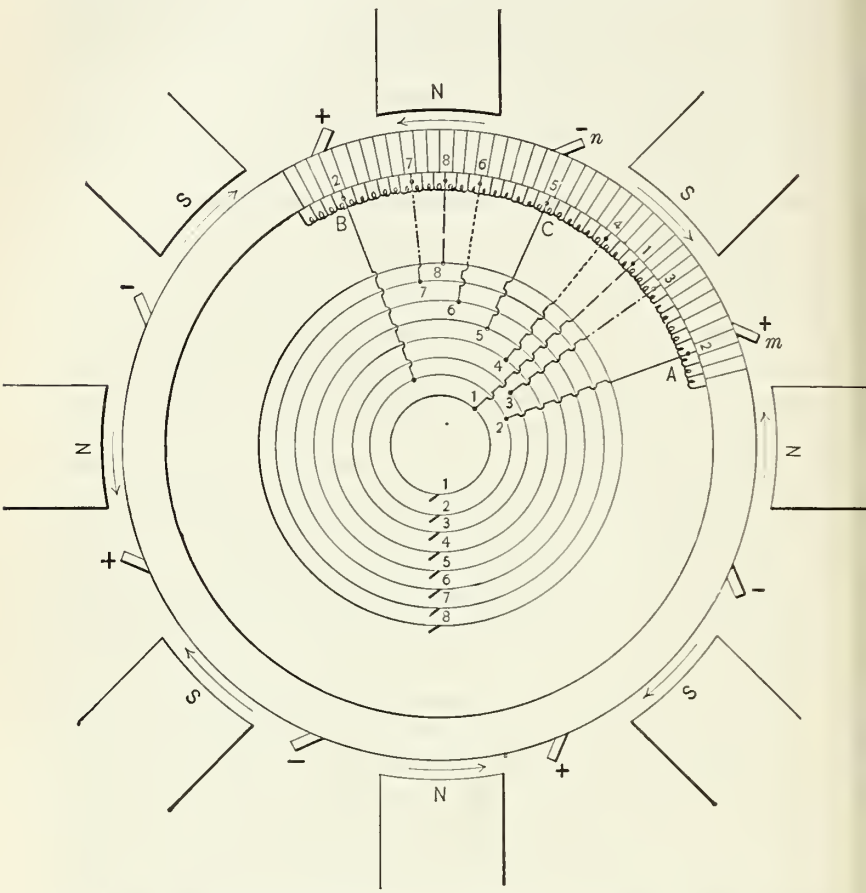


FIG. 448. — Diagrammatic Arrangement for showing Alternating and Direct-current Connections for a Closed Circuit Armature.

frequency of the currents fed to it. Polyphase converters are generally self-starting from the alternating-current end by the effect of eddy currents set up in the pole pieces by the rotary field which exists in the armature when polyphase currents are passed through it. The starting torque may be increased, as in

polyphase synchronous motors, by embedding short-circuited copper bars, or amortisseur windings, across the pole faces. After a converter fed by alternating currents is in synchronism, its field magnet may be magnetized by the direct current produced by itself and collected from its commutator.

In connecting the armature windings of converters to the collector rings, the relative angles corresponding to the current phases must be carefully distinguished as indicated above in the description of Fig. 448. One complete revolution of an armature in a two-pole field corresponds to one complete period of the alternating current, and therefore 360 mechanical degrees corresponds to 360 electrical degrees; but in multipolar machines an extent of rotation of the armature equal to twice the angular pitch of the poles corresponds to one complete period, so that, in general, the relation of electrical degrees to mechanical degrees is $p : 1$, where p is the number of pairs of poles in the field magnet. Two-pole converters evidently utilize the whole of the armature winding with each collector ring connected to a single point on the winding, and the same is true of multipolar machines with two-path armature windings. If single connections to the collector rings should be used in multipolar machines with multiple-path armature windings, a portion only of the armature, corresponding to 360 electrical degrees, would be occupied in utilizing the alternating currents; the armature capacity would therefore be not fully utilized and the machine would be seriously unbalanced. To fully utilize the armature in this case, each collector ring must be connected to the winding at as many points as there are pairs of poles, the points being 360 electrical degrees apart. In Fig. 448 the taps and windings are shown only for 360 electrical degrees. Evidently to complete the connections, the commutator and windings should extend over the entire 360 mechanical degrees and there should be a total of 8 p taps like those shown in the diagram, similar taps in each 360 electrical degrees of circumference being connected to the same collector rings. In this figure the collector rings for single, quarter, three, and six phases are all connected to one armature winding for purposes of comparison. It should be understood that only the number of rings necessary on the system for which the machine is designed to be used are ordinarily placed on the armature shaft.

In general, for any number of taps, the number of collector rings equals the number of phases and the number of electrical degrees between the taps of a phase is equal to $\frac{360}{N}$, where N is the number of phases. In this case N for the single-phase system must be taken as equal to 2, since there are really two phases 180° apart in that system, and for the quarter-phase system N must be taken as equal to 4, as the currents flowing in the four wires differ in phase by 90°, when followed around in order.

176. Ratio of Transformation in Converters. — Referring to Fig. 448, it may be observed that when the machine is driven as a synchronous motor by impressing alternating voltages upon a set of rings, the field magnet, armature windings and commutator constitute a direct-current generator. Consider a theoretical case first, where the armature windings are without resistance and the voltage generated by the revolving conductors is always equal and opposite to an alternating voltage supplied from a single-phase system to the winding section between two rings through which it is supplied, — the two rings considered here being those numbered 2 and 5 in Fig. 448. Then, when the armature is in the position shown in the figure with the points 2 and 5 just under the direct-current brushes, the generated and hence the impressed alternating voltages are a maximum and equal and opposite. This is on the assumption that there are no armature reactions and that the brushes are therefore upon magnetic neutral points. But if there are enough coils in the winding and segments in the commutator to make the direct-current voltage between the brushes of constant value, the voltage generated, at the instant under consideration, is the constant direct-current voltage between each pair of brushes, which is equal to the maximum value of the alternating-current voltage. This latter may be called e_m.

By the assumptions the voltage generated in a section, say AC, is to be opposite to the impressed voltage in the same section. When the middle of this section is under brush m, the voltage is zero because that generated in the two halves of the section neutralize each other. It is again zero when the middle of the section is under brush n. Suppose that the generated and impressed voltages in this section are of sinusoidal

wave shape, then, since the direct voltage is constant, the relation between the direct and alternating voltages is

$$E_2 = \frac{e_m}{\sqrt{2}} = .707\ e_m,$$

where E_2 is the effective impressed alternating voltage. But e_m has been shown to be equal to the direct voltage, which may be called E. Hence $E_2 = .707\ E$.

By the methods used in the construction of Fig. 33 * it was shown that when a wide coil moves in a magnetic field so that a sinusoidal voltage is induced in each conductor, the vector voltages of the individual conductors form a vector diagram, of which the resultant is the voltage between the coil terminals, and is of such shape that the junction points of the component voltages lie in the arc of a circle. The greater the number of conductors, if uniformly distributed, or of slots if they are placed in slots in a given width of winding, the nearer the component vectors coincide with the circular arc, and the wider the coil, the longer the arc, so that with a coil covering 180 electrical degrees it is a semicircle. Let the semicircle *OJGCDKA* of Fig. 449 represent the vector polygon of the component voltages of the coil section *AC* in Fig. 448 and semicircle *ALFBHMO* represent the similar polygon for section *CB* which acts in parallel with *AC* and is therefore in series opposition with respect to induced voltages. It is assumed in this that the windings are so thoroughly distributed and the component voltage vectors are so short that the vector polygon and the circumference of the arc coincide, which for the purpose causes no appreciable error. The diameter *OA* is then the alternating voltage between the terminals of either section and equals $\frac{E}{\sqrt{2}}$.

Suppose now the rings 2, 1, 5, and 8 only are connected in Fig. 448, which results in a quarter-phase machine in which the alternating-current coil sections are one half as long as a single-phase section between 2 and 5. The voltage between connectors 2 and 1 is equal to vector *OC* of Fig. 449 in length and direction; that between 1 and 5 is equal to *CA*; that between 5 and 8 is equal to *AB*; and that between 8 and the

* Art. 20.

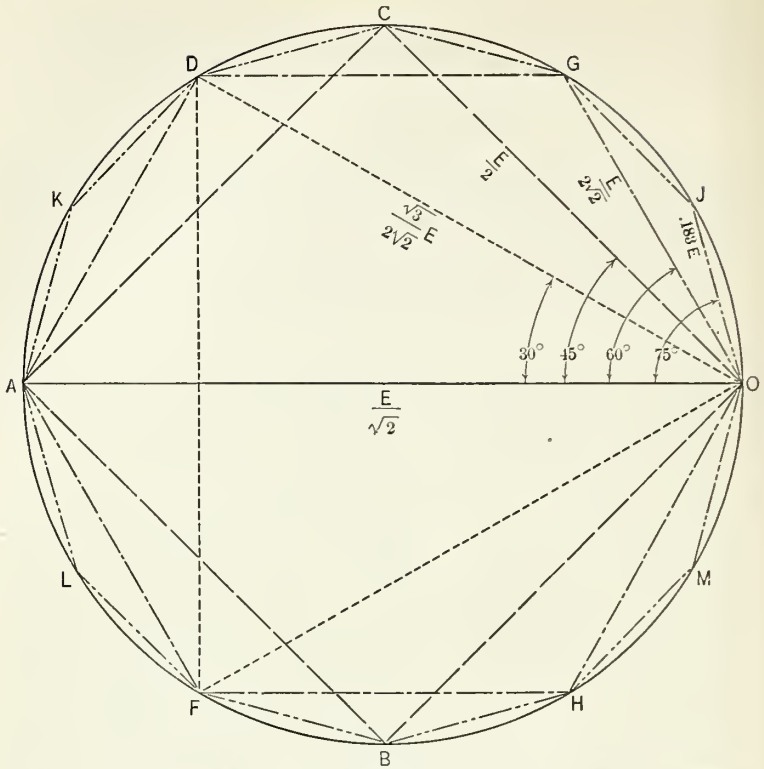


FIG. 449. — Vector Diagram for showing the Relation of Direct and Alternating Voltages in a Rotary Converter.

left hand connection, 2, is equal to BO. Hence the ratio of voltage transformation for a quarter-phase converter is expressed by the relation

$$E_4 = \frac{E}{2} = .5\,E,$$

where E_4 is the effective value of alternating quarter-phase impressed voltage and E is the voltage between direct-current brushes.

Likewise, if the three rings 2, 4, and 6 of Fig. 448 are used, the tri-phase vector voltages are represented by the lines OD, DF, and FO in Fig. 449 and the relative direct and alternating voltages are obtained from the expression,

$$E_3 = \frac{\sqrt{3}}{2\sqrt{2}}E = .612\,E,$$

where E_3 is the effective value of tri-phase alternating voltage.

In like manner the voltage relation in a six-phase converter is found to be

$$E_6 = \frac{E}{2\sqrt{2}} = .354\ E,$$

where E_6 is the alternating six-phase voltage. For twelve-phase converters the relation is

$$E_{12} = \frac{\sqrt{2-\sqrt{3}}}{2\sqrt{2}}\ E = .183\ E,$$

where E_{12} is the alternating voltage, and E as before is the direct-current voltage.

In general, it may be seen, from the construction of Fig. 449 that

$$E_N = \frac{E}{\sqrt{2}} \sin \frac{\pi}{N},$$

where E_N is the alternating voltage per phase for N phases and E is the direct-current voltage, since length

$$OA = E_1 = \frac{E}{\sqrt{2}}.$$

Continuing the assumptions heretofore made and neglecting all losses, then, if the impressed alternating voltage and current are in the same phase, we have

$$2\ EIp = I' \cdot \frac{E}{\sqrt{2}} \sin \frac{\pi}{N} \cdot Np,$$

where E is the direct-current voltage, p is the number of pairs of poles in the field magnet, N is the number of alternating-current phases, I is the direct current in the windings of any one of the armature phase sections, and I' is the component of alternating current in the same section. It will be noted that the actual current flowing in a section is the resultant of I and I'. The two sides of the equation represent respectively the total direct-current output and alternating-current input of the machine. Hence,

$$I' = \frac{2\sqrt{2}\ I}{N \sin \frac{\pi}{N}}.$$

Using the same subscripts for I' as were used with E to designate numbers of phases, we have

$$I'_2 = \sqrt{2}\, I;\ I'_3 = \frac{4\sqrt{2}}{3\sqrt{3}} I;\ I'_4 = I;\ I'_6 = \frac{2\sqrt{2}}{3} I;$$

and $$I'_{12} = \frac{2\sqrt{2}}{12 \times .259} I.$$

Noting the manner in which the parallel currents join in the leads of converters of different numbers of phases, we have the direct current per lead,

$$I_L = 2\, I$$

for converters of all phases. The alternating current per lead in a single-phaser is,

$$I'_{2L} = 2\, I'_2 = 2\sqrt{2}\, I = \sqrt{2}\, I_L.$$

In a tri-phaser, the currents join in a lead at an angle of 120°, and

$$I'_{3L} = \sqrt{3}\, I'_3 = \frac{4\sqrt{2}}{3} I = \frac{2\sqrt{2}}{3} I_L.$$

For a quarter-phaser, since the currents from the two adjacent coil sections join in a lead at the phase angle of 90°,

$$I'_{4L} = \sqrt{2}\, I'_4 = \sqrt{2}\, I = \frac{1}{\sqrt{2}} I_L.$$

In a six-phaser, the junction phase angle being 60°,

$$I'_{6L} = I'_6 = \frac{2\sqrt{2}}{3} I = \frac{\sqrt{2}}{3} I_L.$$

And in a twelve-phaser, the junction angle being 30°,

$$I'_{12L} = 2 \times .259\, I'_{12} = \frac{\sqrt{2}}{3} I = \frac{1}{3\sqrt{2}} I_L.$$

In the actual machine, where there are losses, an additional alternating current must flow in each section of such a value that,

$$P_c = I_c E' N p,$$

where P_c is the watts loss due to the rotation of the armature, I_c is the additional current required, E' is the effective voltage

in an alternating-current winding section, N is the number of phases, and p is the number of pairs of poles in the field magnet. Also as the current flowing in the armature causes an IR drop, E in the formula given above for relations of voltages and currents must be the total direct-current voltage generated. If the impressed voltage and current are not in phase, there will be an additional component of quadrature current, $I'_q = I' \sin \theta = I_a \tan \theta$, where I' is the total component of alternating current, I_a is the active component of current heretofore dealt with alone, and θ is the angle between the impressed voltage and current in the section. The larger current causes a greater IR drop in the direct-current voltage, and as will be seen later causes increased reactions on the converter field magnet.

177. Frequency and Voltage Limitations in the Converter. — Direct-current generators usually have alternating-current frequencies in the armature conductors which are between 8 and 25 cycles per second, while the frequencies of alternating-current systems are usually from 25 to 60 cycles per second. Since the product of number of magnet poles and speed is proportional to the frequency of the conductor current in the usual direct-current machine, and the speed is limited by the permissible circumferential velocity of the commutator, which ordinarily should not run at a velocity of over 3000 feet per minute, the number of magnet poles must in general increase with the frequency. The larger the number of poles, the less is the distance between direct-current brushes on a commutator of given diameter and hence the less the number of commutator bars of particular width that can be placed in the commutator in the arc between adjacent direct-current brushes. As the voltage which a direct-current machine of given design can produce without excessive sparking at the commutator is dependent, among other things, upon the number of commutator bars between adjacent brushes of opposite polarity, it is apparent that the higher the frequency of the alternating current, the less is the possible safe direct-current voltage on a converter. By careful designing it has been found possible to build polyphase converters that operate satisfactorily at a frequency of 60 periods per second with direct-current voltages of from 500 to 600 volts, which is the maximum direct-current voltage ordinarily employed, and for lower

voltages. Difficulty may be experienced in designing very small machines which can be manufactured at a reasonable cost for the maximum frequency and voltage named, on account of the inherently small commutator.

The commercial converter, in its design, is a combination of an alternator and a direct-current generator with a large commutator. Figure 450 represents a typical 60-cycle, 600-volt converter of American design.

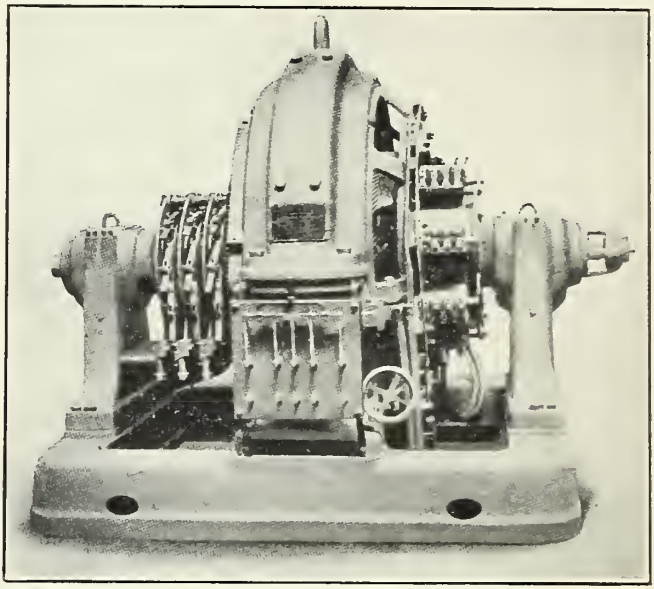


Fig. 450. — A Typical Three-phase Converter.

At one end of the shaft of a converter there is ordinarily installed a small mechanical or magnetic device which will cause the armature to move slowly back and forth lengthwise of the shaft. If this was not done, the converter would rotate without any material movement longitudinally, and grooves would be worn in the commutator and bearings. The commutation of the higher voltage converters is apt to be quite sensitive, and if the machine is not in good order or there are disturbances in the alternating-current line, it may "flash over," that is, short-circuit instantaneously from brush to brush.

When a converter is to be operated inverted (Art. 182) either in regular service or at starting, it is usual to attach an auto-

matic switch actuated by centrifugal balls on the end of the shaft, for the purpose of automatically causing the main switches to open in case the armature races and goes to too high a speed. This is a precaution of great importance, since any defect arising in the field excitation may result in the armature rapidly accelerating to a dangerous speed unless it is instantly disconnected from the circuit.*

178. Heating of the Armature Conductors in Rotary Converters. — Making the same assumptions as in the discussion of the relations of voltages and currents,† the currents in the direct and alternating circuits are as shown in the formulas already developed, but they flow in opposite relative directions in the coils since the alternating current produces, in connection with the field magnet, the motor torque, while the direct current produces the effect of a counter-torque. In Fig. 451 at the top (*a*) is a diagrammatic representation of part of the field magnet and armature of a quarter-phase converter developed. The vertical lines at *C* and *D* represent two quarter-phase taps to two of the four quarter-phase collector rings. For convenience only three armature conductors or coils *C*, *F*, and *D* are shown, though the armature is supposed to be completely wound with distributed windings. Consider first the current which flows in the middle conductor *F* of the section *CD*. The direct current reverses in the conductor as it passes under the brush *B* (the armature is supposed to move to the left), and again when it passes under brush *A*. The current in the outside leads from brushes *A* and *B* is 2 *I*, where the direct-current component in section *CD* is *I*. The line *NNN*, drawn with respect to axis *XX*, Fig. 451 (*b*), is the rectangular curve of this component of the current in conductor *F*, where the abscissas are in electrical degrees having the same scale as in Fig. 451, (*a*). The alternating component of the current in conductor *F*, which is assumed to be in exact opposition to the counter-voltage of the windings and to be sinusoidal, is shown by the curve *M*, with its maximum value occurring when *F* is under the center of pole piece *N*, and the zero values of the cycle occurring when *F* is under the brushes *A* and *B*. It has been shown † that in a quarter-phase system the effective alternating current between *C* and *D* of the section is equal to the current between the

* Art. 182. † Art. 176.

direct-current brushes or $I' = I$. Hence, the maximum value of current M is $\sqrt{2}\, I$, where I is equal to one-half the current entering the external circuit from brush B. Plotting the differ-

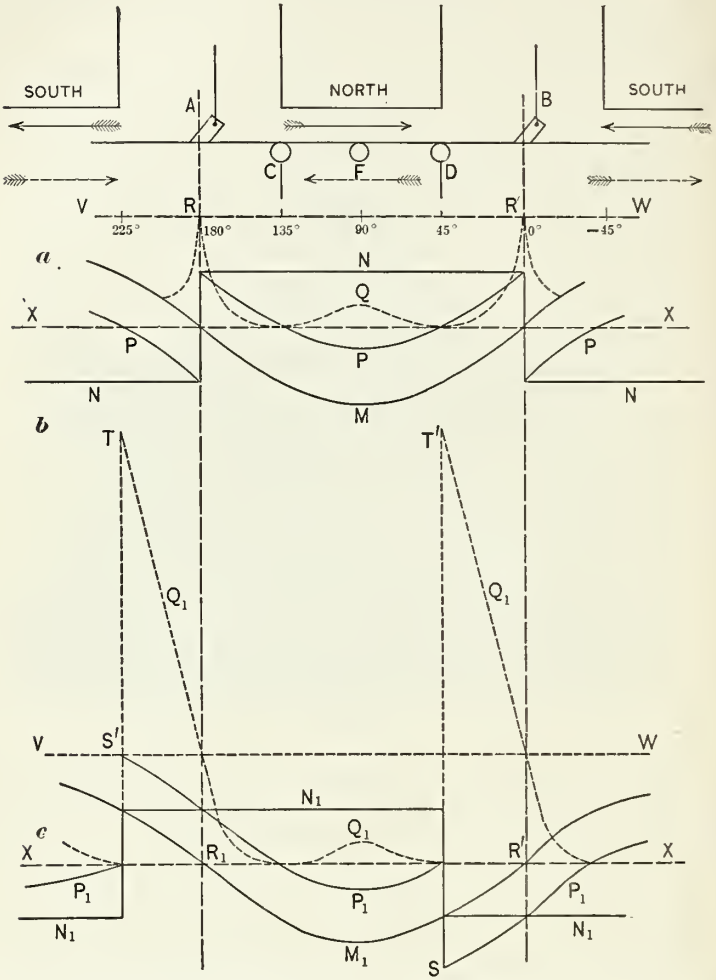


FIG. 451. — Diagrams for Representing the Current Distribution and Heating in the Armature Coils of a Converter.

ence between curves M and NNN for the half period shown in the figure gives curve PPP. This shows the resultant current in conductor F, and it is equal in value and direction to current

NNN just as the conductor approaches or leaves the direct-current brushes but flows in the opposite direction therefrom when the conductor is within the arc from 45° to 135°. Squaring the ordinates of the current curve *PPP*, and the curve *Q* results, the ordinates of which are proportional to the heat produced in conductor *F* as it passes from *B* to *A*, and the area of which is proportional to the average heating in *F*. The height of line *VW* shows on the same scale the heating that would be produced in *F* by the component *NNN* alone. The relative areas of curves *Q* and *VW* for a half cycle, using the *X*-axis as the base line, show the relative heating of conductor *F* when used in a converter and in an equivalent direct-current generator. The arrows in Fig. 451 (*a*) are of some service in obtaining a physical conception of the conditions — if the full line arrows represent the direction of the direct current component of current between the brushes, the dotted arrows represent the direction of the alternating component as the center of the section *CD* passes from brush to brush.

Consider now conductor *D* at the end of section *CD*; the relations are set forth in Fig. 451 (*c*). The alternating voltage line *M*, marked M_1, is as before, but conductor *D* does not reverse its direct current until 45° after the reversal in *F*, so that the direct current curve for *D* is $N_1N_1N_1$. Then from 0° to 45° of the trail of conductor *F* from brush *B* to brush *A*, the alternating current component in *D* is in the same direction as the direct current. This is shown by the curves N_1 and M_1. The same condition is repeated when *D* starts its second loop at 180° and at every other half period. The resultant of the direct and alternating currents, in conductor *D* from −45° to 225°, is given in the curve $S'P_1SP_1$ and the heating curve is $TQ_1Q_1T''Q_1$. In a quarter-phase converter the maximum resultant current in the end conductors of a section is equal to $2I$ as shown at *S* and *S'*, and the heating is four times as great for the instant as that which would be caused by current *I*, the heating effect of which is shown by the height of the line *VW* as before. The average heating in *D* is nearly as great as though *I* flowed in it constantly. The condition of conductor *C* duplicates that of *D*.

Conductors lying between *F* and *D* or *F* and *C* in the section are subjected to heating effects, on account of the I^2R loss,

which are intermediate between those of the edge conductors C and D and the middle conductor F.

By making the width CD correspond with the angle covered by a section between rings for any other number of phases, that is, for tri-phases 120°, six-phases 60°, twelve-phases 30°, or single-phase 180°, and using the currents determined in Art. 176, the relative heating effect in the conductors may be determined by the foregoing process for any converter. By thus obtaining and adding the heat loss for all the conductors of a section, the average heat loss of the section is determined. From this may also be obtained what may be called the apparent resistance. The sum of the areas of the curves of instantaneous resultant current squared for the conductors, which have been referred to as heating curves, made to proper scale and multiplied by the electrical resistance of the section, gives the energy expended for heat losses in the armature section per half cycle of the alternating current. Dividing this by π gives the average power, P', which goes into the heat losses. Then $R'_A = \frac{P'}{I^2}$, where I is one half the current from a direct-current brush, and R'_A is the apparent resistance of the alternating-current section. The apparent resistance of a direct-current armature circuit, R''_A, is manifestly R'_A multiplied by $\frac{N}{2}$, which is the ratio of the winding arc between the direct-current brushes to the arc between taps at the ends of an alternating-current phase section. Dividing R''_A by $2p$, the number of direct current paths through the armature winding, gives the apparent resistance of the entire armature, or

$$R_A = \frac{R''_A}{2p} = \frac{NP'}{4pI^2}.$$

In general, the heating under the conditions can be found in this way:

$$I' = \frac{2\sqrt{2}\,I}{N \sin \frac{\pi}{N}},$$ *

where I' and I are the alternating and direct currents respectively in an armature section, and N is the number of phases.

* Art. 176.

This formula was derived under the assumption that the rotative losses were zero and the angle of lag between the alternating voltage and current was zero. If, however, c per cent of the total current component entering the conductor is used in causing rotation, the left-hand side of the equation must be increased by the ratio $\frac{1}{1-c}$; and if the power factor is not unity, it must be further increased by the factor $\frac{1}{\cos\theta}$, when θ is the angle of lag. Hence the alternating current becomes for such a general condition

$$I' = \frac{2\sqrt{2}\,I}{N \sin\frac{\pi}{N}} \cdot \frac{1}{(1-c)\cos\theta}.$$

From the curves in Fig. 451 it is seen that the component of alternating current at any instant for any conductor between commutator bars at a distance δ from the middle conductor F, when the angle of lag, θ, is zero, is

$$i'_c = \sqrt{2}\,I' \sin(\alpha - \delta),$$

where α is the angular advance of F from 0°. When the lag is not zero this becomes

$$i'_c = \sqrt{2}\,I' \sin(\alpha - \delta - \theta).$$

The component of direct current at any instant is I, where I is one half the current flowing through a direct-current brush to the external circuit. Therefore the instantaneous resultant current in the conductor is

$$i_R = I - \sqrt{2}\,I' \sin(\alpha - \delta - \theta);$$

and substituting the value of I' in terms of I,

$$i_R = I - \frac{4\,I}{N \sin\frac{\pi}{N}} \cdot \frac{1}{(1-c)\cos\theta} \sin(\alpha - \delta - \theta).$$

Placing $\frac{4}{N \sin\frac{\pi}{N}} \cdot \frac{1}{(1-c)\cos\theta} = A$, squaring, and integrating this function of i_R with respect to α between the limits π and

0, and dividing by π to obtain the effective current,* we obtain

$$I_R^2 = \frac{1}{\pi}\sum_0^{\pi} i_R^2 = \frac{I^2}{\pi}\int_0^{\pi}[1 - A\sin(\alpha - \delta - \theta)]^2\, d\alpha,$$

or $$I_R^2 = I^2\left[1 + \frac{A^2}{2} - \frac{4A\cos(\delta + \theta)}{\pi}\right].$$

When $\theta = 0$, this is evidently minimum for $\delta = 0$ or the middle point of the armature section, and maximum for the edges of the section where δ is a maximum. The heating of the conductor is

$$P'_c = R_c I^2\left[\left(\frac{A^2}{2} + 1\right) - \left(\frac{4A\cos(\delta + \theta)}{\pi}\right)\right],$$

where R_c is the electrical resistance of a single conductor of the section. To obtain the mean heating in the conductors of the armature section assume that the single conductors are of negligible width, and then, by integrating the right-hand term in the brackets of the equation with respect to δ between the limits $\frac{\pi}{N}$ and $-\frac{\pi}{N}$ and dividing by the width of the section, $2\frac{\pi}{N}$, the mean heat is found to be

$$P_c = R_c I^2\left[\left(\frac{A^2}{2} + 1\right) - \frac{N}{2\pi}\int_{-\frac{\pi}{N}}^{\frac{\pi}{N}}\left(\frac{4A\cos(\delta + \theta)}{\pi}\right)d\delta\right]$$

and $$P_c = R_c I^2\left[\left(\frac{A^2}{2} + 1\right) - \frac{4AN}{\pi^2}\cdot\sin\frac{\pi}{N}\cos\theta\right].$$

Substituting the value represented by A gives

$$P_c = R_c I^2\left[\frac{8}{N^2\sin^2\frac{\pi}{N}}\quad\frac{1}{(1-c)^2\cos^2\theta} + 1 - \frac{16}{\pi^2}\cdot\frac{1}{(1-c)}\right].$$

When the rotation losses are considered zero and the power factor is unity, this becomes

$$P_c = R_c I^2\left[\frac{8}{N^2\sin^2\frac{\pi}{N}} + 1 - \frac{16}{\pi^2}\right].$$

Since R_c is the true resistance of the conductor, $R_c I^2$ is the heating per conductor that would occur if the machine were run as a direct-current dynamo with an output per brush of $2I$, and as P_c is the mean heating per conductor when the machine gives

* Art. 5.

out $2I$ amperes per direct-current brush when running as a converter, the expression surrounded by brackets in the formulas for P_c is equal to the ratio of the power lost in heat in the two cases. The rated capacity of a machine is inversely proportional to the square root of its temperature rise, and therefore the expression within the brackets may be considered as expressing the square of the ratio of the capacity of the machine when running as a generator to its capacity when running as a converter subjected to sinusoidal voltages. It will be noted that through much the same method of reasoning it is possible to find the ratio of the apparent to the true resistance of the converter armature.

From the graphical constructions and the last formula it may be observed that the capacity of a converter operated at unity power factor, limited by the rise of temperature of the armature, when compared with the capacity of the same machine used as a direct-current generator, is as follows: single-phase .85 to 1; quarter-phase 1.63 to 1; tri-phase 1.34 to 1; six-phase 1.95 to 1; and 12-phase 2.21 to 1. Also, comparing in the same way, the apparent resistances are respectively 1.38 to 1; .38 to 1; .56 to 1; .26 to 1; and .21 to 1.

If the small component of active current required to rotate the armature is included in the calculations, the resistance ratios will be slightly increased over those given. Also, if a quadrature current flows in the armature, the increased current will cause a greater I^2R loss in itself and will cause a further loss by reason of the shifting of the phase position of the current. To solve the problem under those conditions the values of e and θ must be substituted in the general formula for P_c, or if in Fig. 451 a quadrature current flowed, it would have to be combined with components N and M and by reason of its position would cause greater peaks of current and hence greater heating; and to obtain the heating effect of the total current of a converter in which the current is out of phase with the generated counter-voltage it is necessary to add to M an active component equal to that necessary to supply the rotation losses, and a second component equal to the quadrature current. The processes may then be carried through as before. If the impressed and counter voltages are not both sinusoidal, the heating is modified from that computed.

It will be noted both from the formula for P_c and Fig. 451 that the width of the alternating-current coil section directly affects the capacity of the machine. It is evident then that the type of the winding may materially influence the converter capacity. The figure also shows that the heating is not uniformly distributed amongst the armature conductors, but that part of each section is more heated than the remainder.

179. Armature Reactions of a Converter. — A consideration of the basis of the construction of Fig. 451 will show that the magnetic reactions of the armature upon the field magnet in a converter are small when no quadrature current flows. Thus, in the figure, which is for a quarter-phase machine, the effective alternating current component equals and is opposite to the direct current component in any section, so that there can be no resultant reactive effect when integrated through the duration of a cycle. But it is noted that there is none the less a resultant current flowing of irregular double frequency wave shape and irregularly disposed in the individual coils. This varying current causes an alternate skewing of the field magnetic flux forward and back as the directions of the current alternately predominate in the coils. In a single-phase machine, having alternating current coil sections 180 electrical degrees wide, the heavy current that flows in a small part of an alternating current section of the armature when the alternating currents in the coils near the edge of the section are in the same direction as the direct current, makes the reaction of double frequency especially heavy. By reason of their changing the reluctance of the field magnetic circuit these reactions may set up appreciable double harmonic vibrations in the current in the direct-current leads. As the skewing is first one way and then the other way, the brushes cannot be shifted to the point of field strength which gives the least sparking, as in direct-current machines. However, even single-phase converters are built which give excellent satisfaction, while the smaller amount of the resultant current variations in polyphase converters make the effects mostly negligible. Further, induced currents in the pole faces of the field magnet or in the amortisseur windings cut down materially the variations in field flux.

A converter being equivalent to a synchronous motor, in its effect on the alternating current supply circuit, when the

excitation is raised beyond the point giving unity power factor, a leading quadrature current is drawn from the line as explained in Art. 170. This is in such a position, as may be seen by studying the relations shown in Fig. 451, that it tends to weaken the field magnet. Indeed, the amount of the quadrature current that flows in any synchronous machine is such that the field magnet will have a strength which will enable the resultant between the impressed and counter voltages to just drive the armature current through the armature impedance. Hence, changes in converter excitation make only small changes in the direct-current voltage, when the impressed alternating voltage is constant.

When a lagging current is drawn from the line due to under excitation of the converter field magnet, the reactions strengthen the field magnet. In polyphasers, the reaction due to the quadrature component of current is caused by flux from the armature which is stationary with respect to the field magnet, *i.e.* the combined ampere-turns of the several phases cause the armature flux to revolve with respect to the windings against the direction of the revolution of the armature and at equal speed, thus causing this flux to maintain a fixed angular position with reference to the poles of the field magnet.*

180. Voltage Control of Converters. Split-pole Converters. — From the discussion of armature reactions in the previous article it is evident that the direct-current voltage is approximately proportional to the impressed alternating current voltage in any particular converter. The direct-current voltage may then be controlled by any method whereby the impressed voltage is controlled, such as by cutting out or in active turns on the transformers feeding a converter, or by using autotransformers in which the secondary voltage may be varied, or using induction regulators † in the leads.

By placing reactive coils in series with the converter leads, or when the converter is used on lines where there is other inductance, the impressed voltage may be varied somewhat by the same processes as where the synchronous condenser is employed.‡ An illustration of this method is given in Fig. 447. Or this action may be made automatic by winding a few series compounding turns from the direct-current leads of the con-

* Art. 152. † Arts. 159, 160. ‡ Art. 174.

verter on the field spools, and inserting reactances of proper values in the alternator leads. A special alternator may be connected to the converter shaft, with its own field magnet having the same number of poles as the converter, and with its armature coils of each phase connected in series with those on the converter armature, thus composing a **Synchronous regulator**. Such a machine is shown in Fig. 452. The synchronous regulator is shown on the right-hand side of

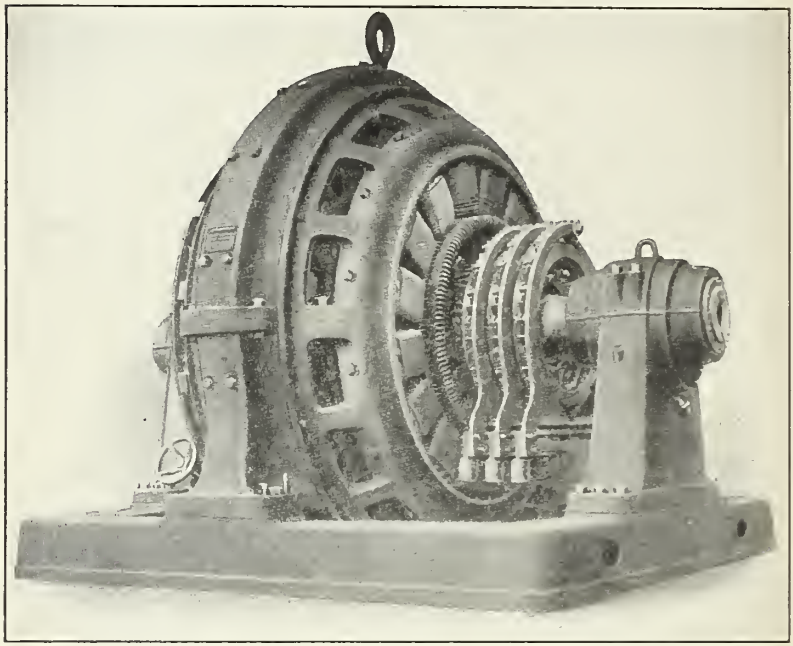


Fig. 452. — Synchronous Regulator attached to a Converter.

the figure. By varying the field strength of the regulator, the voltage delivered to the converter is varied.

As the direct-current voltage depends on the maximum value and wave form of the alternating counter-voltage, any arrangement whereby the wave shape of the counter-voltage can be varied may be used to vary the direct-current voltage. Thus, if each field pole is split parallel to the shaft into three narrow poles, all having separate windings, the magnetic field distribution may be made peaked by strengthening the middle pole part relatively to the others, or may be flattened by weakening the

center part. If the outside pole parts are strengthened when the middle part is weakened, or *vice versa*, the direct-current converter voltage may be varied several per cent. When such an arrangement is made, the machine is called a split-pole converter.

As in any direct-current machine, the direct-current voltage of a converter varies with the shifting of the brushes with reference to the field poles. In order to vary the voltage conveniently by this means, some machines are built with extra poles whereby the magnetic flux instead of the brushes can be shifted. Figure 453 is a diagram of such an arrangement. The auxiliary I may be strengthened when the poles NS are weakened or *vice versa*, thus shifting the center of density of magnetic

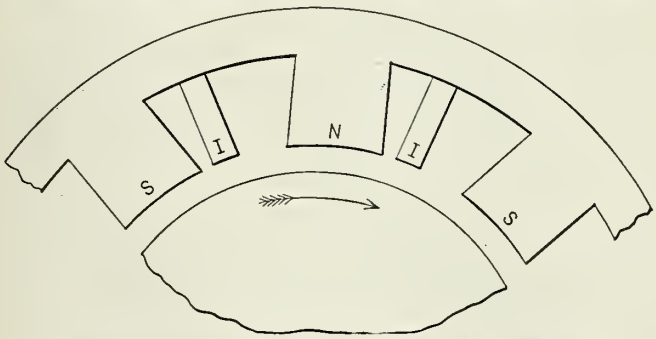


FIG. 453. — Flux shifting Auxiliary Poles on a Converter.

flux. The auxiliary pole piece I is usually somewhat nearer what is intended to be the trailing pole tip, thus leaving a fairly wide space for commutation, and it is not an interpole for commutation.

181. Some Features of Converter Operation. — A converter may be started as a direct-current motor from the direct-current side, or it may be brought to synchronism as a rotary field induction motor, or again it may be brought to speed by some external mechanical starting device. Except for the first-named method, a converter is started like a synchronous motor, and is paralleled in the same way. Figure 454 shows the connections of two six-phase machines arranged to start as rotary field induction motors. A switch for breaking up the length of the shunt field windings is shown at the right hand of each converter, for the purpose of preventing excessive voltages being

induced in them during the starting process; and a switch for short-circuiting the series or compound field winding is at the left hand. The direct-current switchboard panels shown in the

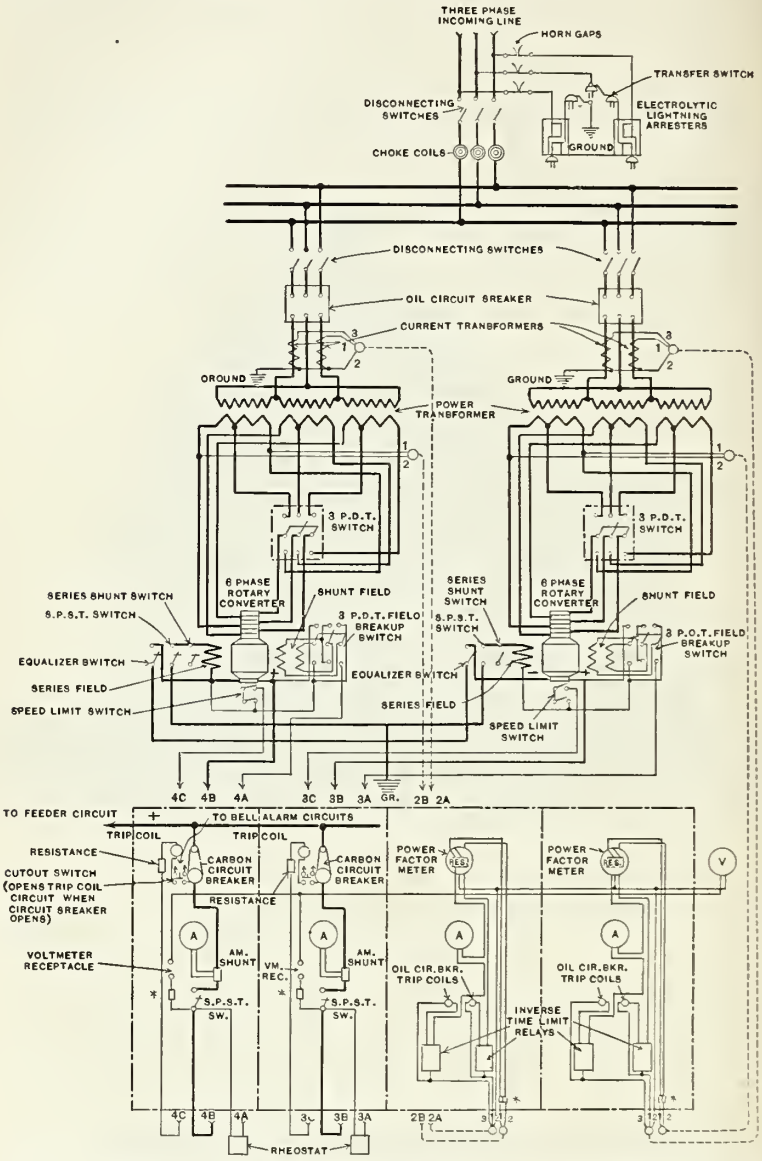


FIG. 454. — Diagram of Connections for Six-phase Converters, including the High Voltage Supply Line, the Transformers with Starting Taps, and the Direct-current Switchboard Panels arranged for Electric Railway Circuits.

figure are arranged for railroad work at from 500 to 600 volts. The neutral point of the polyphase circuit is shown ungrounded. Keeping in mind this figure, the following instructions, taken from those issued by one of the large manufacturers of electrical machinery, give an insight into the methods of operating this kind of machines when arranged for self-starting by the alternating current :

Close the oil circuit breaker on the high voltage side of the transformer. Insert the voltmeter plug for the direct-current voltmeter. Close the double-throw starting switch to the starting position. The rotary converter should come up to synchronous speed in about 30 seconds, and lock into step, indicating this condition by a steady current on the alternating-current side of the converter, and a continuous deflection of the direct-current voltmeter. When the direct-current voltmeter indicates the correct polarity, close the field break-up switch so as to have the field rheostat in the field circuit. When the direct-current voltmeter indicates reversal of polarity, reverse the field break-up switch, thus reversing the shunt field and connecting it directly across the armature. The voltmeter indicator will swing back towards zero. When it reaches zero, throw the field break-up switch to the lower position. If the voltage now comes up with the right polarity, proceed as directed above. If, however, the converter fails to slip a pole, and the voltage again comes up with reversed polarity, it is evident that the field induced by the alternating currents in the armature is too strong. The starting switch should then be opened for a moment, thus permitting the converter to slow down somewhat. The starting switch should then be closed again in the starting position. When the machine is up to synchronous speed, and the direct-current voltmeter shows correct polarity, throw the starting switch to the running position. Adjust the direct-current voltage to the proper value. Close the series shunt switch (when there is a series shunt circuit). Close the direct-current equalizer and negative switches ; then close the direct-current circuit breaker and the positive switch.

The following instructions for connecting up rotary converters for the first time after they have been installed come from the same source :

"Connect the alternating current leads from the same bus bars to similar switches and collector rings as compared with the converters already installed. Place the direct-current brush holder in the same position with respect to the field poles as that on the other converters, and run the positive, negative, and equalizer leads through their respective switches to the positive, negative, and equalizer bus bars of the other converters. See that the field wires are brought out to the corresponding terminals of the other converters and connected in the same way. Be sure that the voltmeter lead from the positive terminal goes to the positive voltmeter bus, and the negative to the negative bus. After carefully checking all wiring and seeing that all is clear, start the converter by closing the alternating-current switch in the starting position. If the armature revolves in the wrong direction, shut down and change the alternating-current cables to the converter. If the converter is two-phase, reverse the two leads of either one of the phases, if it is three-phase, reverse any two leads. After the converter has come to correct speed, proceed as above for self-starting rotaries. In first starting a converter it is necessary to synchronize all phases. This can be done in a two-phase converter by placing a bank of lamps equal to the voltage of the converter across the jaw and blade of the switch on each side of each phase. If the converter is three-phase, place a bank across each of the three switches. Now synchronize as before, noticing the lamps. If they all indicate the proper phase relation at the same time, the phases are wired correctly and the switch may be closed at the proper time. If, however, one set of lamps is bright, while the other is dark, it will be necessary to change the main alternating-current leads, either at the converter terminals or at the point of the switch running directly to these terminals. If the converter is two-phase, reverse the leads of either phase; and if it is three-phase, interchange any two leads. After this change is made, check it by means of the lamps when starting again. After synchronizing the converter, put load on the direct-current circuit and test the series field coil by short-circuiting it and noticing the direct-current voltmeter. If this is in opposition to the shunt field, as shown by the voltage increasing with the series coil short-circuited, reverse the lead connections to these coils. The converter should now be ready for paralleling on both sides."

The windings of a converter are of the same general type as the windings on direct-current machines with reëntrant armature coils. As there are likely to be many sets of direct-current brushes on account of the multiplicity of poles on high frequency converters, there are many direct-current paths through the armature winding. Any inequality of the resistances in

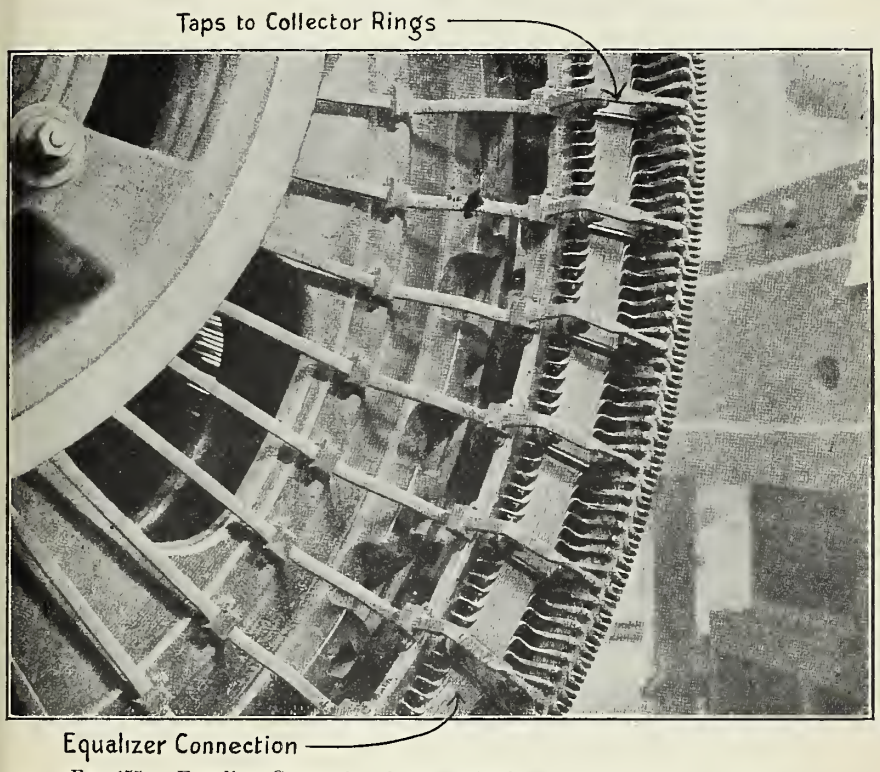


Fig. 455. — Equalizer Connections between Sections of the Armature Windings of Converters.

these paths or in the strengths of the individual magnet poles is apt to unbalance the currents in the paths and cause undue heating. It is thus necessary to use equalizer connections between points of equal potential in the windings, as shown in Fig. 455.

The direct-current characteristics of a converter are very much like those of a direct-current generator. Figure 456 shows typical curves of regulation, commercial efficiency, and losses for a machine of this type. Since there is little armature

reaction, the field and all other losses except the armature copper loss may be considered constant.

The efficiency of the converter is

$$\eta = \frac{IE}{IE + P + I^2R},$$

where I and E are the direct current and voltage, P is the fixed power loss, including the field and rotation losses, and R is the apparent resistance of the armature. The efficiency may be measured by placing wattmeters in the alternating and direct current leads respectively, and loading the machine to the

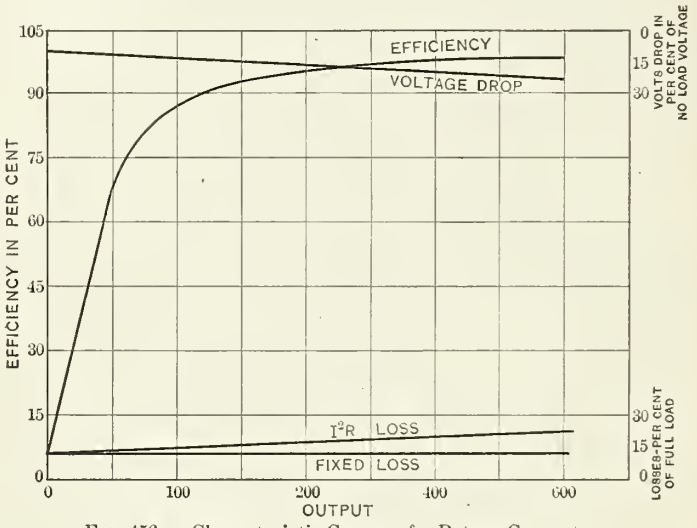


Fig. 456. — Characteristic Curves of a Rotary Converter.

amount and at the power factor desired. Or the losses may be determined by running the motor unloaded at the excitation giving minimum current, when the power input measured by the wattmeters in the alternating-current lines plus the field losses approximately equals P. The apparent resistance may be obtained by measuring the actual resistance of the armature and multiplying by the ratio of apparent to true resistance for the number of phases of the machine tested.* From this the I^2R losses can be calculated. Otherwise tests of this kind of apparatus conform with that of other electrical machines.

The efficiencies of converters are higher than the efficiencies

* Art. 178.

of direct-current or alternating-current generators of corresponding size, on account of the greater output obtained for the converters with given conditions of loss, as set forth in Art. 178.

When converters are feeding a three-wire direct-current system, the neutral conductor of the direct-current side may be connected to the neutral point of the alternating-current side. For a quarter-phase converter, the secondary windings of the supply transformers can be connected to the direct-current neutral conductor at their neutral points. For a tri-phase converter, the neutral point can be readily obtained from a wye or tee connection of transformers and for a six-phase converter the connections may be conveniently either wye or tee.

182. Inverted Converters: Double-Current Generators. When a converter is driven by direct-current power and delivers alternating current to the line, it is termed an inverted rotary converter. When so run, the machine has features similar to that of a direct-current shunt motor. Thus, the speed varies inversely as the field strength and directly as the impressed voltage. The alternating-current end acts like an alternating-current generator, a lagging current delivered by the machine causing a weakening of the field magnet, and a leading current delivered by the machine causing strengthening of the magnet. Therefore, if an inverted rotary finds variable alternating inductive loads, the speed and hence the frequency are apt to be dangerously variable. It is therefore common to use an individual shunt-wound exciter for such a machine, which is driven at a speed proportional to that of the inverted converter. The field magnet of the exciter is designed with low magnetic saturation, so that with slight increase of speed the exciter voltage is largely increased, thus sending an increased current through the inverted converter field windings and thereby maintaining approximately normal speed. If the armature of an inverted converter has little reaction on the field, and if the field magnet is highly saturated, very little trouble need be experienced. Mechanical and other devices may be used if necessary to control the speed. The distribution of current among the armature conductors and other characteristics for inverted converters may be determined in the same manner as that developed in the preceding discussion of the non-inverted converter.

In the use of inverted converters, and indeed in all alter-

nating-current generating plants, it is desirable to have a **Frequency indicator** attached to the lines. Such an instrument can be made on the principle of a rotary field induction motor, where a pointer is attached to the armature, which is restrained by springs. The torque of the armature varies as the frequency and hence a scale in frequency can be made for the pointer to travel over. The frequency may also be determined by an instrument depending upon the vibration of reeds and in other ways.

Double-current generators are sometimes of use in plants connected to both direct and alternating current systems. The double-current generator differs from the converter in that it is driven by mechanical power and therefore both the alternating and direct currents are in relatively the same direction in the armature windings. If curve M in Fig. 451 is reversed, its resultant with N gives the current distribution in the armature conductors of a double-current generator when the power factor is unity and equal amounts of power are being delivered to the direct current and the alternating-current circuits. It is evident from the construction of Fig. 451 that reversing curve M must result in a relatively great I^2R loss, and a further investigation will show a larger I^2R loss for a given output in a double-current generator than in an equivalent converter for the same output.

183. The Alternating-Current Motor-generator. — The motor-generator consists of an alternating current motor driving a direct current generator or *vice versa*, or another alternating current generator, the two machines being mounted on a common bed plate and shaft. In the latter case the generator in commercial practice is usually of different frequency from that of the motor, and the machine is called a **Frequency changer**. A problem of importance is to be met in connecting up and paralleling the latter machines when they are to run in parallel with each other. Suppose, for instance, that the motor of a synchronous motor-generator set is fed in parallel with the motors of other motor-generators with currents having a frequency of 25 cycles per second, and the generator delivers currents in parallel with the generators of the same motor-generator sets, the currents having a frequency of 60 cycles per second. The number of poles on the two machines of each motor-generator set must have a ratio equal to the ratio of their respective frequencies; thus if the

25-cycle motor has 10 poles, the 60-cycle generator must have 24 poles. With the ratio of frequencies incommensurable, like $25 : 60 = 5 : 12$, there is much embarrassment to conjointly meet the mechanical and electrical requirements of good commercial machines. Each field magnet must contain an even number of poles, and the number of pairs of poles must be in the ratio of 5 : 12. This gives 10 poles and 24 poles for the respective field magnets as the smallest numbers that will fit. The speed of the motor-generator set with these numbers is 300 revolutions per minute. The next polar relation is 20 : 48, which gives a speed of 150 revolutions per minute and no exact ratio intervenes. The choice of two incommensurable numbers like 25 and 60 for the standard frequencies therefore causes inconvenience in this situation. A modification to 24 and 60 would improve the relations here and also elsewhere.

After the motor is running properly in synchronism and the voltages of both motor and generator are right, the generator phases may be out of step with those of the mains to which it is to be connected, and it may become necessary to reverse the polarity of the motor field magnet by the field switch and possibly continue opening or reversing the field circuit a number of times in order that the motor may fall back successive angles of 180 electrical degrees until the counter-voltages in the motor and the terminal voltages in the generator hold the proper step relations coincidently in their respective circuits; or the required step relations may sometimes be obtained by reversing the polarity of the generator field magnet. Even when the best practicable point is reached, a large synchronizing current may flow if the motor-generator does not have quite its designed ratio of frequencies, *i.e.* 25 to 60, or if the angular relations of the parts of the motor and the generator are not adjusted to accurately give the proper time phase between their two voltages.

When a motor-generator set is being brought into parallel operation with others, it must be brought up to speed (the motor usually starting as an induction machine) and the motor be connected to the supply circuit when the indications of the synchroscope are favorable. The generator is then in synchronism, but the synchroscope must still be used to show whether the generator is in proper step with the generator bus bars before it is connected to its circuit.

CHAPTER XII

ASYNCHRONOUS MOTORS AND GENERATORS

184. Asynchronous Motors and Generators Defined. — Motors called **Synchronous** are those in which the moving parts rotate at such a velocity with reference to the fixed parts that an angular distance equal to that of the polar pitch is passed over during the time of a half cycle of the impressed alternating voltage, *i.e.* the motor runs in **Synchronism** with the generating apparatus which drives it, and it has the same speed for all loads provided the impressed voltage is of constant frequency.

Motors which do not have this characteristic, but vary the relative speed between the fixed and movable parts more or less independently of the frequency of the impressed voltage, are called **Asynchronous motors.** Alternating current **Induction motors, Repulsion motors, Series motors,** and others in which there is no magnetic field set up by direct current are usually of the latter class. **Asynchronous generators** are similar to asynchronous motors in possessing no definite relation between the armature speed and main circuit frequency.

185. Rotary Field Induction Motors. — The well-known principles which cause the rotation of a disk of copper pivoted above a rotating horseshoe magnet have been put into use through the discoveries of Ferraris, Tesla, Haselwander, Dobrowolsky, and many others. The arrangements proposed by Tesla were doubtless the first direct applications of these principles to commercial use, in which they now play a primary part in the transmission and distribution of power.* An almost simultaneous publication of a series of scientific experiments by Ferraris shows the operation of similar apparatus, † and various experiments of a similar nature or for a similar purpose are on

* A New System of Alternate-Current Motors and Transformers, *Trans. Amer. Inst. E. E.*, Vol. 5, p. 308.

† Electro-dynamic Rotation by Means of Alternating Currents, *London Electrician*, Vol. 21, p. 86.

record. Each of these experiments caused an iron or copper armature to rotate when placed within the region of a rotating magnetic field, and these are the foundation of the now well-known asynchronous induction motors.

186. A Rotating Magnetic Field. — If two pairs of coils are placed at right angles on a laminated iron ring (Fig. 457), with the connections so arranged that the coils of each pair are in magnetic opposition in the ring, or again, if the two pairs of coils are placed on two pairs of salient poles, or embedded in the inside face of the ring, so that the current in each pair of coils tends to send magnetic flux directly across a central core C, the magnetic flux set up in the core when a current is passed through the coils may be considered as the resultant of the magnetization due to the two coils or pairs of coils. As in a transformer, if the resistance of the coils is relatively small, the magnetic flux threading those of each phase when alternating currents flow in them must be such that it will create a counter-voltage in the coils practically equal and opposite to the impressed voltage. This must always be the case, whatever may be the form of the impressed voltage wave; consequently, if the voltage wave is sinusoidal, the wave of magnetic flux within the coils must be sinusoidal, because its rate of change is sinusoidal. On the other hand, the wave shape of the exciting current takes the form requisite to set up the required core flux, as in transformers,* and its wave form is, therefore, dependent upon the variation in the reluctance of the magnetic core caused by saturation and hysteresis.† The resultant flux in any fixed direction across the central core C will vary in value as in the common core of a quarter-phase transformer, but the special arrangement of the magnetic circuit causes it also to vary in direction in case the currents in the two circuits are not in the same phase. Considering a more or less conventional case where four coils, as in Fig. 457, are supposed to uniformly cover the whole ring, which is

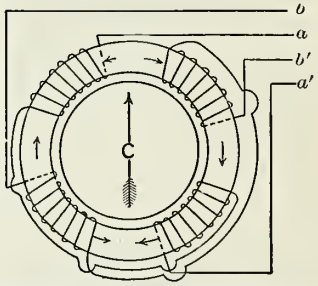


FIG. 457. — Apparatus for setting up a Rotating Magnetic Field.

* Art. 118. † Art. 126.

equivalent to an elementary arrangement in a rotary field induction motor winding, the magnetic reluctance of the ring and core C being assumed constant for all paths the flux may take parallel to the plane of the paper, the following propositions are approximately true.

If two equal sinusoidal voltages 90° apart in phase are impressed upon the respective equal pairs of coils on the ring of Fig. 457, the exciting currents must be sinusoidal to create the required sinusoidal magnetic flux, the magnetic reluctance being assumed to be constant, and the magneto-motive forces must, therefore, be sinusoidal. Under these conditions at any instant, the magnetizing force due to one pair of the coils is $H_1 = H_m \sin \alpha$, and the magnetizing force due to the other pair of coils is,

$$H_2 = H_m \sin (\alpha - 90°) = -\mathrm{H}_m \cos \alpha,$$

where H_m is the maximum magnetizing force of either pair of coils. Thus the two magnetizing forces differ in phase by 90°, and the resultant magnetizing force of the two pairs of coils is then $H_r = \sqrt{H_1^2 + H_2^2} = H_m$, as illustrated in Fig. 458, and is, therefore, constant in magnitude. The direction in which this constant magnetizing force acts across the core C varies with α. When $\alpha = 0°$, H_r tends to lie in the plane of one pair of coils, and when $\alpha = 90°$, it tends to lie in the plane of the other pair of coils. The magneto-motive force of each pair of coils has a sinusoidal or harmonic variation, and the resultant magnetizing force is the resultant of two harmonic variations with 90° difference of phase. Such a resultant tends to have a uniform magnitude and a uniformly rotating direction in one plane. The instantaneous values of the resultant may, therefore, be diagrammatically represented, with approximate accuracy under the conditions assumed, by the instantaneous positions of a line of fixed length, rotating at a uniform rate around one end, such as OH_r in Fig. 458. In this figure the vector of magneto-motive force H_1 of one pair of the coils of Fig. 457 is considered to lie in the line AB, and to vary sinusoidally in magnitude, while the magneto-motive force H_2 of the other pair of coils lies in the line CD, and also varies sinusoidally in magnitude. The combination of these two magneto-motive forces in space causes the resultant H_r, which as ex-

plained is, under the conditions, approximately constant in length and of uniform coplaner rotation. In Fig. 457 the small arrows show the direction of the magnetic fluxes at the instant when the currents, and hence the magneto-motive forces, are equal in the two branches of the two-phase circuit. As the current in the coils of one phase dies out, and the other increases to a maximum, the flux in C swings around 45° on the center of C as an axis. When that point is reached, the current is zero in one pair of coils, and the other pair furnishes the entire flux in C. As the current now rises in the reverse direction in the first pair of coils and falls in the other pair, the flux in C is still caused to rotate by virtue of the rotating direction of the resultant magneto-motive force. In this way the direction of the flux in C uniformly rotates through 360° during the time of each cycle of the current. This may be readily illustrated by a series of diagrams similar to those shown in Fig. 457, in which the magneto-motive forces are represented by arrows for the range of angular advances during a cycle of the currents. The arrows must be reversed in direction in a pair of coils when the current reverses, and should be made approximately proportional in length to the instantaneous currents. The instantaneous current for each value of α may be taken from a pair of sine curves, drawn with one displaced 90° from the other.

If the maximum value of the ampere-turns of one pair of coils is greater than that of the other pair of coils instead of the two being equal, as above assumed, the magnitude of the resultant magnetizing force varies. The rotating field, in this case, may be diagrammatically represented by a uniformly rotating line, which varies in length so that its tip approximately traces an ellipse

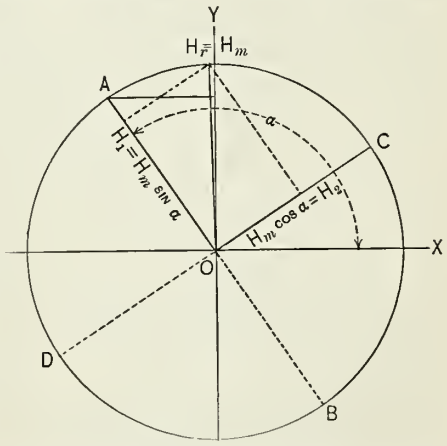


FIG. 458.— Locus Diagram of a Uniform Constantly Rotating Magneto-motive Force created by Two Equal Harmonically Vibrating Magneto-motive Forces spaced 90 Mechanical Degrees Apart.

whose minor and major axes are respectively in the planes of the stronger and weaker coils. If the windings of the coils are similar, and the currents equal, but the phase difference is not 90°, a variable field again results.

If the phases of the two currents are in unison instead of in quadrature,

$$H_r = \sqrt{H_1^2 + H_2^2} = \sqrt{2}\, H_m \sin \alpha.$$

This shows that when the two currents are in unison the magnitude of H_r varies with $\sin \alpha$, and therefore varies from $-\sqrt{2}\, H_m$ through zero to $+\sqrt{2}\, H_m$, but the direction of H_r remains constant, since the instantaneous values of its two components are always equal. Its direction evidently lies in a line corresponding to the arrows in Fig. 457 (or 90° therefrom depending upon the connections) if the two currents are equal, and turned more or less from that position if the two currents are unequal. The diagrammatic representation of the resultant here is a line of fixed direction which harmonically varies in length, the total range of variation being from $-\sqrt{2}\, H_m$ to $+\sqrt{2}\, H_m$.

For any difference of the current phases between zero and 90°, both the magnitude and direction of H_r vary, and the diagrammatic representation is a rotating line with its tip approximately tracing an ellipse. The ratio of the two axes depends upon the phase difference of the currents. If the currents have 90° phase difference, but the planes of the coils are not 90° apart, the effect on the resultant magnetizing force is evidently the same as if the conditions were reversed.

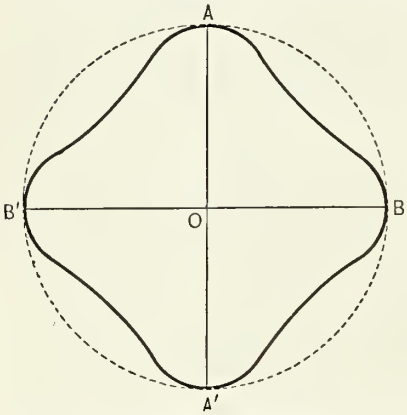


FIG. 459.—Locus of Rotating Field Vector when the Exciting Current Wave is of Distorted Form.

If the impressed voltages are not sinusoidal, the value of the resultant magnetizing force H_r varies in a more or less irregular manner, inasmuch as the different harmonics combine in different ways. For instance, in a quarter-phase circuit the third harmonics of the two magneto-motive forces produce a result-

ant that rotates in the opposite direction from the rotation of the resultant produced by the fundamentals, and the rotation is three times as fast. The fifth harmonics produce a resultant which rotates in the same direction as the resultant produced by the fundamentals, but rotates five times as fast. The heavy line in Fig. 459 illustrates the locus of the tip of the rotating field vector produced by two magneto-motive forces in time and space quadrature, when each consists of a fundamental and a third harmonic, the latter located to cause peakedness of the wave of magneto-motive force. Figure 460 shows a corresponding locus when the third harmonic causes a flattening of the top of the wave of magneto-motive force. The value of flux for $\alpha = 0°$, 90°, 180°, and 270° is alike for the two figures, and the dotted circles show the locus for sinusoidal magneto-motive forces.

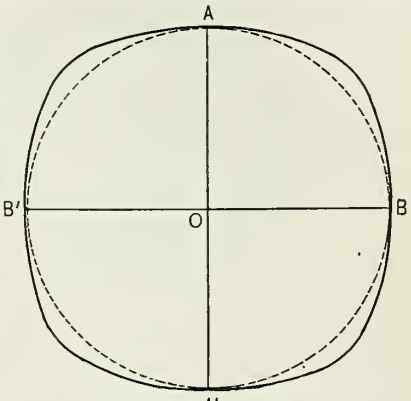


FIG. 460. — Locus of Rotating Field Vector when the Exciting Current Wave is of Distorted Form.

The same argument may be readily seen to apply to the resultant magnetizing force due to any number of coils surrounding a core. Thus, an arrangement similar to that of Fig. 457, except that it is applicable for obtaining a rotating field from a tri-phase circuit, may be constructed by winding three coils upon the ring and properly connecting them to the three phases of a triphase supply circuit. When equal coils are at equal angular distances, and equal currents in the individual coils differ in phase by an amount equal to the angular distances of the coils from each other, the resultant magnetizing force is always approximately uniform in magnitude under the conditions assumed, and rotates at a uniform rate, provided the currents are sinusoidal. The magnitude of the resultant is $H_r = \frac{m}{2} H_m$, where m is the number of phases, in accordance with the theorems of simple harmonic motion. The correctness of these deductions has been proved by experiment.

The Germans call the rotating magnetic field **Drehfelde**, and the polyphase currents which set up a rotating magnetic field the **Drehstrom**, or rotating current. In the actual rotating field, the magnetic flux does not necessarily vary with absolute constancy in accord with the propositions given above, but it varies even if the windings are uniformly distributed. This variation is dependent upon the variation of the instantaneous values of magneto-motive force and reluctance of each elementary width of the magnetic circuit through the central core. Indeed, it is possible for the resultant magnetic flux to vary in instantaneous value as much as 40 per cent, as it rotates, even in the case of properly distributed coils and balanced voltages, when there is no closed secondary circuit acted on by the flux. The extent of this variation depends upon the number of phases of the windings and other variables. However, when secondary circuits on the central core are closed, the reactions are found to smooth out the irregularities that occur, and the wave of resultant magnetic flux may be considered approximately sinusoidal in distribution and of approximately constant value in magnitude and rate of rotation in those cases in which care has been exercised to properly place the windings on the core, and when approximately sinusoidal impressed voltages are used.

187. Action of a Short-circuited Armature Winding within a Rotating Field. — If a drum core of laminated iron is suitably pivoted within a ring on which coils carrying alternating currents are so situated that the resultant magnetic field rotates, the pivoted core will be dragged into rotation by the magnetic pull. If the pivoted core is a cylinder of copper, it will be dragged into rotation by the reactions between the rotating magnetic flux and the eddy currents which are developed in the cylinder. The latter is directly analogous to the classic experiment of Arago with a disk of copper pivoted before the poles of a rotating permanent magnet.

In the case of either a solid core or Arago disk, the eddy currents are not constrained in position and therefore take the paths of least resistance. The result is that much of the effectiveness of the currents in bringing about a rotation is lost, and the efficiency of the device is small. If, in the disk experiment, the disk is cut up into an indefinitely large number of fine radiating wires which are connected together at their inner

and outer ends, the losses due to parasitic eddies may in a large measure be done away with, and the efficiency of the device is considerably raised. In the same way the drum core may be made of laminated iron in order that the magnetic circuit shall be of small reluctance, and embedded in this may be copper wires which cross the face of the core and are all short-circuited by copper rings at the ends (Fig. 461), making a cage of conductors like a squirrel cage. These make constrained paths for the induced currents, and, if the core is sufficiently laminated and the copper conductors are not too thick, the parasitic eddies are largely done away with and the efficiency of such a motor may be made quite large. This construction is the essential construction of what are known as induction motors.

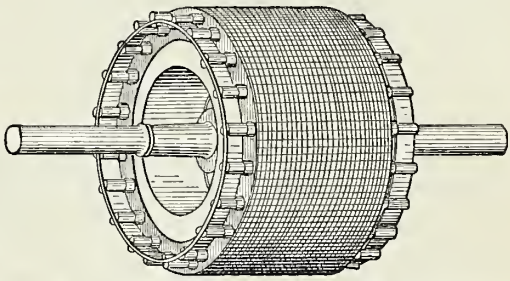


Fig. 461. — Induction Motor Armature having a Short-circuited (Squirrel Cage) Winding.

There has been more or less ambiguity in applying the designations of field and armature windings to the primary and secondary windings of induction motors, since either may rotate, and both windings carry alternating currents. The following definitions avoid all ambiguities in simple machines. The **Field magnet** is the core upon which are placed windings connected to the external circuit. The currents in the field windings are, therefore, due to the impressed voltage of the external circuit. The **Armature** is the part carrying conductors in which current is induced by the revolving magnetism of the field magnet. Since the current is inductively set up in the armature conductors, these motors are called **Induction motors**. It is readily seen that the induction motor is a transformer as well as a motor, the primary winding of which is on the field magnet, and the secondary winding on the armature, and it is convenient to call the windings receiving power conductively from the power supply circuit the **Primary** windings, and those in which current is induced the **Secondary** windings. The part that rotates in any alternating current machine having a rotating part is often called the

Rotor, while that which is stationary is called the **Stator**, irrespective of which is the primary or secondary part.

188. Counter-voltage induced in Primary Circuit. Exciting Current. — Treating the primary winding of an induction motor in the same manner as the primary winding of a transformer,— which is permissible, — the following formula* gives the relation between counter-voltage, frequency, magnetism, and the turns of the coils:

$$E'_1 = \frac{\sqrt{2}\,\pi n_1 \Phi f}{10^8},$$

in which E'_1 is the induced voltage in the coil, n_1 the number of turns in the coil, Φ the maximum flux including the leakage flux from one pole of the magnetic field, f the frequency of the magnetic cycles, provided all the magnetism is included within all the turns of the winding. This proviso, however, must be modified in an ordinary induction motor, since the magnetic density in the air gap may be assumed to vary as a sinusoid,† and that condition requires that the number of lines of force passing through the different turns of the coils shall also vary as a sine function. This is illustrated in Fig. 462. In this figure, overlapping distributed polyphase windings are laid in slots in the internal face of the field magnet; and polyphase currents of a corresponding number of phases flowing in the coils produce a resultant rotating magnetic flux distribution in the air gap between field magnet and armature similar to that shown in the figure for a particular instant by the dotted lines in the air gap. The figure pertains to a four-pole field magnet.

Secondary windings are laid in slots in the external surface of the cylindrical armature core, and both primary and secondary windings are laid down so as to uniformly cover the winding surface of the core on which it is laid, as is usual in commercial practice. The broken lines in the polar space and the curves indicate the approximate magnetic distribution. Consequently we have for the induction motor,

$$E_1 = \frac{\sqrt{2}\,\pi n_1 \Phi f}{10^8\,\theta} \int_{-\frac{1}{2}\theta}^{\frac{1}{2}\theta} \cos \alpha\, d\alpha$$

where $\frac{\theta}{2}$ is the value of α corresponding to the sine ordinate

* Art. 135. † Art. 186.

which is proportional to the number of lines of force passing through the extreme turns of the winding concerned, when the center of the magnetic field is exactly over the center turns of the coil.

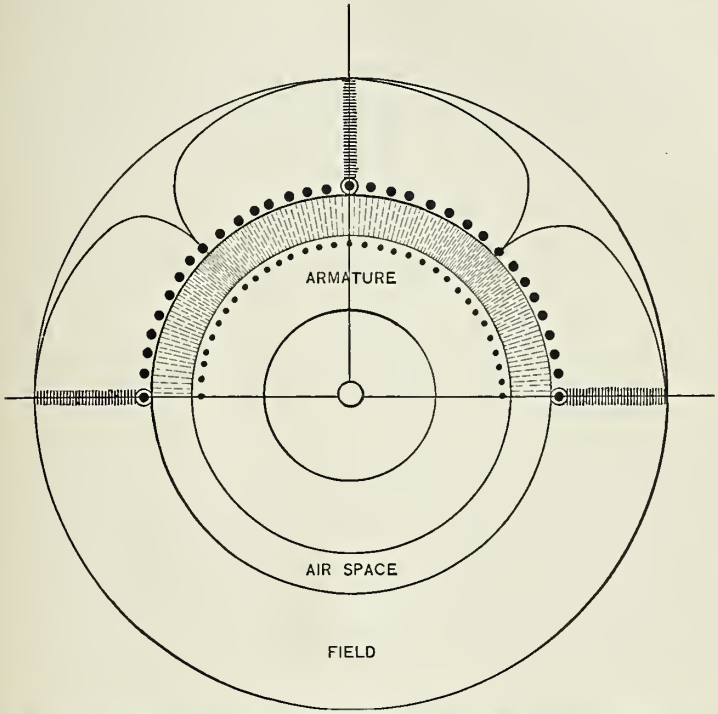


FIG. 462. — Diagram showing Primary and Secondary Windings of an Induction Motor evenly distributed over Internal and External Surfaces of Field Magnet and Armature Core respectively, and also showing the Approximate Distribution of Magnetic Flux for a Given Instant.

For uniformly distributed windings in two phases, $\theta = 90°$; and in three phases, $\theta = 60°$; while for coils on salient poles $\theta = 0°$. The values of E' for the field winding of the induction motor then become

$$E'_{1,2} = \frac{4\, n_1 \Phi f}{10^8} \text{ and } E'_{1,3} = \frac{3\sqrt{2}\, n_1 \Phi f}{10^8},$$

where $E'_{1,2}$ is the induced voltage in the coils of one phase of a quarter-phase motor field winding and $E'_{1,3}$ is the induced voltage, likewise, for a tri-phase motor field winding. Hence, the voltage set up in a uniformly distributed field winding of a

quarter-phase motor is, other things being equal, about 10 per cent less than if the windings were in narrow coils ; and in a tri-phase motor the deficit is nearly 5 per cent. The exact ratios are $\pi : 2\sqrt{2}$ and $\pi : 3$. To give the same counter voltage in the uniformly distributed field windings of an induction motor arranged for two phases, requires about 6 per cent more turns in the windings than when the same machine is arranged for three phases. If the windings are placed on salient poles, as was at one time done in quarter-phase motors, all the lines of force pass through the windings, and the coils therefore act as though they were very narrow; but the increased and irregular reluctance of the magnetic circuit caused by this construction more than destroys any advantage for ordinary motors pertaining to this form of the winding.

The formulas given above may also be derived by considering the primary windings as stationary windings of an alternating-current generator with rotating field magnet in which the magnetic flux from the poles has approximately a sinusoidal distribution. For some purposes this method is useful.* It is evident that the voltage set up in the windings by the rotating magnetism is exactly the same whether the magnetic flux moves with the field core as in a synchronous machine or moves through the core by reason of the varying resultant of two or more magneto-motive forces as in the induction motor. The voltage induced in a conductor depends, in other words, upon the relative angular velocity of conductor and magnetic flux, and it makes no difference how the relative velocity is obtained. Therefore, the generator formulas apply to this case, of

$$e'_1 = \frac{2\pi n_1 \Phi VP}{10^8 \times 60} \sin \alpha, \dagger$$

in which e'_1 is the instantaneous induced voltage in the coil, V the revolutions per minute, α the instantaneous angular position of the coil in the magnetic field, and Φ the flux per magnet pole. In the case under consideration,

$$\Phi = \frac{2}{\pi} \frac{2\pi rl\phi}{2p} = \frac{2rl\phi}{p},$$

* Art. 20.

† Blondel, Notes sur la théorie élémentaire des appareils à champ tournant, *La Lumière Électrique*, Vol. 5, p. 351 ; Jackson, Three-phase Rotary Field. *Electrical Journal*, Vol. 1, p. 185.

where r and l are the inner radius of the field core and its length parallel to the motor shaft, ϕ the maximum magnetic density in the air space, and p the number of pairs of poles in the magnetic field. Since $\frac{V}{60} = \frac{f}{p}$, and the magnetism per magnetic circuit in a multi-polar machine must be multiplied by the number of pairs of poles to get the total number of lines of force cut per conductor per revolution, the formula may be written, generally, in the form

$$e'_1 = \frac{2\pi n_1 \Phi f}{10^8} \sin \alpha;$$

whence

$$e'_m = \frac{2\pi n_1 \Phi f}{10^8},$$

and

$$E'_1 = \frac{\sqrt{2}\,\pi n_1 \Phi f}{10^8},$$

which is as in the transformer. This is the value of the voltage developed in a narrow coil, but if the coil is spread over a considerable area, the maximum voltage is less, as shown earlier in the article.

For various reasons it is more convenient to study induction motors from the transformer standpoint, and we may consider them as transformers with relative motion between the primary and secondary windings.

Since an air space must be made in the magnetic circuit to allow such a motor to operate, it is evident that the quadrature element of the exciting current of induction motors must be materially greater than that of ordinary transformers. In fact, the no load current of some comparatively small motors of this type, which show quite a high full load efficiency, is entirely comparable to the full load current. To reduce the quadrature current to a reasonable limit, every effort must be bent to decrease the reluctance of the air space. As the armature conductors are embedded in the primary and secondary cores, it is possible to make the air space simply that required for mechanical clearance; and, by care in the workmanship, this may be made quite small compared with the air space of dynamos built according to the ordinary methods.

The exciting current for an induction motor may be calculated for each circuit in exactly the same manner as that for a transformer. It is composed of two components in quadrature:

(1) The active component, which is equal to the sum of the no-load losses in the circuit measured in watts, divided by the volts per circuit. The fixed losses, as in the transformer, vary but little from no load to full load in well-designed motors. The number of circuits is equal to the number of phases. The total losses entering into the exciting current (or the no load losses) are the core losses in the field magnet and armature; the I^2R loss in the field winding which is due to the exciting current; a small I^2R loss in the armature conductors which is due to the secondary current required to run the armature against friction and armature core losses; and the friction loss. The watts represented in the exciting current per electrical circuit are equal to the total no load losses divided by the number of phases.

(2) The magnetizing ampere-turns, which are in quadrature with the active component of the exciting current, may be calculated, as in the case of transformers, from the formula $nI = \frac{P\Phi}{1.25}$, where P is the reluctance of the magnetic circuit and nI represents the resultant ampere-turns. But the magneto-motive force per phase is equal to $\frac{2}{m}$ times the resultant magneto-motive force.* Therefore, if n_1 is the number of turns per phase which link each magnetic circuit, the actual magnetizing component of current per circuit is

$$I_\mu = \frac{P \times \Phi}{n_1 \times \sqrt{2} \times 1.25} \times \frac{2}{m},$$

where m is the number of phases. Which for a quarter-phase machine becomes approximately

$$I_{\mu 2} = \frac{P\Phi}{1.75\, n_1},$$

and for a tri-phase machine, approximately

$$I_{\mu 3} = \frac{P\Phi}{2.65\, n_1}.$$

* Art. 186.

Since mechanical clearance between the armature and field magnet is an essential feature of a motor, the reluctance of the motor magnetic circuit is much greater than that of a transformer of corresponding capacity. This makes the magnetizing current greater, as stated, increases the exciting current, and reduces the power factor.

The total exciting current is equal to the square root of the sum of the squares of its two components, or $I_e = \sqrt{I_c^2 + I_\mu^2}$, where I_c is the active component of the current.

The ampere-turns on each magnetic circuit of an induction motor are the resultant of the ampere-turns due to all the phases. It is therefore evident, from previous deductions, that the resultant ampere-turns in the magnetic circuit of an induction motor is $\frac{m}{2}(nI_e\sqrt{2})$, where n is as before the number of turns belonging to each phase which link each magnetic circuit, m is the number of phases, and I_e is the no load or exciting current in each phase.* Consequently if y ampere-turns are required in the magnetic circuit, the winding in each phase must furnish $\frac{2}{m}$ times the whole, as was assumed earlier in the paragraph in obtaining a value for I_μ. For quarter-phase motors, $\frac{2}{m} = 1$, and for tri-phase motors, $\frac{2}{m} = \frac{2}{3}$.

189. Motor Speeds, Slip, and Secondary Induced Voltage. — The velocity of rotation of the resultant primary magnetic flux depends upon the frequency of the current supplied to the motor, and the number of pairs of poles in the field. In two-pole machines, the number of rotations which the magnetic flux makes per second, or the **Field frequency**, is equal to the frequency of the primary voltage; and, in multi-polar machines, the field frequency is equal to the frequency of the primary voltage divided by the number of pairs of poles, or $\frac{f}{p}$. The number of pairs of poles which is referred to by the symbol p is the number in the rotating magnetic field. This is equal to the number of pairs of poles set up by the windings in the primary cores with a smooth magnetic surface, but is equal to $\frac{1}{2m}$ times

* Art. 186.

the number of polar projections which would be necessary if each phase coil was wound upon a projection, though in this latter case the number of resultant magnet poles is the same as where the smooth core is used.

The velocity of rotation of the secondary winding of an induction motor can never equal the velocity of rotation of the rotating magnetic flux (when the machine is running as an induction motor), since the secondary conductors must be cut by the lines of force of the rotating flux in order that voltage may be developed in the armature conductors; that is, the rotating magnetic field must always have a relative velocity of rotation with reference to the armature conductors. In any machine, the relative velocity given in terms of revolutions per minute is $v = V - V'$, where V and V' are respectively the number of revolutions per minute of the magnetic field and the secondary conductors. This relative velocity measured as a fraction of the rotating field velocity is called the secondary **Slip**, and is small, seldom exceeding 5 or 10 per cent of the speed of the motor for ordinary so-called constant speed machines, and reaching these values only in small motors. Slip is usually for convenience named as a fraction or percentage of the synchronous velocity V of the machine, in which case slip is $s_2 = \frac{v}{V}$; and as in a machine with a given number of poles, the frequency of the current in the primary conductors must be to the frequency in the secondary conductors as $\frac{V}{v}$, therefore $s_2 = \frac{f_2}{f}$, where f and f_2 are the two frequencies, respectively. Since the current in the secondary winding must be proportional to the work done by the motor, it must vary with the load, if the secondary impedance is constant, and v must increase as the load is increased. A little consideration shows that, if the rotating field flux remains constant in value, the variation of v with the load must be just sufficient to counterbalance the drop of the secondary voltage caused by the current flowing in the secondary conductors.

A variation of v demands a variation of V' of equal magnitude, since V is fixed by the frequency of the current delivered to the motor; consequently, the speed regulation of a rotary field motor is directly dependent upon the loss of voltage in the

secondary conductors, if we neglect the effect of variable armature reactions and drop of voltage in the primary windings. This is analogous to the case of direct-current shunt-wound motors.

At starting, the relative velocity of the rotating field flux with respect to the secondary conductors is evidently V, since V' is zero. The secondary current therefore may be very great, and the starting torque may also be great provided the armature reactions do not too greatly disturb the phase position of the rotating magnetic flux. To avoid injury to the secondary windings, in large motors, by the current at starting, means must be taken to prevent its becoming excessive, exactly as in the case of direct-current motors worked on constant voltage.

From the relation $V - V' = v$, it is evident that the value of v determines the frequency with which the rotating magnetic flux, Φ, cuts the secondary conductors, and it will vary from a value $v = V$ to $v = 0$ as the motor armature changes from a condition of rest to a speed of synchronism. Slip is frequently named in per cent of motor speed. In ordinary practice, the slip varies, at full load, from a fraction of 1 per cent to 10 per cent of V, having the smaller value in large machines.

As in a transformer, the voltage induced in the turns of the secondary winding when at zero speed ($V = 0$) by the mutual, rotating flux, is the same per turn as that induced per turn in the primary winding, and its value is

$$E_2 = \frac{\sqrt{2}\,\pi\, A n_2 \Phi f}{10^8},$$

where Φ is the flux passing through both primary and secondary coils, *i.e.* the mutual flux per magnetic circuit, and A is a factor depending on the width of the coils per phase, being unity for very narrow coils and less than unity when the coils are distributed over material width, as explained in the previous article. n_2 is the number of secondary turns and f the frequency of the primary supply circuits. In a squirrel cage armature the conductors within twice the width of the polar pitch may be conveniently considered as composing one turn which is wound of parallel conductors distributed over 180 electrical degrees of the core surface and wound under two contiguous

poles of opposite sign. For this case A becomes equal to $\frac{2}{\pi}$. Under these circumstances the formula becomes

$$E'_2 = \frac{2\sqrt{2}\, n_2 \Phi f}{10^8}.$$

For other armature speeds than zero, the voltage evidently becomes $s_2 E_2$, as the rate at which the secondary conductors are cut by the rotating field is proportional to s_2, *i.e.* at standstill the secondary voltage is proportional to the speed V, while when running it is proportional to the speed v, but $s_2 = \frac{v}{V}$, hence for any speed the secondary voltage is $s_2 E_2$. E_2, it will be noted, is the standstill secondary voltage.

It is sometimes more convenient to know the voltage for a conductor, rather than the resultant voltage for all the conductors in series or parallel; in such case the instantaneous maximum voltage is, for any slip s_2,

$$e_{2mc} = \frac{2\pi\Phi f s_2}{10^8} = \frac{2\pi\Phi p v}{10^8 \times 60},$$

where Φ is the maximum value of the mutual rotating flux per pole. The effective voltage per conductor is

$$E_{2c} = \frac{\sqrt{2}\pi\Phi f s_2}{10^8} = \frac{\sqrt{2}\pi\Phi p v}{10^8 \times 60},$$

since the voltage curves in the conductors must be sinusoidal if the magnetism has a sinusoidal distribution in the air space and the velocity is uniform, as has been assumed.

190. Currents, Torque, Impedance, and Magnetic Leakage in a Rotary Field Induction Motor. — The reactance of an induction motor is that due to the magnetic leakage between the primary and secondary windings, as in a transformer. The reactance due to one phase of the secondary windings when the armature is at a standstill and the slip equals unity is

$$X_2 = 2\pi f L_2,$$

where L_2 is the self-inductance of the secondary winding due to leakage flux; and where s_2 equals any fractional value other than unity, $X'_2 = 2\pi f_2 L_2$, but $f_2 = s_2 f$, and

$$X'_2 = s_2 X_2 = 2\pi f L_2 s_2.$$

Therefore, the impedance of the secondary winding for any slip is

$$Z_2 = (R_2^2 + s_2^2 X_2^2)^{\frac{1}{2}},$$

where R_2 is the resistance of the secondary winding. The current which flows in the secondary winding is then

$$I_2 = \frac{s_2 E_2}{Z_2} = \frac{s_2 E_2}{\sqrt{R_2^2 + s_2^2 X_2^2}} = \frac{\sqrt{2}\,\pi A n_2 \Phi f s_2}{10^8 Z_2}.$$

The symbols for current, resistance, reactance, impedance, etc., have the same subscripts and meaning as used when in dealing with the secondary windings of a transformer, but it must be remembered that the impedance Z_2 varies as the speed of the motor changes.

The primary current has a component, of which the magneto-motive force is equal and opposite to that of the secondary current, as in a transformer; therefore, the primary current is composed of the exciting current and the component required to neutralize the mutual magneto-motive force of the secondary current.

The angle of lag in the secondary circuit is

$$\theta_2 = \tan^{-1}\frac{s_2 X_2}{R_2} = \cos^{-1}\frac{R_2}{\sqrt{R^2 + s_2^2 X_2^2}}.$$

The torque between the field magnet and armature is proportional to the integrated product of the rotating mutual magnetic flux with the secondary current, or

$$T = K\Phi I_2 \cos\theta_2 = K\Phi \frac{s_2 E_2}{\sqrt{R_2^2 + s_2^2 X_2^2}} \times \frac{R_2}{\sqrt{R_2^2 + s_2^2 X^2{}_2^2}},$$

where K depends upon the construction of the machine and the distribution of the flux and current at the surface of the armature. From this the torque is found to be

$$T = K\Phi \frac{s_2 E_2 R_2}{R_2^2 + s_2^2 X_2^2}.$$

From the formulas given it is seen that the $I_2^2 R_2$ loss is

$$P_c = \frac{s_2^2 E_2^2}{R_2^2 + s_2^2 X_2^2} R_2 = \frac{T s_2 E_2}{K\Phi}.$$

Hence there results

$$T \propto \frac{P_c}{s_2},$$

when K can be considered to be constant, since E_2 is a direct function of Φ.

From the last formula it is seen that, if the motor is attached mechanically to a load having a constant resisting moment, the copper loss and slip will bear the same ratio whatever may be the changes in the secondary resistance or the primary voltage. Thus, if the secondary resistance is lowered, the armature speed increases slightly and the slip becomes less, the current becomes somewhat less, but the cosine of the angle of lag decreases proportionately so that the electro-magnetic torque continues to just balance the mechanical torque of the load.

In a short-circuited secondary winding, $s_2 E_2$ volts are all used in driving the current through its impedance. The electrical power converted into mechanical power by the motor can be expressed by $S_2 E_2 I_2 \cos \theta_2 = E_2 I_2 \cos \theta_2 - s_2 E_2 I_2 \cos \theta_2$, when $S_2 E_2$ is a voltage that would be generated if the armature ran at a slip of $V - v = V'$ and $S_2 = 1 - s_2$. An analogous situation is created if the rotating flux is imagined to be stationary as in an alternator with rotating armature and the armature is mechanically driven at a speed V'; then there results a voltage $S_2 E_2$ generated in the secondary windings if E_2 is the voltage accompanying a speed V and $S_2 = \frac{V'}{V}$, as in the foregoing.

In the case of the motor the power delivered to the shaft equals the torque times the velocity of rotation in angular travel per unit of time, so that the mechanical power in watts is $P = 2\pi \frac{V'}{60} T$. Transposing gives $T = 2\pi \frac{60 P}{V'}$. The voltage $S_2 E_2$ does not actually appear in the motor as an electrical quantity, but as a mechanical torque, and it is represented in the primary circuit by the voltage drop whose active component, multiplied by the component of primary current flowing because of the secondary current, equals the motor's mechanical output in watts.

In a transformer the secondary external circuit provides a path for a current which, by setting up a counter-magneto-motive force, causes a current of equal and opposite-magneto-motive force to flow in the primary windings. The vector

product of this current by the impressed voltage equals the transformer output plus the copper losses of this component of current. In an induction motor the resisting moment of the load, by lowering the velocity of rotation of the armature, is the cause of the flow of a current in the secondary windings which, as in a transformer, demands current of equal and opposite magneto-motive force in the primary windings.

191. Vector Relations in the Rotating Field Induction Motor. — From the preceding discussions it is evident that a diagram of current and voltage loci may be drawn for an induction motor similar to the corresponding diagram for a transformer. In the transformer diagram showing the effect of internal reactance and a non-reactive load (Fig. 277), the secondary current locus is drawn upon the basis of a circuit containing fixed inductive reactance and variable resistance.* In the rotary field induction motor the resistance of the secondary circuit is usually constant when the motor is in normal operation, but the frequency and voltage vary together, which results in an effect equivalent to varying the resistance as in the case of a transformer.† Therefore, the induction motor diagram is similar in principle to that of Fig. 277 for the transformer, but in the motor diagram the diameter of the current locus is relatively much shorter, on account of the greater magnetic leakage caused by the air space; and the motor current varies less in the windings of small-sized motors from standstill, which is equivalent to the condition of secondary terminals short-circuited in the transformer, to nearly synchronism, which is equivalent to secondary circuit open in the transformer, than is the case for transformers. Figure 463 shows a diagram for an induction motor which is similar to the transformer diagram of Fig. 277, in which as before the ratio of transformation is for convenience taken equal to unity, and reference should be made to the method of construction of that figure.‡ The exciting current SO is taken to be constant though it varies somewhat more in both length and angle than in a transformer because of the greater portion of the circumference of the current locus which comes within the limits of no load and full load. The primary power factor is a maximum when the primary current vector FS, closing the triangle of which the other two sides are the exciting

* Art. 70 (*a*), Case 1. † Art. 70 (*b*), Case 5. ‡ Art. 129.

current SO and the secondary current FO, is tangent at F to the secondary current locus $OFMX$. For smaller primary currents the exciting current component SO causes the lag angle between the primary current and the vertical impressed voltage line OE to be larger than at F, the point of tangency, and for larger

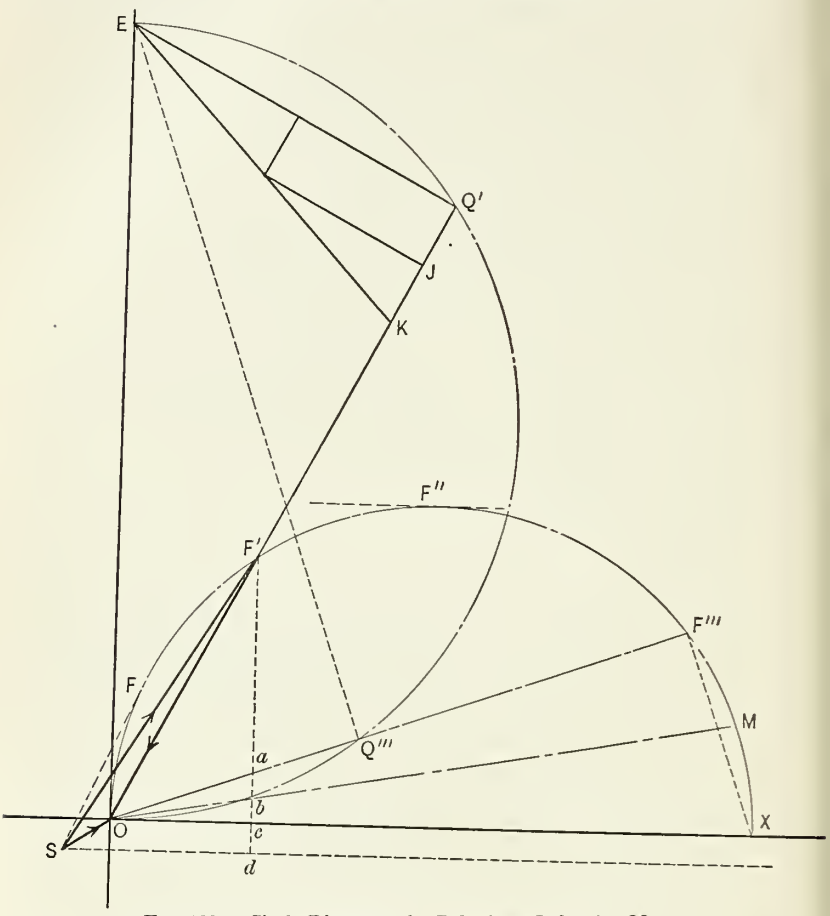


Fig. 463. — Circle Diagram of a Polyphase Induction Motor.

currents than FS the magnetic leakage of the primary and secondary circuits causes the angle to be greater than at tangency. At F'' on $OFMX$ is shown the current locus point of maximum power input, as at this point the projection of the primary current on the voltage vector OE has its maximum value. The primary current at no load is SO in the diagram;

the exciting component, which may be considered constant without serious error, and the component due to the secondary current are at that time of very small value, the latter having an active component just sufficient to drive the armature against bearing friction and windage and to supply magnetization losses. The component of the primary current required to oppose the magneto-motive force of the no load secondary current is added vectorially to the true primary exciting current and is considered part thereof. The standstill or starting current, when starting resistance or other current-reducing devices are not used, is OF'''. This is located on $OFMX$ at the point where the active voltage OQ''' shown in the voltage locus construction is all absorbed in the resistance of the primary and secondary windings. At this point the drop of impressed voltage due to magnetic leakage is $Q'''E$. This is the same as the short-circuit point in a transformer diagram.* The power input, output, and all conditions of working may be obtained from this locus, as has already been explained for the transformer,† or a different method described fully by McAllister, which is sometimes more convenient, may be used for the purpose.‡ Using Fig. 463, the triangle of primary voltages OEQ', for any secondary current as OF', is similar by geometrical construction to $OF'c$, where $F'c$ is a line drawn from F' to c at right angles to OX. The side OQ' of OEQ' is proportional to $F'c$. The points a and b are laid off on $F'c$ so that $F'a$ is proportional to OK, the impressed voltage drop due to the torque of the motor load; ab is proportional to KJ, the drop due to the secondary resistance; and bc is proportional to JQ', the drop due to the primary resistance. The distance cd is equal to the active component of the no load current. Then, since $F'd$ is the active component of the primary current, it is proportional to the power input; also, $F'a$ is proportional to the power output; ab is proportional to the I^2R loss in secondary resistance; bc is proportional to the I^2R loss in primary resistance; and cd is proportional to the no load power loss. The lengths of these lines respectively multiplied by a value representing the impressed voltage OE (that is, taken to a proper scale) are there-

* Art. 129. † Art. 130.

‡ McAllister, *Sibley Journal of Engineering*, Nov., 1904, and *Alternating Current Motors*, 3d Ed., p. 105 *et seq.*

fore equal to the motor input and output and the several losses. The angle $Q'KE$ is considered constant for all positions of Q; hence the angle Oac is constant for all values of F, and the two straight lines drawn from O to F''' and O to M, through a and b respectively, divide every vertical line dropped from any point on the current locus to the line Sd extended into segments proportional to the power quantities.

From the construction it is seen that $\frac{F'a}{F'd}$ equals the efficiency when the secondary current is OF'. The corresponding ratio decreases to a value of zero when the motor is standing still or running at no load, and increases to a maximum when F is at a point on the locus where the tangent to the locus is parallel to OF'''. The length $F'b$ is proportional to the torque measured in terms of watts received by the secondary circuit divided by the theoretical synchronous speed, and it grows from a value of zero when the secondary current is zero to a maximum value when F is at a point on the locus where the tangent is parallel to OM. The speed is proportional to $\frac{F'a}{F'b}$, since the speed equals the output divided by the torque. The slip for varying loads is proportional to $\frac{ab}{F'b}$, since, as shown by the formulas,* the slip is proportional to the secondary copper loss divided by the torque, provided K of the formulas remains constant.

When the impressed primary voltage, the no load primary current, the no load primary angle of lag, the resistances and standstill reactances of the primary and secondary windings, and the synchronous speed of an induction motor are known, it is possible to construct accurate curves of its performance from no load to full load. In fact, test curves of motor performance closely approximate those that are derived from the circle diagram. In using the diagram it should be noted that the exciting current is assumed to be the same in all the phases, and that all currents represented are for a single phase, so that the power and losses obtained from the products of current and voltage vectors must be multiplied by m, where m is the number of phases. The exciting current line SO should have an active component equal to the no load losses divided by m and a quadra-

* Art. 190.

ture component equal to $\frac{1}{n_1 \times \sqrt{2} \times 1.25} \times \frac{2}{m}$ times the magneto-motive force necessary to create a flux equal to that of the rotating field, as shown in Art. 188.

Prob. 1. Construct the circle diagram and from that draw the curves — from no load speed to standstill — of efficiency, power factor, torque, power input, copper loss, slip, speed, and primary and secondary currents (considering the ratio of transformation unity) as functions of the power output used as the abscissas, for a rotary field two-phase induction motor having the following characteristics: full load capacity rating 25 horse power; pairs of poles 6; connection quarter-phase; impressed volts per phase 440 between loads; no load losses 2 kilowatts, which are assumed fixed for all loads. The no load current is 15 amperes per phase and the no load power factor is 20 per cent. The standstill, or short-circuit current, is 125 amperes, and the standstill power factor is 50 per cent. The secondary winding is also assumed to be quarter-phase and the resistances and standstill reactances of the primary and secondary windings to be equal. Figure 463 indicates the process. The circle diagram may be drawn and the currents laid out for a single phase, as suggested above, using one-half of the total power as the full load output. The standstill reactance for either the primary or secondary winding may be obtained by dividing one half the product of the voltage and standstill induction factor ($\sin \theta$) by the standstill current. The resistance of either winding is equal to one-half of the product of the voltage times the standstill power factor ($\cos \theta$) divided by the current. The diameter of the locus is $D = \frac{E}{X_1 + X_2}$, where X_1 and X_2 are the standstill reactances of the primary and secondary windings. The standstill current is very large, so that some error is introduced in finding the diameter by using the reactances as here determined. More accurate methods will be given later.

192. Substituted Impedance for the Rotary Field Induction Motor. — From the circle diagram and the formulas of the preceding article it is evident that equivalent impedance can be substituted for the rotary field induction motor, as was done in the case of the transformer. In this case the load must be represented by non-reactive resistance having a value equal

to the watts output per phase divided by the square of the secondary current reduced to the primary equivalent. This latter is equal to the primary current per phase after the no load current per phase has been vectorially deducted from it. Figure 464 shows the arrangement. In this figure R_f and X_f represent an impedance which will permit the exciting current to flow; R_1 and X_1 are the resistance and reactance of the primary coils; R_2 is the resistance of the secondary coils; X_2 is the reactance of the secondary coils at standstill; s_2 is the slip; and R is the resistance that would be required when the slip is s_2 to dissipate power equal to the mechanical load of the motor when connected into the circuit as indicated in the figure. The

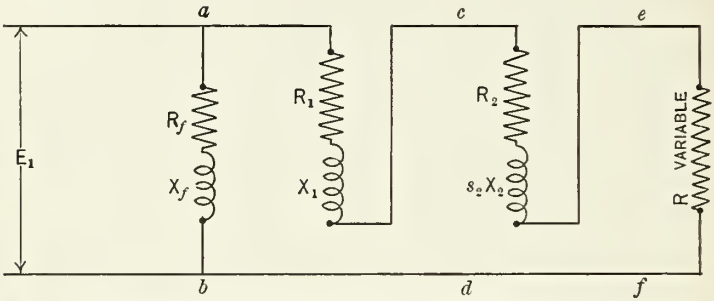


FIG. 464. — Arrangements of Impedance which are Approximately Equivalent to the Circuits and Load of a Rotary Field Induction Motor.

impedances of the primary and secondary windings should be the impedances of one phase for each, and the secondary quantities should be properly reduced to primary equivalents. The coils arranged to absorb the exciting or no load current should absorb a current comprising a quadrature component equal to the $\frac{1}{\sqrt{2} \times 1.25\, n_1}$ times the magneto-motive force of the coils of a single phase, combined vectorially with an active component representing $\frac{1}{m}$ times the total no load losses, where m equals the number of phases; this is of course equal to the no load current measured in one phase.

The solution of these circuits by impedance and conductivity formulas is the same as in a simple transformer,* except that

* Art. 133.

the total motor output and losses are equal to the values given by the formulas multiplied by the number of phases. The no load current varies in value and phase position more than in the case of a well-designed constant potential transformer, so that it is necessary, where minute accuracy is for any reason demanded, to transfer the no load circuit (Fig. 464) from *ab* to *cd*.

The secondary induced voltage is all absorbed in the impedance of the secondary windings. In terms of Fig. 464 in which the secondary quantities are all reduced to terms of the primary circuit, it is the voltage measured from *c* to *e*. It varies directly with the slip and has a value at any slip s_2 which is $s_2 E_2$, in which E_2 is the induced voltage corresponding to a frequency of f periods per second for the cycles of inducing magnetism and is the voltage measured from *c* to *d* in Fig. 464. The induced voltage in the primary winding is also E_2. From Fig. 464 it will be observed that

$$\bar{I}_1 = \bar{I}_2 + \bar{I}_f = \frac{\bar{Z} + \bar{Z}_f}{\bar{Z}\bar{Z}_f} E_1,$$

in which I_1 is the total current in the primary winding, I_2 is the component in opposition to the induced secondary current, I_f is the exciting current, E_1 is the primary impressed voltage, Z is the impedance of the motor circuit from *a* to *b* through *c*, *e*, *f*, and *d*, and Z_f is the impedance from *a* to *b* through the parallel path.

Also, obviously,

$$s_2 \bar{E}_2 = I_2 (R_2 + j s_2 X_2)$$

and $$\bar{E}_1 = \bar{E}_2 + I_2 (R_1 + jX_1) = I_2 \left[\left(R_1 + \frac{R_2}{s_2} \right) + j (X_1 + X_2) \right].$$

From which,

$$s_2 E_2 = I_2 (R_2^2 + s_2^2 X_2^2)^{\frac{1}{2}},$$

$$E_1 = I_2 \left[\left(R_1 + \frac{R_2}{s_2} \right)^2 + (X_1 + X_2)^2 \right]^{\frac{1}{2}}$$

and $$s_2 E_1 = I_2 [(s_2 R_1 + R_2)^2 + s_2^2 (X_1 + X_2)^2]^{\frac{1}{2}}.$$

Therefore

$$E_2^2 = E_1^2 \frac{R_2^2 + s_2^2 X_2^2}{(s_2 R_1 + R_2)^2 + s_2^2 (X_1 + X_2)^2}.$$

Also, the torque per phase is equal to

$$T = K'\Phi n_2 I_2 \cos\theta_2;$$

but $$\Phi = \frac{10^8 E_2}{\sqrt{2}\pi A n_2 f},$$

$$I_2 = \frac{s_2 E_2}{(R_2^2 + s_2^2 X_2^2)^{\frac{1}{2}}}$$

and

$$\cos\theta_2 = \frac{R_2}{(R_2^2 + s_2^2 X_2^2)^{\frac{1}{2}}}.$$

Therefore,

$$T = K\frac{s_2 E_2^2 R_2}{R_2^2 + s_2^2 X_2^2} = K\frac{s_2 E_1^2 R_2}{(s_2 R_1 + R_2)^2 + s_2^2 (X_1 + X_2)^2},$$

in which K is a numerical constant.

The torque is therefore a function of the slip. If the resistance of the armature circuit of a motor is varied, while E_1, R_1, X_1 and X_2 are maintained unchanged, the torque will pass through a maximum value when $dT/ds_2 = 0$, which value is

$$T_{\max} = \tfrac{1}{2} K\frac{E_1^2}{R_1 + [R_1^2 + (X_1 + X_2)^2]^{\frac{1}{2}}}.$$

This equation shows that the maximum torque of any induction motor is proportional to the square of the impressed voltage, and that its value is independent of R_2. The primary current producing it is also independent of R_2; but the slip at which it occurs,

$$s_2' = \frac{R_2}{[R_1^2 + (X_1 + X_2)^2]^{\frac{1}{2}}},$$

is directly proportional to R_2.

If the armature core losses are relatively large and vary with the slip, as would be the case if the armature core were made of unlaminated hard cast iron, the apparent resistance of the armature winding might be mostly the effect of core losses, which vary with the slip, and the motor might then produce a uniform torque (like a series-wound, direct-current motor supplied with a constant current) over a considerable range of speeds.

The resistance of the armature conductors is usually such that the maximum torque comes at from 1 to 10 per cent slip,

or s_2' equals from .01 to .1; and in practice an external resistance is usually introduced into the armature windings at starting,* which serves both to increase the torque at starting and to avoid the excessive rush of current which might otherwise occur while the armature is stationary. Figure 465 shows the relation of torque to slip for a motor when the armature circuits have resistances of .02, .045, .18, and .75 ohm.† This shows plainly that the torque can be caused to have a maximum value at different slips from 100 per cent to 10 per cent of the field

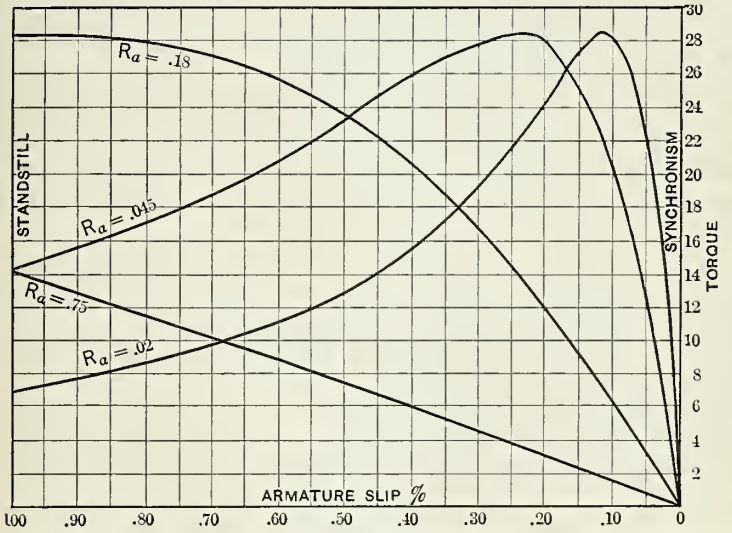


FIG. 465.— Torque Curves for a Rotary Field Motor with Different Resistances introduced into the Secondary Winding.

frequency by gradually reducing the resistance of the armature circuit from .18 to .02 ohm as the speed of the armature increases. An increase of armature resistance above .18 ohm brings the torque to a maximum value when the armature is driven backwards by external mechanical power, thereby increasing the slip beyond 100 per cent.

Induction motors are usually designed to run normally at a speed which is between synchronism and the speed giving the greatest torque. In designing them L_2 is made as small as practicable, by reducing the magnetic leakage to a minimum, and R_2 is then given such a value that the slip at normal full

* Art. 196. † Steinmetz, *Trans. Amer. Inst. E. E.*, Vol. XI, p. 760.

load is sufficient to give a value of the torque which is from one fourth to three fourths of its maximum value. Such motors can therefore carry considerable overloads, as they normally operate at speeds between synchronism and the speed of maximum torque ; but if the resisting moment of the load is increased beyond the maximum torque, the motor stops. In this respect, induction motors differ from shunt-wound direct-current motors operated at constant voltage, in which the torque increases in direct proportion with the armature current and therefore with the resisting moment of the load, provided the total magnetic flux passing through the armature remains constant and does not shift. Increasing the load on a well-designed shunt-wound direct-current motor will not ordinarily stop it until the armature windings are melted, or in small motors until the drop of voltage due to current flowing through the armature conductors is equal to the impressed voltage.

Near synchronism the leakage reactance of induction motors becomes negligible, and from the formula it is seen that the torque varies almost directly as the slip. This is approximately true through the ordinary range of slips from no load to full load in commercial induction motors.

The torque of the armature when the speed is V' revolutions per minute and the output P is given in watts is equal to

$$T' = \frac{60 \times P \times 10^7}{2\pi V'} \text{ in dyne-centimeters,}$$

$$T'' = \frac{P \times 10^7}{2\pi V' 16.3} \text{ in gram-centimeters,}$$

$$T''' = \frac{P \times 10^4}{2\pi V' 226} \text{ in pound-feet.}$$

Since the power equals 2π times the product of torque and speed, and it has already been shown that

$$T = K\Phi \frac{s_2 E_2 R_2}{Z_2^2},$$

therefore,

$$P = \frac{S_2 V}{60} K\Phi \frac{s_2 E_2 R_2}{Z_2^2},$$

in which $S_2 = \frac{V'}{V}$ and $s_2 = \frac{v}{V}$, and consequently, since

$$S_2 = \frac{V'}{V} = \frac{V - v}{V} = 1 - \frac{v}{V} = 1 - s_2,$$

$$P = \frac{V}{60} \Phi K \frac{s_2(1 - s_2) R_2 E_2}{R^2 + s_2^2 X_2^2},$$

which reaches a maximum value at a lesser slip than the torque.

The sum of the output of the motor plus the armature core losses and friction is equal to the continued product of (1) the number of armature conductors, (2) the effective motor load current in each conductor for the given output, (3) the voltage which would be developed respectively in each conductor if the armature were driven at its running speed in an equal stationary field, and (4) the cosine of the angle of lag of the armature current with respect to the induced voltage, or,

$$P = n_2 I_{2c} S_2 E_{2c} \cos \theta_{2c},$$

but

$$I_{2c} = \frac{s_2 E_{2c}}{\sqrt{R_{2c}^2 + s_2^2 X_{2c}^2}};$$

where n_2, I_{2c}, $s_2 E_{2c}$, R_{2c}, X_{2c}, and θ_{2c} are the number of conductors and the current, voltage, resistance, reactance, and angle of lag for any conductor, the subscript c indicating that the quantities are for a single conductor; and

$$\cos \theta_{2c} = \frac{R_{2c}}{\sqrt{R_{2c}^2 + s_2^2 X_{2c}^2}}.$$

Therefore,

$$P = \frac{n_2 E_{2c}^2 R_{2c} S_2 s_2}{R_{2c}^2 + s_2^2 X_{2c}^2};$$

also

$$E_{2c} = E_{1c} = \frac{1}{n_1} E_1,$$

where n_1 is the number of primary conductors, and therefore

$$P = \frac{\frac{n_2}{n_1^2} E_1^2 S_2 s_2 R_{2c}}{R_{2c}^2 + s_2^2 X_{2c}^2} = \frac{\frac{n_2^2}{n_1^2} E_1^2 S_2 s_2 R_{2c}}{n_2 (R_{2c}^2 + s_2^2 X_{2c}^2)}.$$

This formula is not one which is ordinarily needed in the design of a motor, but it plainly shows the effect on the output

of a motor which is caused by varying any one of its constructive details while the others remain unchanged. A very important deduction from the formula is that the torque and output of an induction motor vary as the square of the primary voltage, so that a machine which will carry an overload of 50 per cent on its normal voltage will barely run at full load if the voltage is reduced 20 per cent. The formula also shows that the slip is inversely dependent on the square of the primary voltage.

The motor torque is shown from the equivalent impedance circuits to be approximately, in gram-centimeters,

$$T = \frac{RI_2^2 10^7}{2\pi V' 16.3},$$

where V' is the revolutions of the motor secondary per minute and RI_2^2 is the motor rotative output. The slip is

$$s_2 = \frac{R_2}{R_2 + R}.$$

The number of revolutions per minute the motor armature speed is less than that of the rotating field is

$$v = V\frac{R_2}{R + R_2} = \frac{60f}{p}\frac{R_2}{R + R_2},$$

since the speed of the rotating field in revolutions per minutes, V, is equal to $\frac{60f}{p}$, where f is the frequency of the primary circuit and p is the number of pairs of motor poles. Likewise, the actual speed of the armature in revolutions per minute is

$$V' = V(1 - s_2) = \frac{60f}{p}\frac{R}{R + R_2}.$$

Methods of determining other characteristics are apparent from the study of the induction motor formulas and Figs. 464 and 465. The processes used in dealing with substituted impedances for the induction motor are very similar to those required for the transformer. It will be remembered that to reduce secondary resistances and reactances to primary equivalents, they must be multiplied by s^2 where s is the ratio of transformation.

Prob. 1. Determine the substituted impedances for the motor given in the Problem at the end of Art. 191.

Prob. 2. From the substituted impedances obtain the primary current and power factor for full load, half load, and quarter load for the motor of Prob. 1.

Prob. 3. From the substituted impedances of Prob. 1 find the torque and slip at full load, half load, and quarter load.

193. Additional Formulas for Torque and Slip of a Polyphase Induction Motor. Circle Diagram of Magnetic Fluxes. — In the last article, the slip was expressed thus,

$$s_2 = \frac{R_2}{R_2 + R}.$$

Multiplying this by the secondary current I_2, squared, there results

$$s_2 = \frac{I_2^2 R_2}{I_2^2 R_2 + I_2^2 R},$$

where the numerator represents the armature copper losses and the denominator represents the same copper losses plus the power output (R should be of such value that this will include frictional losses); therefore,

$$s_2 = \frac{P_c}{P_i},$$

where P_c equals the armature copper losses and P_i equals the power transformed from the primary circuit into the secondary circuit. The iron losses may be considered as all absorbed from the primary circuit.

The value of torque stated in pound-feet is

$$T = \frac{P_o 10^4}{2\pi V' \cdot 226},$$

where P_o is the output.

$P_o = I^2 R$, where R is the equivalent secondary substituted load impedance and $P_i = I^2 R + I^2 R_2$. Therefore,

$$P_o = P_i \frac{R}{R + R_2};$$

but $V' = V \frac{R}{R + R_2} = \frac{60 f}{p} \frac{R}{R + R_2}$. Substituting these values of P_o and V' in the formula for torque gives

$$T = \frac{p}{f} \cdot \frac{P_i 10^4}{2\pi \cdot 226 \cdot 60} = .117 \frac{p P_i}{f}.$$

The secondary current of the motor when the armature has no external load, expressed in primary equivalents, may be obtained with approximate accuracy by vectorially subtracting the primary current at no load from the primary current which flows at any input under consideration. Likewise, the resistance of a closed circuit secondary winding can be calculated by observing the primary power input and current when a low voltage is applied to the primary windings and the armature is locked in a stationary position. The difference between the calculated primary I^2R loss and the observed power is approximately the secondary I^2R loss. Dividing this difference by the square of the observed current gives the approximate armature resistance in primary equivalents. The current used should be less than that for full load in order that the magnetization losses may be negligible. However, correction can be made as in a transformer.*

In an earlier article it was stated that the leakage reactance of a polyphase motor decreases on account of the saturation of the core teeth with the heavy currents of short circuit. In order to obtain a value of $X_1 + X_2$ to be used in laying out the diameter of the circular locus of currents, it is well to experimentally determine the value when the current is at or near its value for full load. This can be done by locking the armature in a stationary position and applying a sufficiently reduced voltage to the primary winding so as to cause the desired current to flow. Then, if the observed voltage, current, and watts are E', I', and P', we have

$$\theta' = \cos^{-1} \frac{P'}{E'I'},$$

where θ' is the angle of lag. From which the reactive component of the voltage is

$$E' \sin \theta' = E'_x = I'X' \text{ and } X' = \frac{E' \sin \theta'}{I'},$$

where E'_x is the component of the voltage produced by the change of the leakage flux, and $X' = X_1 + X_2$ is the combined leakage reactance of the primary and secondary circuits. Then the diameter of the current locus for a machine having leakage reactance X' when current I' flows is

$$D = \frac{EI'}{E' \sin \theta'},$$

* Art. 147.

where E is the normal motor voltage; or, if the assumption is made that the reactance of the primary and secondary circuits are equal, — which will not ordinarily introduce a very large error, — and we know the current, voltage, watts, and slip, and hence θ when the machine is running at the working load desired, we can calculate the combined leakage reactance at that load. Thus,

$$\frac{E \sin \theta}{I} = (X + s_2 X),$$

where X is the leakage reactance of either winding; hence, where s_2 is the slip

$$X = \frac{E \sin \theta}{I(1 + s_2)},$$

and the diameter of the current locus circle is

$$\frac{EI(1 + s_2)}{2E \sin \theta} = \frac{I(1 + s_2)}{2 \sin \theta}.$$

In laying out the circle diagram, it is possible to consider the diameter of the locus extended to the origin of the primary current vector as being proportional to a magnetic flux, if the power component of the exciting current is neglected. Thus, consider $OFCA$, Fig. 466, the parallelogram of currents in a motor when the primary current is $OF = \bar{I}_1$, the secondary current is $OA = \bar{I}_2$, and the exciting current is $OC = \bar{I}_\mu$ (neglecting the power component of the exciting current for the present). The impressed voltage is OE_1. This is the phase diagram of currents for a transformer with magnetic leakage * which is similar in voltage and current relations to a polyphase induction motor. For convenience, consider that the primary current sets up a magneto-motive force which sets up the mutual flux through the primary and secondary windings, the leakage flux about its own windings, and neutralizes any tendency of the secondary current to set up leakage flux. The secondary current and the component of voltage produced by the mutual flux thus considered are in the same phase, while the leakage flux about the primary conductors is due to the magnetic paths not only across the teeth in the field core, but also across the teeth of the armature core. This assumption is not necessary, but it

* Art. 124.

simplifies the conception without introducing error. Draw OD upon OC extended so that

$$OD = \frac{4\pi n_1 I'_1}{10\,R} = \Phi_t$$

where n_1 are the primary turns, Φ_t the total flux, and R the reluctance of the magnetic leakage and mutual magnetic paths in parallel, or where

$$R = \frac{R_M R_{1L} R_{2L}}{R_M R_{1L} + R_M R_{2L} + R_{1L} R_{2L}},$$

where R_M is the reluctance of the path of flux which passes through both windings, R_{1L} is the reluctance of the path for

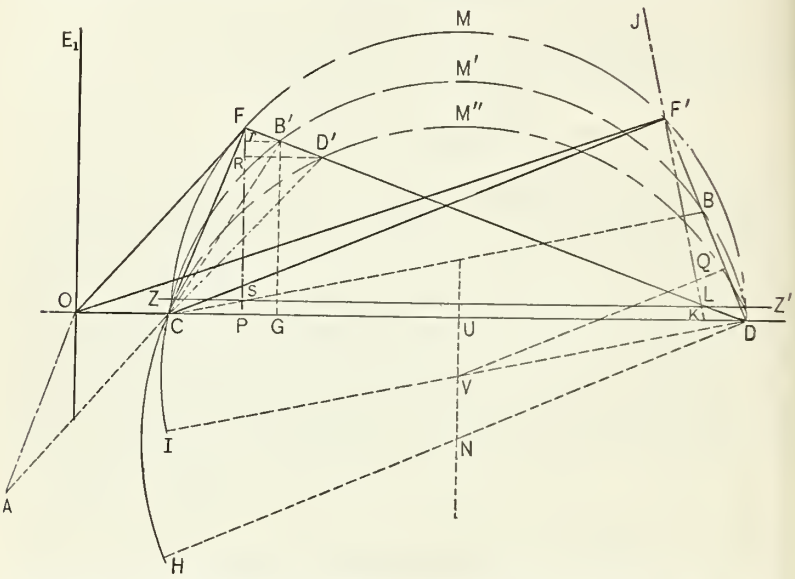


Fig. 466. — Circle Diagram for showing Relation of Fluxes, Slip Line, and Circle Loci for determining other Characteristics of a Polyphase Induction Motor.

primary leakage flux which passes across the teeth of the field core, and R_{2L} is the reluctance of the path for leakage flux around the primary windings which passes across the teeth of the armature core and around the air gap. OD then equals the total flux set up in the magnetic circuits of the machine, and it is produced by the magnetizing current OC. The leakage component of this flux must be in phase with the total primary current, if the iron loss angle is neglected; and by properly

adjusting the flux scale in which OD and other fluxes are laid out it may be represented in phase and magnitude by OF, where

$$OF = \frac{4\pi n_1 I_1}{10 R_L} = \Phi_L$$

where Φ_L is the leakage flux and R_L is the combined reluctance of the leakage paths in parallel. The mutual flux must be the vector difference of the two noted, or

$$\bar{\Phi}_M = \bar{\Phi}_t - \bar{\Phi}_L = FD.$$

But the mutual flux Φ_M is cut by the secondary conductors and sets up the secondary voltage E_2, which must be at right angles to it. As the secondary voltage is, by the assumption, in phase with the secondary current I_2, the latter also must be in quadrature with Φ_M. Therefore, as FC in Fig. 466 equals OA, the angle DFC is always a right angle. Therefore, as F moves, due to changes in the load and hence the primary and secondary currents, it must describe an arc, $CFMD$. If there were no losses this locus would be a complete semicircle.

The output when the secondary neutralizing component of the primary current is CF is evidently $E_1 \times FP$, or at proper scale equals FP if there are no losses. A primary resistance loss will reduce this because the induced primary and secondary voltages will be smaller by the effect of the I_1R_1 drop in the primary windings. This means that FD, the mutual flux, must be shorter in equal proportion. In the same way the secondary output must be less by reason of the resistance drop in the secondary windings, and this may also be considered as equivalent to the shortening of the mutual flux line. Suppose, now, the motor is loaded until it comes to a standstill, and that the currents have increased until F has moved to F'. Since the armature is doing no work, the secondary voltage set up by the mutual flux FD is now all expended in driving the current $F'C$ through the secondary windings. The proportion of the inductive effect of the mutual flux used in overcoming IR drop per ampere is then measured by the ratio $\frac{F'D}{CF'}$ if the copper loss of the exciting current OC is neglected and the currents are reduced for the ratio of transformation of unity as shown in the figure. Since the same relations must hold for all cur-

rents if the reluctances of the magnetic circuits are considered to be constant, the proportion of the inductive effect of the total flux which is absorbed by IR drop for any other current, such as CF, is FD' where the triangle CFD' is made similar to $CF''D$. But F is any point on the arc CFF'' and angle $FD'C$ is constant, and therefore angle $CD'D$ is constant for any position of F on the locus CMD. Hence the point D' must trace the arc of a circle $CM''D$, having the chord CD. If U is the center of the chord, the perpendicular UN lies on the radius of the circle $CD'D$, and, D being one point of the circumference, the center of the circle is the intersection of a normal to $F'D$ at D with UN, since $F'D$ must be tangent to the circle at D, as the inductive effect of the mutual flux represented by $F'D$ is just used up in furnishing the IR voltages. The power lost in the resistance of primary and secondary windings when any current CF flows is then FR. If now the ordinates of the line ZZ' are equal to the assumed constant waste of power in iron losses, the losses in the primary winding by reason of the copper loss caused by the flow of the magnetizing current OC, and the mechanical frictional losses, then the net power given to the shaft by the armature when the secondary current CF flows is represented by the line RS.

The relative primary and secondary copper loss can be found in this way: divide $F''D$, the standstill mutual flux which is all used in generating the primary and secondary IR voltages, into two parts, such that $F''B$ is proportional to the flux used in generating the primary resistance drop and BD to that used in generating the secondary resistance drop. Then, the mutual flux used in generating the IR drop for any other current such as CF must be divided in the same proportions, and hence, FD' is divided into FB' and $B'D'$. Now a triangle drawn between the points C, B, and D is similar to a triangle drawn between the points C, B', and D', since triangles $CF'D$ and CFD' are similar. Hence, as F varies in its positions, the angle made by lines connecting C and B', and B' and D must be equal whatever may be the position of F. The locus of B' is then the arc of a circle upon the chord CD. The center V is the intersection of the perpendicular UN with the normal to BD erected at center point Q. This is evident, since B and D both lie on the arc and the line BD is therefore a chord of the arc between B and D.

For any primary current $OF = I$, supposing the scales in which the lines are measured to be properly adjusted, we have the primary input equal to FP; the secondary input nearly equal to TS; the secondary output equal to RS; the primary copper loss equal to FT; the secondary copper loss equal to TR; and the excitation and frictional losses (assumed constant) equal to SP.

The total armature torque for any primary current OF may be scaled as approximately equal to TS, since $CF \times FD = I_2 \times \Phi_M$, which is proportional to the torque. But, if the perpendicular $B'W = TS$ is dropped from B', the similar triangles $B'WD$ and CFD give the proportion $CF : B'W :: CD : B'D$, or $CF \times B'D = B'W \times CD$. Therefore, as CD is fixed in length, $B'W = TS$ is proportional to the torque and can be scaled to read torque directly. The load torque equals TS minus the frictional and windage torque of the armature itself.

The slip for any primary current may be found as follows: the secondary current $I_2 \propto \Phi_M s_2$, where s_2 is the slip, or $s_2 \propto \frac{I_2}{\Phi_M} \propto \frac{CF}{B'D}$. But, as the triangle CFB' is similar for all positions of F, the slip may be taken as approximately proportional to $\frac{CB'}{B'D}$. Therefore lay out the angle DKJ equal to $CB'D$, then triangles $CB'D$ and LKD are similar and $\frac{CB'}{B'D} = \frac{LK}{KD}$. As KD is constant in length, the slip is approximately proportional to the intercept LK made from the fixed line JK by the line FD, wherever F lies on the locus.

This diagram, in so far as the current locus is concerned, is similar to that of Fig. 463, and the various quantities may be determined in the same way in either, with the exception that in Fig. 466 the fixed losses are subtracted from the measuring line dropped from the end of the current vector, while in Fig. 463 the voltage is made a little larger and the fixed losses are added to the same line. The circle diagram of Fig. 466, with the approximate method of finding the losses, etc., was discovered by Heyland, Behrend, and others early in the art of induction motor manufacture, but it is only approximate at the best and in many instances is quite inaccurate. The diagram of Fig. 463 is more accurate and is therefore preferable to use.

The point F' in both Fig. 463 and Fig. 466 shows the stand-

still point; the locus from F' to C represents the machine as a motor for the entire range of speeds from standstill to theoretical synchronous speed; and the locus from F' to D represents the machine as a frequency transformer, driven backwards (with respect to the rotating magnetic field) from a speed of standstill to infinite speed.

Either of these circle diagrams given for the induction motor is lacking in absolute accuracy, since both the phase and the scalar value of the induced voltage of the induction motor change when the current changes, and this changes the phase and scalar value of the exciting current as well as the radius and position of the locus circle. Attention has already been called to the fact that the reluctances of the leakage paths vary with different currents. Suffice it to say, however, that the diagrams given are sufficiently accurate for many practical purposes, and if they are constructed upon the basis of the magnetic path reluctances to be found at or near full load, they will ordinarily approximate to accuracy from no load to well over full load.

194. The Rotary Field Induction Motor as an Asynchronous Generator. — If a rotary field induction motor is driven above synchronous speed by a source of mechanical power attached to the shaft, while it is electrically connected to a conductor system in which there are voltages which would run the machine as a motor, it no longer acts in the capacity of a motor, but the armature generates electric power and delivers it to the supply lines. In this case the primary circuit of the machine continues to receive its quadrature current from the external supply system, and the rotating magnetic flux is the same as if the machine was operating as a motor. But the relative motion between the primary and secondary circuits having been reversed, — the secondary circuit now rotating faster than the rotating flux, — the active component of current generated in the windings of the armature is reversed, which results in the opposing component of the primary current being reversed. This gives the locus shown in Fig. 467. The upper half of the locus $OTXT'$ is the induction motor and frequency transformer locus and the lower half the asynchronous generator locus. When sufficient power is applied to the pulley to drive the machine at theoretical synchronism, only the primary circuit exciting current SO flows. When the armature speed is raised above synchronism,

current flows in the armature windings, inducing an equal and opposite component of current in the primary windings, which passes out to the external circuit to which the machine is attached. Since the magnetic flux has not been reversed, however, the primary circuit of the machine must continue to draw from the external circuit a component of lagging quadrature current

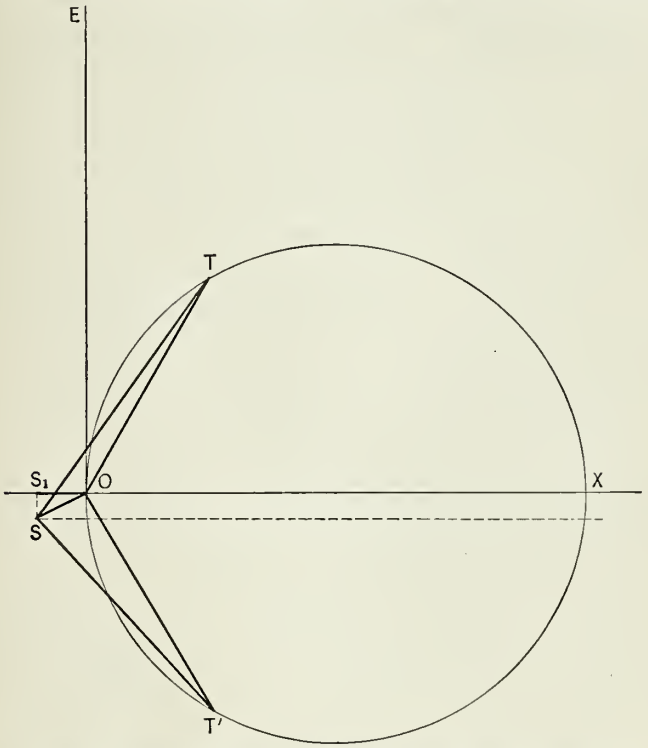


FIG. 467. — Locus of Currents in Induction Machine when Run as a Motor and as an Asynchronous Generator.

sufficient to furnish the requisite magneto-motive force. This is equivalent to a flow from the machine of a leading quadrature current. Such a machine will evidently work in parallel with synchronous alternators or other apparatus which will furnish this exciting current, but it will not work when connected to circuits containing only resistance and self-inductance. The exciting current may be obtained by placing condensers of sufficient capacity across the circuit or by obtaining the required leading current from a synchronous motor connected in the

circuit.* The asynchronous generator function of the polyphase induction machine is sometimes made use of on traction systems in which the cars are driven by such motors. In this case, when the motors are properly connected in tandem,† they can be caused to pump power back into the line when running down grade or being brought to a stop.

When an induction generator receives its quadrature exciting current from synchronous generators of fixed speed, the current delivered from the asynchronous generator is of the same frequency as that of the synchronous generators, regardless of the speed of the armature of the asynchronous machine, but the output changes with the speed in the manner shown by the locus diagram. However, when the quadrature exciting current is provided by a condenser or a synchronous motor, no synchronous machine of fixed speed being connected with the circuit, the frequency of the voltage and current delivered by the asynchronous machine varies with its armature speed. The locus diagram of Fig. 467 is not applicable to the latter case.

If a circuit receiving power from an induction generator becomes short-circuited, the generator at once ceases delivering power on account of the fact that the magnetic flux disappears when the impressed voltage disappears. This is contrary to the action of synchronous generators, which are self-reliant in magnetic field and continue to deliver power to the short-circuited lines.

195. Some Features of Construction of Rotating Field Induction Motors. — The secondary windings of induction motors are wound upon laminated drums or equivalent rings, which, with the windings, usually constitute the rotating part or rotor. The arrangement of the windings may be of several forms:

1. *Squirrel-cage Form*, in which single embedded bar conductors are placed on the armature core and all connected together at each end by a copper ring, thus making a conductor system similar in form to the revolving cylinder of a squirrel cage (Fig. 461). The conductors are insulated from the core.

2. *Independent Short-circuited Coils.* — In this form of winding the secondary conductors are of insulated wire wound in independent short-circuited coils, or of insulated bars connected by end connectors in such a way as to make independent short-circuited coils.

* Art. 174. † Art. 196 (6).

3. *Independent Coils short-circuited in Common.* — Here the coils are wound as in the preceding form, but instead of being short-circuited independently, all the ends are brought to a common point, or pair of points, one of which may be at the front end and the other at the back end of the armature.

4. *Coil-wound Armatures connected to External Devices.* — The windings of paragraphs 2 and 3 can be arranged for connection to external terminals or collecting rings, so that external resistances may be included in series for varying the torque or slip. The connection may be so arranged that the windings form single or any polyphase system of circuits.

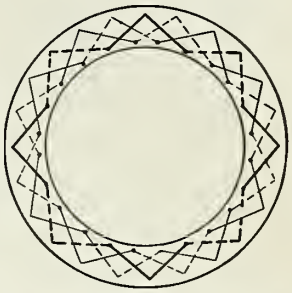


Fig. 468.—Elementary Diagram of a Tri-phase Armature Winding for a Secondary Stator arranged to surround a Primary Rotor having Eight Resultant Poles.

It is evident that the pitch of the coils of the second and third forms of drum windings must be equal to an odd number of times the pitch of the rotating field poles, in order that the voltage set up in the conductors may be additive, and coils may therefore be diametral or chordal in machines with an odd number of pairs of poles, but cannot be diametral in machines with an even number of pairs of poles. The actual number of coils is a matter of perfect freedom, provided it is a multiple of two or three, and the connections of conductors is properly made so that the surface of the armature core may be uniformly covered. A tri-phase winding for a stationary drum armature, which is intended to surround a rotating primary structure having eight poles, is shown diagrammatically in Fig. 468, and a three-coil armature which is intended to revolve within a six-pole primary structure is shown in Fig. 469.

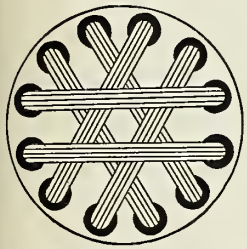


Fig. 469. — Elementary Diagram of a Wound Secondary Rotor to be Surrounded by a Six-pole Rotating Field of the Primary Stator.

As said earlier, it is a matter of indifference from the electrical standpoint whether the primary or the secondary part rotates. From the mechanical standpoint it is usually desirable to have the secondary part rotate, as the primary windings, which are usually of higher

voltage than the secondary, are then subject to less mechanical vibration and can be connected directly to the supply mains without the interposition of collector rings. Coil-wound secondary windings are in general of the same type as the armature windings of synchronous alternators, and, as explained, may have their terminals short-circuited, or they may be connected (through collector rings if the armature rotates) to outside devices suitable for starting or for regulation purposes.*

The primary windings of induction motors are almost always arranged to produce more than two poles, in order to bring the machine to a reasonable speed. The rotating field frequency in revolutions per second is equal to $\frac{f}{p}$, where f is the frequency of the alternating current and p the number of pairs of poles in the magnetic field, and in revolutions per minute this becomes $V = \frac{60\,f}{p}$. We therefore have the following table of motor speeds for the frequencies common in this country.

Relation of Number of Poles to Synchronous Speed in Induction Motors

Number of Poles of Motor	V, when $f = 25$	V, when $f = 60$
2	1500	3600
4	750	1800
6	500	1200
8	375	900
10	300	720
12	250	600
16	187.5	450
20	150	360
24	125	300

This table shows the futility of attempting to build satisfactory induction motors, intended for use on even the lowest frequency commonly used in this country, with less than four poles ; and on the higher frequencies not less than six to ten poles are required to give reasonable speeds. Motors of greater output than ten horse power should have a sufficiently

* Art. 196.

large number of poles to give a field velocity which does not exceed 750 or 800 revolutions per minute.

The primary windings, in the very early history of the art, were sometimes placed directly upon polar projections of the field frame, as in Fig. 470, which shows a four-pole two-phase machine. In modern machines the windings are arranged as embedded uniformly distributed conductors in a frame of uniform magnetic surface, as in Fig. 471, which shows a four-pole three-phase machine. Conductors embedded in open slots give the most approved arrangement, since embedding serves to reduce the

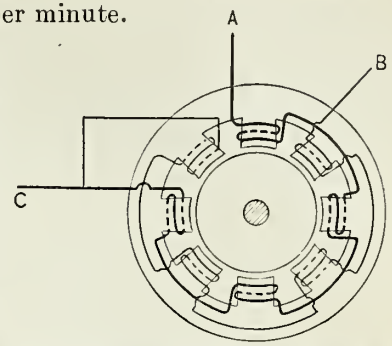


FIG. 470. — Diagram of Salient Pole Induction Motor Primary Winding as sometimes used in the Early History of the Art.

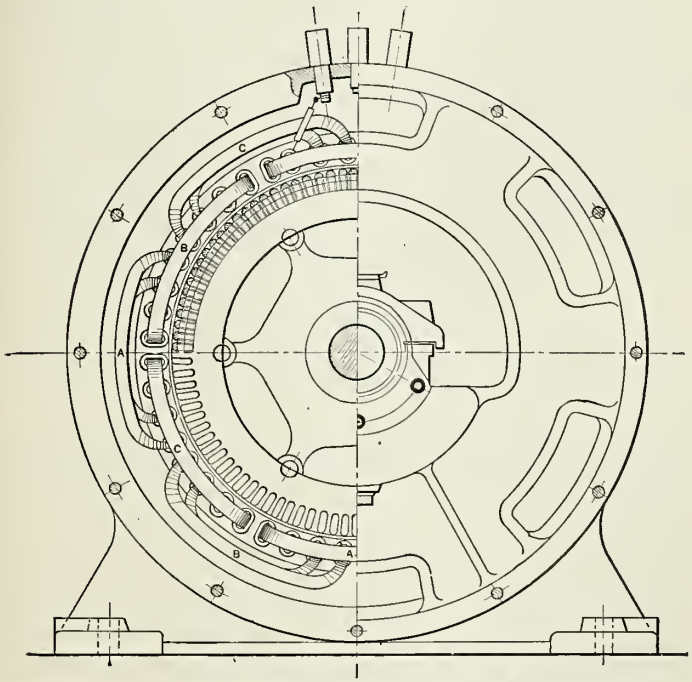


FIG. 471. — Diagram of Rotating Field Induction Motor, in Which the Stator has a Tri-phase Four-pole Primary Winding, and the Rotor may be wound either as a Squirrel Cage or Coil-wound Secondary.

reluctance of the magnetic circuit and reduce magnetic leakage, and therefore serves to increase the power factor of the motor. Open slots are better than closed slots, as they offer greater reluctance to the magnetic leakage paths and permit the use of form-wound coils. Since the actual magnet poles are produced by the resultant effects of the polyphase currents, it requires two coil sections to produce each magnet pole in a two-phase ma-

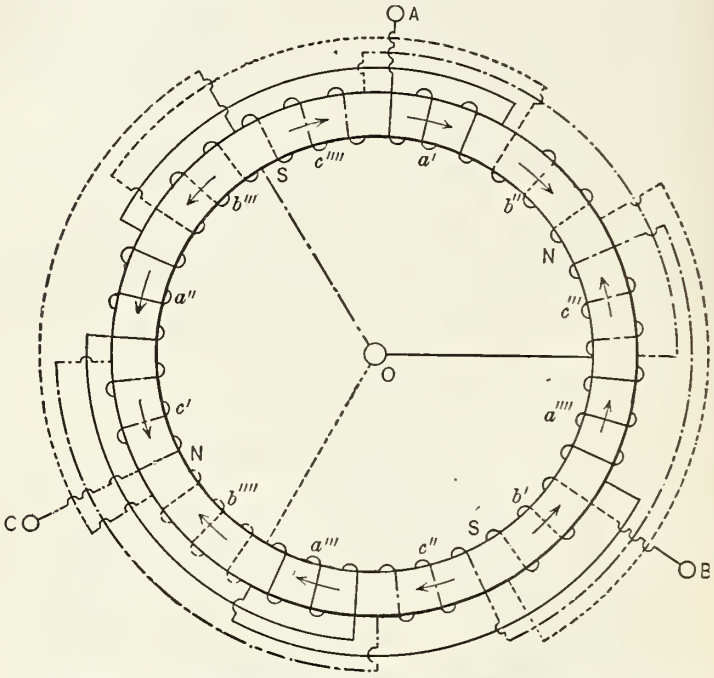


FIG. 472. — Simple Ring Diagram for showing the connections of a Four-Pole Tri-phase Wye-Connected Winding.

chine, and three coil sections in a three-phase machine. The connections of the field coils may be traced out according to the instructions given for connecting the armature coils of poly-phase generators.*

The connections for a three-phase field winding are illustrated diagrammatically by Fig. 472, which shows a wye-connected, four-pole ring field. The arrows indicate how the magnetic fluxes combine at a particular instant under the impulsion of the magneto-motive forces of the tri-phase currents. It should be

* Art. 22.

noted that the ring type of winding is used in this illustration for the sake of simplicity in showing the connections. Windings embedded in the polar surfaces are now used almost altogether, but the principle of the connections is the same as in the simple diagram of the figure. The wye connection is often preferred for three-phase field windings, as less voltage is impressed on a coil, so that fewer turns of wire are required, and the strain on the insulation of each coil is less. The circulation of internally induced third harmonic currents is also prevented. The total weight of copper is equal in wye and delta connections. Figure 473 shows a development of the eight-pole, wye-connected winding such as that of Fig. 474.

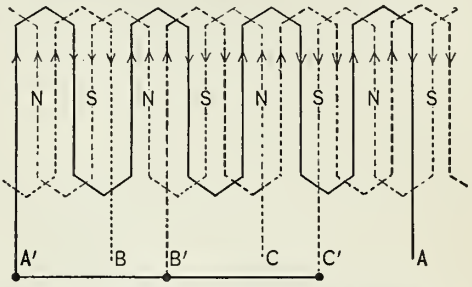


Fig. 473. — Diagram of a Simple Progressive Triphase Eight-pole Winding such as is to be found on the Primary and Secondary Windings of Fig. 474.

In the case of Fig. 474, the secondary winding terminals are connected to rings so that regulating or starting resistances may be inserted.

It is seen from the above discussion that the primary windings of rotating field induction motors are of the same forms as the armature windings of synchronous alternators. The secondary windings may be of the short-circuited squirrel cage type or be similar to the primary windings, but the numbers of slots in the field and armature cores should differ, and should be of numbers which are prime to each other, to prevent recurrent variations of the resultant reluctance of the mutual magnetic circuit.

The frequency of the magnetic cycles in the iron of the primary core is equal to the frequency of the current flowing in the magnetizing coils, but in the armature it is equal to the motor slip; and the hysteresis and eddy current losses per pound of iron are therefore many times greater in the primary core than in the secondary core when the motor is operating at ordinary speeds. In this respect the characteristics of an induction motor are the reverse of those of a synchronous alternator, in which the field loss consists in most

part of the I^2R loss, while the armature loss is the sum of the I^2R and core losses, of which the latter may be the larger portion. In the induction motor, the field core losses are large and the

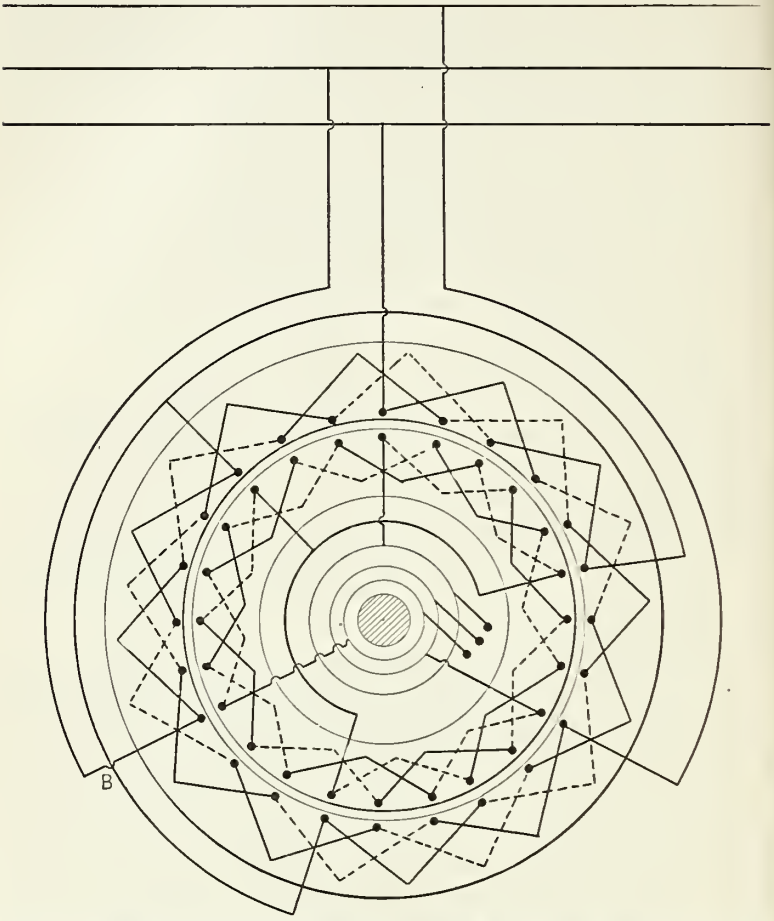


Fig. 474.—Diagram of a Rotary Field Induction Motor having Eight Poles. The Field and Armature Cores are both wound for Three Phases Wye connected and the Secondary Winding has its Free Terminals connected to Collector Rings for Use with Starting and Regulating Resistances.

armature core losses are scarcely appreciable in a well-designed machine; it is therefore desirable to reduce the amount of iron in the primary core to a small volume, keeping in mind the effect of the several variables such as weight of copper, density of flux, etc., which enter in the problem as they do in the design of a

transformer.* It will be noted that with a rotating armature the winding space is smaller for the secondary than for the primary windings, but the voltage of the former is usually lower than that of the latter, so that less space is used for insulation. The smaller the number of slots per coil, the smaller is the reluctance of the leakage paths and hence the greater is the leakage reactance with its attendant disadvantages. It is therefore common practice to make the rotor diameter large and the length correspondingly short so that from 3 to 6 slots can be used on the stator per coil side and from 3 to 7 on the rotor — the number depending upon the given conditions that must be fulfilled in each particular motor design, such as voltage, speed, rated load, etc. To reduce reactance the air space is of course made as small as is mechanically practicable.

In all cases, especially where squirrel cage secondaries are used, the number of primary core slots should be an uneven multiple of the number of secondary core slots, as intimated, in order that there may be no dead points in starting. If the teeth of the primary core lie exactly opposite the teeth of the secondary core, the magnetic reluctance of the paths through the teeth will be very low compared with that of the winding spaces between the teeth. This causes exceedingly dense tufts of magnetic flux to be set up between opposing teeth at the instant when they are facing each other. The tendency, then, of each tooth of an opposing pair is to strongly attract the other as the magnetic poles created by the flux at their ends are of opposite sign. If at starting the tooth to tooth attraction is stronger than the pull between the secondary current and rotating magnetic flux, the motor will not move. Using unlike numbers of slots on the two members avoids this trouble. In a somewhat similar manner when the armature is running unloaded and nearly at synchronism, there is a tendency for the secondary core to jump into synchronism with the rotating magnetic field and convert the machine for the time into synchronous motor relations, the armature becoming the mechanically rotating field magnet with permanent poles magnetized by the synchronously running resultant rotating field. This is because the secondary current when the speed nears synchronism at no load is exceedingly

* Arts. 135–137.

small, only enough to drive the armature against the resisting moment of the friction of bearings and air, and the attraction between teeth under these circumstances sometimes shows a larger torque than that between the rotating flux and the bands of armature current. When this occurs, the former prevails and the secondary core takes the same speed of rotation as the rotating flux. While this condition continues, the secondary current becomes zero and the windings act somewhat to maintain the condition of synchronism as will be seen later. The condition is not necessarily undesirable, though it calls for a sudden change in primary current. Where a load of any material size comes on to the motor, the secondary core drops out of synchronism and the machine resumes its action as an induction motor. This peculiarity is of material value when synchronous motors are started as induction motors.*

196. Starting and Regulating Devices. — Several different arrangements may be used for starting polyphase induction motors.

1. *Small machines* are commonly connected directly to the circuit without the intervention of any special starting devices. This is not a safe proceeding for large machines, as when the secondary circuit is at rest and the primary windings are directly connected to the supply circuit, the machine is in the condition of a transformer with the secondary windings short-circuited, and is liable to burn up before getting under way. Some of the older small motor armatures were arranged with two rows of conductors, making two independent squirrel cages (Fig. 475), one considerably farther from the armature surface than the other for the ostensible purpose of reducing the starting current of the machines.

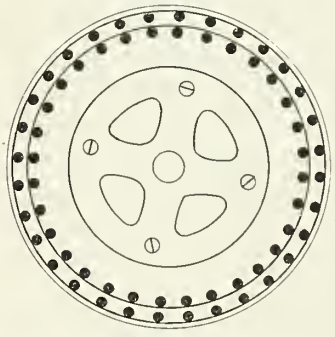


Fig. 475. — Squirrel Cage Secondary with Double Row of Conductors.

2. (*a*) *Resistance in Field Circuit.* — Resistance may be inserted in the circuits leading to the primary windings to be used in much the same manner as starting rheostats are used in starting direct-current, constant voltage motors. Rheostats

* Art. 167.

arranged in this way in each circuit must be manipulated simultaneously in all the circuits, and therefore must be mechanically coupled. Besides starting rheostats similar to starting boxes for direct-current motors, liquid rheostats, arranged to be varied by dipping plates in a bath, have been used with early motors. Three rheostats are required for a tri-phase motor, and two for a quarter-phase motor operated on independent circuits. Quarter-phase motors on three-wire circuits may be started with a single resistance device inserted in the common wire, or by two resistance devices inserted respectively in the independent wires.

The insertion of resistance in the field circuits of induction motors serves to restrain the starting current by reducing the voltage at the primary terminals. This, however, reduces E_1 in the numerator of the expression *

$$T = K \frac{s_2 E_1^2 R_2}{(s_2 R_1 + R_2)^2 + s_2^2 (X_1 + X_2)^2},$$

and thereby greatly lowers the available torque per ampere and the motor takes an excessive current from the supply mains to accelerate a heavy load. In the past this plan was used quite extensively by European manufacturers, especially for large machines which could be started with the belt on a loose pulley. This method has the disadvantage of wasting much energy through the dissipation of heat in the rheostats, and has, therefore, except in special cases been abandoned in favor of the autotransformer and transformer methods described below.

(*b*) *Variable Compensator or "Autotransformer."* — The voltage at the terminals of the primary circuits may be reduced at starting by introducing an impedance coil across the supply circuits and feeding the motor from variable points on its windings. This arrangement may be caused to supply a large starting current to the motor without interfering with the supply circuits, but it has the same effect on the motor torque as resistance in series with the primary windings. The losses by this method of starting are much reduced, for the power required by the motor for starting is delivered at the proper reduced voltage by transformer action at comparatively high efficiency. Figure 476 shows a compensator connected to a tri-phase motor with a no-voltage release, which automatically opens the circuit when, for

* Art. 192.

any reason, the supply voltage disappears. It contains also an overload release. The lower row of terminals, designated the starting side, are connected by proper switch blades or contactors

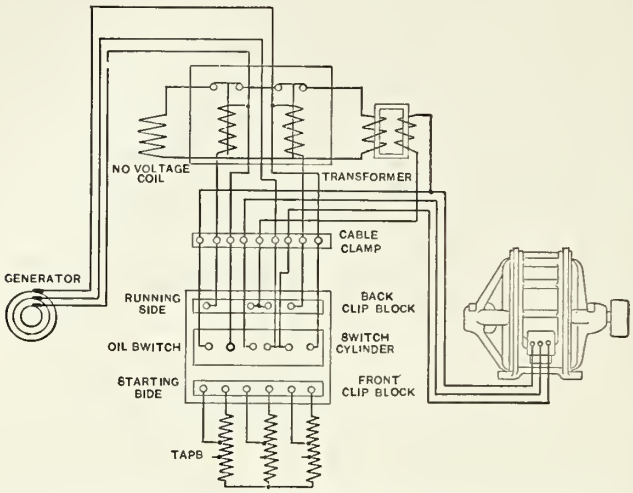


FIG. 476. — Diagram of Compensator Starting Device for Tri-phase Induction Motor, with No-voltage and Overload Release.

with the middle row marked "cylinder switch" when the motor is to be started. This connects the motor with the low voltage terminals of the autotransformer or compensator and the supply

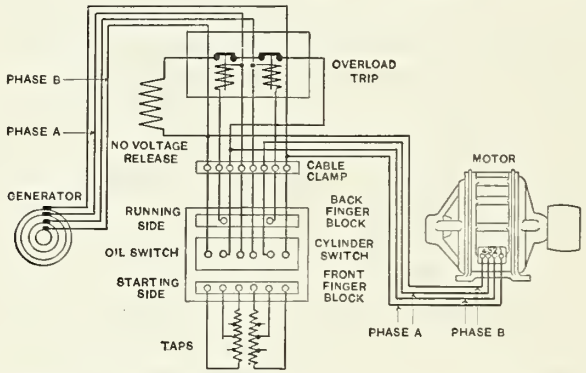


FIG. 477. — Diagram of Compensator Starting Device for a Quarter-phase Motor with No-voltage and Overload Release.

mains with the primary terminals thereof. When the motor has reached its full speed for this connection, the central switch cylinder blades or contactors are thrown over to the running side

and the motor is connected directly to the supply lines. When very large motors are in use several steps can be made in this operation by having several sets of taps on the compensator coils and stepping from one to the next successively. Figure 477 shows a similar arrangement for a quarter-phase motor.

It is evident that these arrangements are purely starting devices and not regulators, since from the motor formulas * it is seen that the running torque is changed with the square of the voltage; and, therefore, if it should be desired to drive a load of given torque at reduced speed, reduction of the impressed voltage by means of the compensator could only change the speed through very narrow limits before the maximum motor torque would be reduced to the opposing moment of the load. Further reduction of voltage would result in the motor coming to a standstill.

3. *Resistance in Armature Circuits for Speed Regulation.* — The torque of the armature at starting may be made equal to the maximum running torque by inserting a definite amount of resistance in the secondary circuits which increases the total armature resistance at starting in the ratio $\frac{R_e + R_2}{R_2}$, where R_e is the external or starting resistance. The value of R_e may be determined as follows: As already shown,† $s_{2m} = \frac{R_t}{[R_1^2 + (X_1 + X_2)^2]^{\frac{1}{2}}}$, where s_{2m} is the slip at maximum torque, and R_t is the total resistance through which the armature current flows. Therefore to get a maximum torque when $s_{2m} = 1$, requires that $\frac{R_2 + R_e}{[R_1^2 + (X_1 + X_2)^2]^{\frac{1}{2}}} = 1$ or $R_e = [R_1^2 + (X_1 + X_2)^2]^{\frac{1}{2}} - R_2$. The reactances are determined from the circle diagram or by calculation. The total resistance used in a starting rheostat should be much larger and arrangements should be made to reduce it gradually in order that the motor may not start with a jerk.

As the maximum torque is usually designed to occur at a slip between one and one third and two times that corresponding to the normal full load torque, when running with the winding resistance only in the secondary circuits, the armature and field currents at maximum torque do not exceed twice the full load currents, so that resistance inserted in the armature circuits

* Art. 192. † Art. 192.

serves the double purpose of increasing the starting torque and keeping the starting current within bounds.

This plan is largely used by many commercial companies. The arrangement of the starting rheostat depends largely upon the type of the secondary windings to which it is applied.

In secondary windings of the squirrel cage type the conductors may be tipped at one end with tapered high resistance metal strips, which come in contact with a sliding copper ring. At starting, this ring may just touch the high resistance tips, and as the machine speeds up the ring may be slid along until the tips are cut out and the copper armature conductors are directly connected together through the ring. If the armature revolves, it is evident that the ring must be arranged to slide on a spline on the shaft, and to be controlled by a grooved sliding collar and loose lever. The same device may be used for secondary windings with coils having a common short-circuiting point. In this case one set of coil terminals are permanently connected together, and the other set are connected into a rheostat of high resistance metal strips, which may be short-circuited by a sliding ring, as already explained.

If the secondary windings are arranged so as to have but one coil for each phase, the introduction of resistance is very simple, since only one resistance coil for each phase is required. In this case, if the secondary core is stationary, the connections of the rheostat are made directly into the secondary circuits, or between the secondary circuits and one point of common connection; while if the secondary core revolves, collector rings may be placed on the shaft, and stationary rheostats may be used to control the resistance of the coils which are properly connected to the rings, or the resistance may be placed inside the armature spider and be controlled by a sliding collar and loose lever. Figure 478 shows a tri-phase motor having three collector rings for connecting up external starting and regulating resistance to the secondary windings. The rheostat coils can be connected in wye or delta. A quarter-phase winding would preferably use four rings and two resistance devices.

By using this method the motor can be regulated for any torque up to its maximum value for all speeds, from standstill to the speed at which the machine drives its load when the secondary winding is short-circuited. The method is effective for

speed control, but has the disadvantage of absorbing much power in the external resistance when the speed is made much below that normal for the given load when the secondary windings are short-circuited without external resistance. Also, with a given resistance inserted, the speed varies with the load. These conditions are somewhat analogous to those of a direct-current shunt-wound motor when the speed is controlled through the introduction of external resistance into the armature circuit.

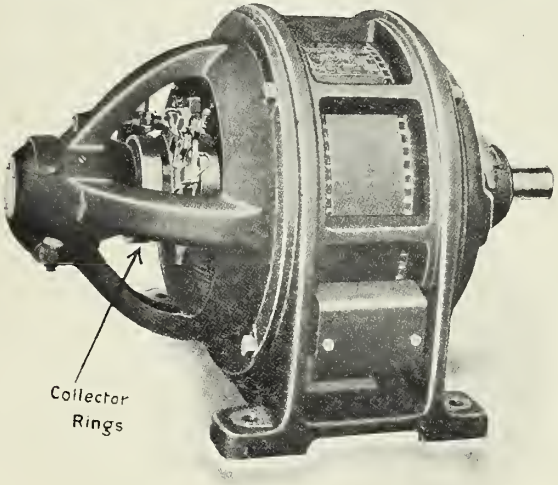


FIG. 478. — A Rotary Field Motor with Tri-phase Secondary Winding arranged with Collector Rings for the Introduction of External Starting or Regulating Resistances.

4. *Commutated Armature.* — The armature may be wound with the coils so arranged that, instead of starting with all conductors in cumulative series in each coil, a portion of the conductors are connected in opposition to the others at the start, and are then reversed and connected properly in series with the others after the machine is in operation. The opposition arrangement affects the current both of the secondary and primary circuits.

Again, the connectors, by means of which the armature conductors are placed in series, may be made of high resistance material, and these connectors may be excluded from the circuit when the conductors are short-circuited together after the

motor is running at full speed. Devices of this character can be used for giving two motor speeds through the range of torque for which the motor is designed, but any additional speeds would make cumbersome connections.

5. *Commutated Primary Windings.* — In order to enable a motor to run at two speeds without loss in external secondary resistance, the number of poles of the rotating magnetic field may be changed by properly arranging the primary windings. Thus, a motor having a primary winding for eight poles may have its speed doubled by so connecting the primary coils that the resultant flux has four poles. This, as may be seen by a study of primary windings, requires quite a little shifting of coil terminals, which becomes burdensome if the method is used for obtaining more than two speeds. To obtain the same motor torque at the double speed requires one half as many primary coils per phase, since the number of poles of the rotating field is cut in half, while the primary and secondary magneto-motive force must remain constant to supply the necessary torque, since the flux per pole is doubled but the polar area is also doubled. Changing from wye to delta connection for starting purposes is referred to on page 857.

6. *Concatenation Control.* — Where two like motors are connected rigidly to the same mechanical load, as when two poly-

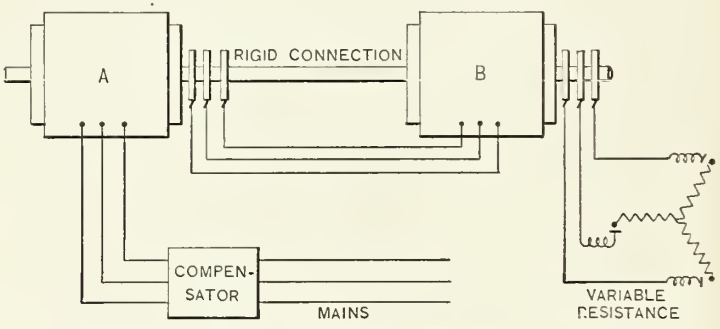


FIG. 479. — Diagram of Connections of Tri-phase Motors for Concatenation Control.

phase motors are used in driving an electric car, a speed control may be obtained which is somewhat analogous to the series-parallel control of two direct-current electric traction motors. Figure 479 shows two motors, both having unity ratio of transformation between their primary and secondary windings, con-

nected in this way, which is called the **Tandem** or **Concatenated** connection, ready to start. Motor A is connected directly to the supply mains ; its secondary windings are connected to the primary windings of motor B, which thereby is supplied with current of frequency corresponding to the slip of motor A; the secondary windings of B are connected to an external set of rheostats ; and the shafts of the two motors are rigidly joined mechanically. The ratios of transformation of the motors are equal, and the motors are otherwise alike. When motor A begins to rotate, motor B is caused to rotate through the mechanical connection. As the resistance in the secondary circuits of motor B is cut out, the speeds rise until, when the resistance is short-circuited, the motors are running at something under one half the synchronous speed pertaining to each when supplied with voltage at the frequency of the mains. They cannot run above half speed and carry any considerable load, for in that case motor B would run above synchronism with the current supplied to its primary windings from the secondary windings of A, and this would convert motor B into an asynchronous generator * tending to force current back into the secondary circuits of A. The faster the motor armatures rotate, the smaller is the slip of A and the lower the frequency of the current it supplies to B. At a speed of revolution equal to one half synchronous speed for motor A, the rotating flux of B and its armature conductors would be in exact synchronism. Then the only current that could flow from the secondary circuits of motor A would be that delivered to the primary circuits of B, which, under the conditions named, would be the exciting current necessary to set up the rotating magnetic field of B. Therefore neither motor would produce useful torque at this speed. Since the resisting moment of the load must be equaled by the electrical torque, this state could not exist when any considerable load is to be driven ; and, therefore, the speed of rotation of the armatures when driving a load must be less than one half normal speed by an amount which results in currents flowing in the windings sufficient to produce the necessary torque. At this point the slip in B (with respect to the field frequency provided by current from the secondary circuits of A) is sufficient to induce enough secondary voltage to drive

* Art. 194.

current through its secondary winding. The slip in A (with respect to the field frequency provided by current from the mains) is sufficient to induce enough secondary voltage in its secondary windings to drive the current through the impedance of those windings plus as much more as is required to provide the primary terminal voltage at B.

The primary current in each phase of A is the sum of a component equal and opposite to the secondary current of B, the exciting current of B, and the exciting current of A. The currents in the secondary circuits of A are equal to those in the primary circuits of B. The primary voltage is absorbed by the drop due to the mutually induced voltage of A, the impedance of the primary and secondary windings of A, the mutually induced voltage in B, and the impedance of the primary and secondary windings of B.

The induction motor locus diagram given in Fig. 463 can be readily applied to this case by letting the current locus which represents the current in the secondary windings of B have a diameter equal to the voltage of the mains divided by the sum of the leakage reactances of the two motors, and the exciting current have a value equal to the combination of the exciting currents of the two motors. The quadrature voltage in the voltage locus diagram of Fig. 463 is then the sum of the leakage drop voltages; and the active voltage is the sum of the drops through the counter-voltages and resistances of the two motors. If the ratios of transformation of the motors are not both exactly alike or the impedances of the two motors are not equal, one motor will "rob" the other of load with danger of its becoming overloaded; which is comparable to the results in the case of two rigidly connected series direct-current motors which have different armature resistances or field strengths.

As a result of this discussion the conclusion can be safely drawn that the rotary field induction motor does not, in its present form, lend itself readily to wide, economical variation of speed. Where such speed variation is demanded, series or repulsion motors, which are described in later pages, are more naturally fitted.

197. Reversing Polyphase Motors. — Polyphase motors may be reversed by reversing the direction of rotation of the rotating magnetic field.

In quarter-phase motors with two independent circuits, reversing the terminal connections of either circuit will effect the

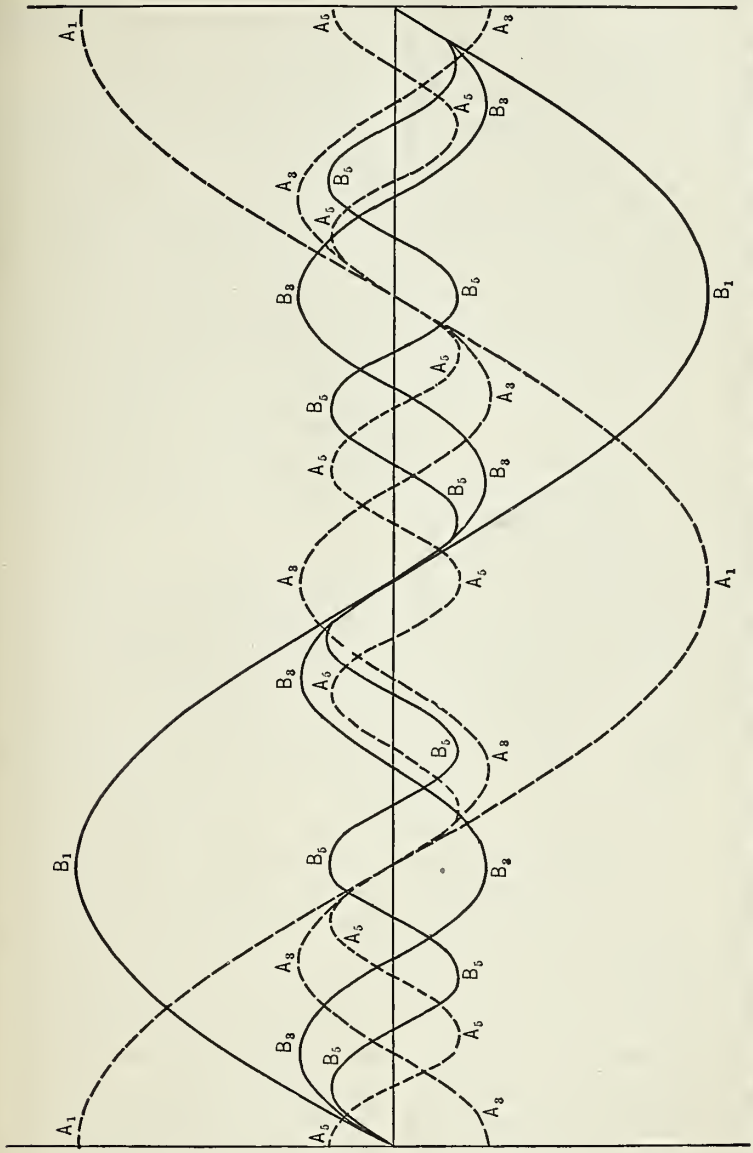


FIG. 480. — Curves for showing the Effect of Third and Fifth Harmonics on the Operation of a Quarter-phase Induction Motor.

reversal of rotation, but reversing the terminals of both circuits will not alter the direction of rotation. Quarter-phase motors

with three-wire connections may be reversed by interchanging the connections of the outside conductors, leaving the common return conductor unchanged.

The direction of rotation of tri-phase motors may be reversed by interchanging the connections of any pair of leads.

198. Effect of the Form of Curves of Voltage. — The effect of distorted curves of voltage upon the operation of induction motors depends upon the number of phases. The harmonics of three and five times the fundamental frequency are the only ones which need be considered; and indeed, that of three times the fundamental frequency is the only one which as a rule has an appreciable influence. In single-phase motors the harmonics should affect the magnetic field as they affect that of a transformer, so that peaked voltage curves should cause a decrease in core losses, and the operation of the motor should not be otherwise greatly influenced. In polyphase motors, however, the harmonics may set up a rotating field of their own, which is superposed upon the regular field, and may interfere with the operation of the machine.

The harmonics with three times the fundamental frequency belonging to the two circuits of a quarter-phase system have a phase difference of 90° (Fig. 480), and these set up a superposed rotating field in the induction motor which has a field velocity of three times that of the main field. The figure shows that the harmonics of triple frequency, belonging to the two phases, are reversed in relative position compared with the fundamental waves. The field due to these harmonics rotates in the reverse direction from that of the main field, and therefore tends directly to decrease the torque of the motor and to increase the slip. The field due to the harmonics of five times the frequency rotates in the same direction as the main field, and its only disadvantageous effect is in causing eddy currents which may slightly decrease the efficiency of the motor. The frequencies of the harmonic curves are indicated in the figure by subscripts.

In tri-phase circuits the harmonics of triple frequency belonging to the different currents are directly superposed in phase as is shown in Fig. 481, and therefore the superposed field which they cause in tri-phase induction motors is a stationary one whose influence is only to decrease the efficiency by setting up extra losses. The figure also shows that the harmonics of five

frequencies have 120° difference of phase and are in reversed orders, so that they set up a reverse rotating field, and if they are of much strength they may affect the torque.

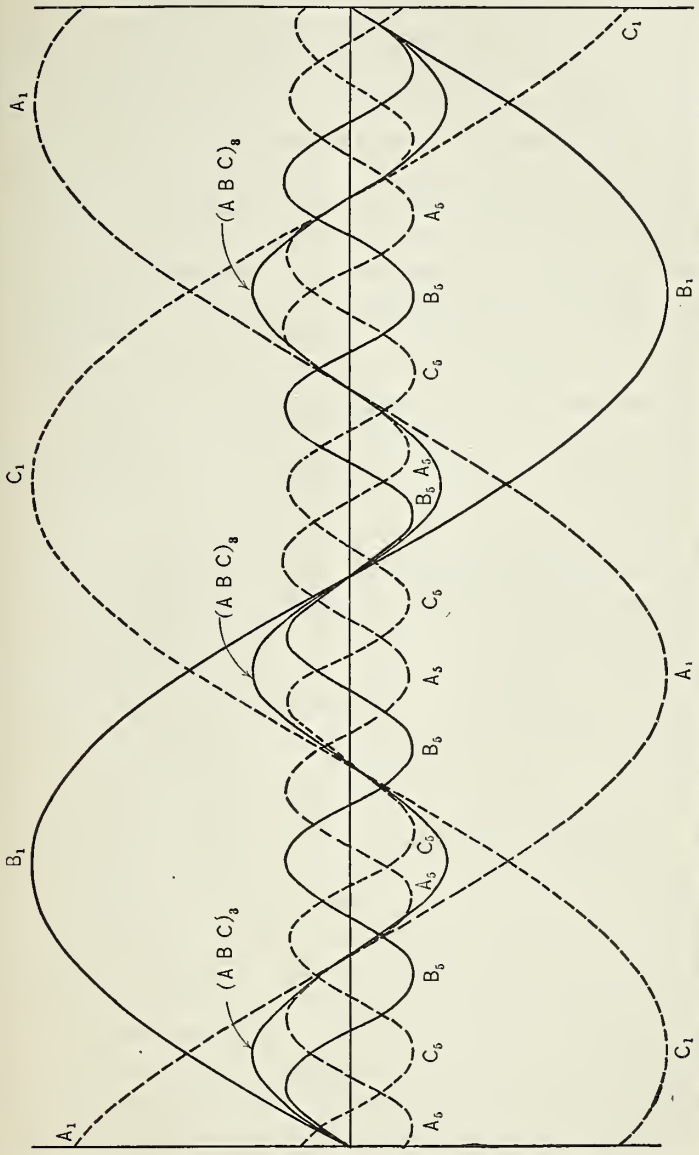


FIG. 481. — Curve showing the Effect of Third and Fifth Harmonics on the Operation of a Tri-phase Induction Motor.

199. Single-phase Induction Motors. — If the field magnet of an induction motor is wound with one set of coils so that the

field poles are set up by a single alternating current flowing in the coils, the poles will be stationary but alternating, and the effects of electro-magnetic repulsion may be utilized for the purpose of causing the armature to rotate. Thus, if a uniformly wound short-circuited armature (such as is used for polyphase induction motors) is started to revolving in a single-phase alternating magnetic field, the balance of repulsions which exists when the armature is at rest is disturbed, and the armature tends to continue its motion. To illustrate this, the condition of two coils in complementary positions with reference to one of the poles may be considered. As the armature revolves, one coil moves toward a position where it includes more lines of force from the pole, and the other coil moves so as to exclude lines of force. If the strength of pole is rising, the first coil will have the larger current induced in it, since the rate of change of lines of force through the first coil is proportional to the sum of the rate of change in the strength of field and the rate of change of flux linkages due to the motion, while the rate of change of lines of force through the second coil is the difference of these two rates. Thanks to the lag in the coil circuits, the currents in both coils are in such a direction as to result in an attractive force on the pole, but a much stronger force is experienced by the first coil than the second. When the field is falling, the magnetic condition of the second coil is changing the more rapidly, but the direction of the induced currents in the coils is reversed with respect to the direction of the inducing field, and the coils experience a repulsive force with reference to the pole. The effect during one complete period of the magnetic field therefore tends to cause the armature to rotate in the same direction in which it was started. The torque is a maximum when the positive product of current and magnetic flux is a maximum, which is when the current lags behind the induced voltage by an angle between 45° and 90°. The torque at any speed is equal to the torque which would be given by a polyphase induction motor of similar construction at the slip $v = V - V'$ minus the torque which the polyphase induction motor would give in a field of double the frequency, and with a slip proportioned to $V + V' = 2V - v$; where V is synchronous speed, V' actual speed, and v relative speed in revolutions per minute; but with the ordinary ratio of the resistance and inductance in the arma-

ture winding, the torque due to the latter slip is negligible at such full load slips as are satisfactory in practice. In this case, V need not be looked upon as a speed of rotation of a magnetic field, but as the speed of the armature which keeps each conductor at the same position with reference to a pole for any fixed instant in each period of the magnetic flux. Hence $V = \frac{60f}{p}$, exactly as in rotary field machines. When the armature is stationary, $v = V$, and the two torques are equal and opposite. A single-phase induction motor may therefore be designed in a manner similar to the manner of designing a polyphaser in respect to its operation after it has reached its normal speed, but it requires special treatment in the design for the purpose of making it self-starting, and it is both heavier and less efficient for a given output.

The principal voltages induced are indicated in the diagram of Fig. 482 for a bi-polar machine. The alternating magneto-motive force impressed by current flowing from the external circuit through the primary or field coils is represented by the arrow *CD* and the designation *N–S* and *S–N*. The resulting alternating magnetic flux set up through the armature core sets up counter-voltages in the secondary or armature windings, as in any transformer. The direction and position of the voltage in the armature conductors are indicated by the arrow points (dots) and feathers (crosses) marked in the figure within the cross-sections of the conductors which are indicated by small circles. This voltage is in time quadrature with the primary flux. These conditions are as in a transformer with large magnetic leakage and may be indicated by the ordinary

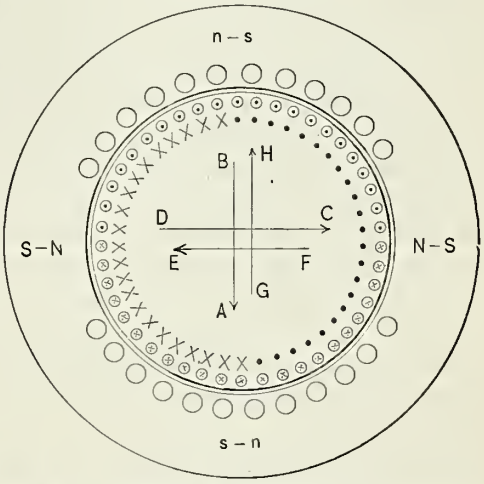


FIG. 482. — Diagram for indicating the Voltage generated in a Single-phase Induction Motor.

transformer diagram.* The turns across the vertical diameter have the highest voltage induced in them. (The armature, for convenience, is assumed to be of the squirrel cage type and the primary windings to be distributed windings, passing through the holes from the top to the bottom of the figure.) This voltage decreases more or less sinusoidally until the horizontal turns are reached, which are in a neutral position with regard to N–S. When the armature revolves, voltages as shown by the arrowheads and feathers indicated in the inner circle are set up in the conductors by their motion through the primary field. These voltages are proportional in strength to the field N–S and hence are in time phase therewith. They reach a maximum value in the conductors under the poles N–S and S–N and decrease more or less sinusoidally to zero in the turns 90° therefrom, which do not cut the field flux. There are, therefore, two sets of voltages in the conductors, the first created by ordinary transformer action caused by variation of the field N–S, and the other by the motion of the conductors through the lines of force of the same field. These two voltages are at ninety degrees displacement with reference to each other, both in time phase and space position. Two components of current, conjointly composing the armature current, will flow under the influence of these two voltages. The first reacts on the primary circuit like the secondary current of a transformer, and has a time lag due to leakage flux only; while the second or "speed" current sets up magnet poles at n–s and s–n, and as the reluctance of the magnetic path is low, this component tends to take a time quadrature relation with its inducing voltage. Hence N–S and n–s tend to rise and fall nearly in time quadrature and exactly in space quadrature.

When the armature is running at the speed corresponding to theoretical synchronism for the frequency of the primary current, each turn of wire on the armature in each revolution cuts the same number of lines of force as are comprised in the maximum value of the field N–S in one cycle of the magnetic flux, so that the transformer and speed voltages are equal. Therefore, the fields N–S and n–s are then of approximately equal magnitude. Under these conditions N–S and n–s may be

* Arts. 124 and 129.

considered to form an ordinary rotating field acting upon a short-circuited armature, as in polyphase motors. When the speed is lower than the theoretical synchronous speed, the speed voltage is proportionately less; and at standstill the speed voltage is zero.

The foregoing shows that it is possible to consider the single-phase motor in the light of a motor-generator in which the transformer component of current is a torque producing current acted on by the poles n–s, and the speed current is a generated exciting current which sets up the motor fields n–s. The counter-voltage caused by the conductors cutting flux n–s tends to oppose the effective transformer voltage in driving current through the armature resistance. The product of counter-voltage with transformer current equals the output of the motor. If the load is changed, the motor speed must change so that the difference between the transformer and counter voltages will permit a transformer current to flow which will give the requisite torque.

Again, the single-phase alternating field may be treated in a different manner to get the same result more simply. An alternating field stationary in position may be resolved into two rotary fields, revolving in opposite directions, having the same frequency as the stationary field, and of one half its magnitude in strength.* This is similar to the principle of mechanics by which a simple harmonic motion may be resolved into two uniform opposite circular motions of one half the amplitude. The torque diagrams of each of these fields acting alone are shown in Fig. 483, where O is the point of armature rest, and armature speed is counted from that point along the horizontal axis. The curves A and A' are the torque curves that would be given by either field acting alone, the torque due to one being in one direction, and that of the other in the opposite direction. It is evident that when the armature is at rest it has no tendency to revolve, as the slip of the armature with respect to the two fields is equal, and torques Ot and Ot' created by the two fields are equal and opposite; but if the armature is started in one direction, for instance toward the right, the slip with respect to field A decreases, the torque caused by it increases and tends

* Ferraris, A Method for the Treatment of Rotating or Alternating Vectors, *London Electrician*, Vol. 33, p. 110.

to continue the rotation, while the slip with respect to field A' increases, and the torque caused by A' decreases. When the armature speed becomes V', the torque caused by A is T, which is due to a slip $V - V' = v$; while the torque caused by A' is T', which is due to a slip in relative speed between armature and field of $V + V' = 2V - v$. From the relations of torque to slip, which have already been discussed,* it is evident that the torque caused by A' decreases as the relative speed increases. If the differences between the corresponding ordinates of the curves of torque A and A' are plotted in a curve, the actual

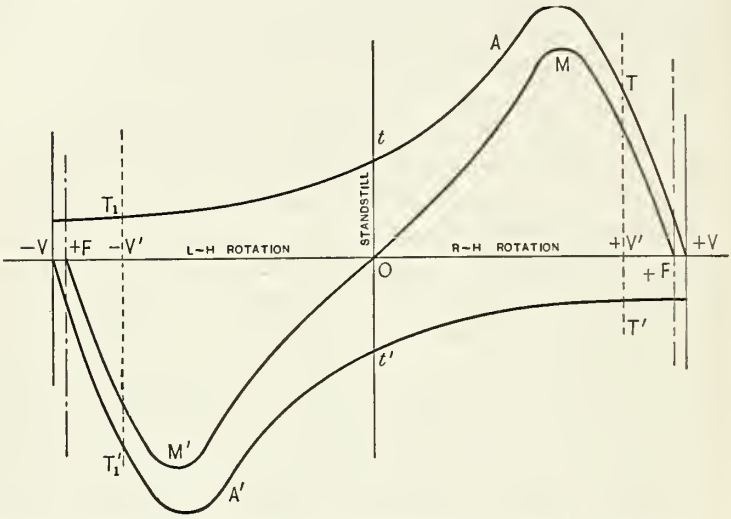


FIG. 483. — Motor Torque Diagram of a Single-phase Motor with the Field resolved into Two Component Rotating Fields having Opposite Directions of Rotation.

torque curve M is given. The ordinates of this will give the actual motor torque with respect to slip. From this curve it is seen that the motor will work at no load at an almost synchronous speed, and may then be loaded until the speed has dropped to a point where the torque is at a maximum. If the load exceeds this, the motor will stop. The curve M falls to zero a little to the left of $+V$, because of the torque T'. However, as the backward field is, at synchronism, cutting the armature conductors at a rate of $S''_2 = 2f$, the reactance $(2fX_2)$ against that field is very great and the ordinates of T' (exaggerated in the figure) are ordinarily negligible for working speeds. From

* Art. 192.

the point $+F$ to $+V$ there should be a double frequency component of current in the armature and a slight negative torque. The ordinary values of resistance and inductance which are required in an efficient and economical design, make the effect of A' so small at the speed of normal full load that the action of A only need be considered.

If the armature should be started toward the left, instead of the right, as here assumed, the conditions would be reversed and the motor would operate under the torque line M'. Evidently the formula from which the torque curve M (Fig. 483) can be plotted is

$$T = K \frac{s_2 E_1^2 R_2}{(s_2 R_1 + R_2)^2 + s_2^2 (X_1 + X_2)^2}$$

$$- K \frac{(2f - s_2) E_1^2 R_2}{(2fR_1 - s_2 R_1 + R_2)^2 + (2f - s_2)^2 (X_1 + X_2)^2}.$$

The second term becomes small for armature speeds approaching synchronism. It will be appreciated, on account of the high reactance $(2f - s_2)X_2$, that the magnetic flux of this field which actually passes through the armature windings is very small when s_2 is small, so that if Φ' represents this flux, $\Phi' = \Phi - \Phi_L'$, when Φ is the backward rotating component of the flux in the field, and Φ_L' is the leakage component of that flux. Evidently Φ_L' is always large and approaches closely to the value of Φ when s_2 is small.

From these considerations it is seen that a single-phase motor may be designed in the same manner as a polyphaser, and that for equal output the ampere-turns upon the field magnet must be equal to the resultant number on a polyphase motor. The field winding may be arranged as in polyphase motors, covering the entire polar surface, as is shown in Fig. 484, for a four-pole machine; but the differential action in this case reduces the effectiveness of the winding in the proportion of $1 : \frac{2}{\pi}$, as has already been shown in Art. 188. Consequently, the equation from which the field windings are determined becomes

$$E' = K \frac{\sqrt{2}\, \pi n' \Phi f}{10^8} = \frac{2}{\pi} \times \frac{\sqrt{2}\, \pi n' \Phi f}{10^8} = \frac{2\sqrt{2}\, n \Phi' f}{10^8}.$$

The value of K may be increased and material saved by leaving

space between the primary coils, as in Fig. 485, which shows the windings for a two-pole field.

The efficiency of single-phasers is less than that of poly-phasers, since the armature core losses are proportional to the

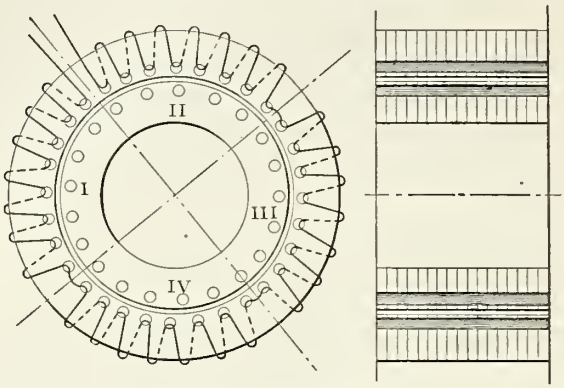


Fig. 484. — Diagram of a Single-phase Induction Motor with Primary Winding distributed over the Entire Surface of the Field Core.

frequency of the main field instead of to the slip, and the I^2R losses are greater; their slip for a given load and similar design is greater; and their maximum torque is slightly less than that of polyphasers, as is shown by Fig. 483; but these

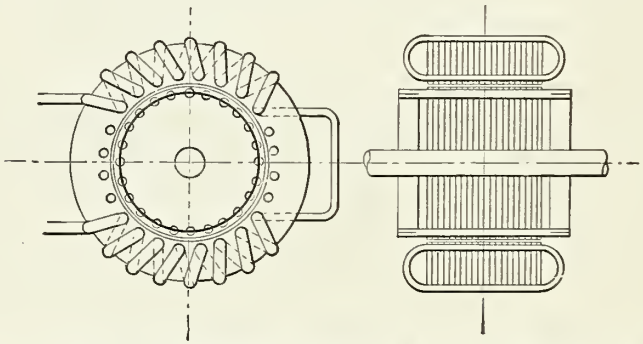


Fig. 485. — Diagram of a Single-phase Induction Motor with the Primary Winding Grouped in Coils covering Part of the Field Core.

differences in well-designed machines should not be great. The weight of single-phasers is larger than that of equal poly-phasers, because the value of K is smaller.

200. Locus Diagram of the Single-phase Motor and Substituted Impedance. — As the single-phase motor is equivalent to

a transformer having constant reactance and varying load of unity power factor, it may be represented by a current locus diagram similar to that of a transformer or polyphase induction motor. Thus Fig. 486 shows such a diagram, constructed and lettered similarly to the diagram of Fig. 463, which represents the condition of one branch or phase in a polyphase induction motor. The only important element of difference is found in

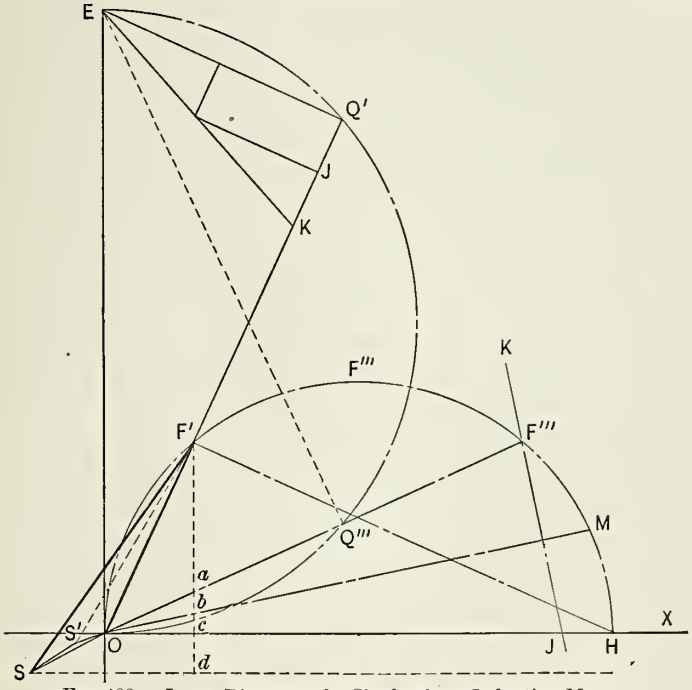


FIG. 486. — Locus Diagram of a Single-phase Induction Motor.

the no load current SO. This shows an active component which is greater in proportion, due to the larger armature iron losses, and a wattless current which is greater in proportion, due to the backward rotating components of the field.

The line OF'' is as before the added component of secondary current due to the load, but the total secondary current is greater than this by some component SS'' because of the armature current set up by the backward rotating component of the magnetic flux. Referring to Fig. 483, the total armature current found by adding the forward and backward components of the current may be roughly considered, for a given load, to have

the value $F'S'$ (Fig. 486), and to have the relations to the impressed voltage OE, the total primary current SF', and the phase angles which are shown in the diagram. This is under the assumption that the backward rotating component of the primary current remains constantly of uniform value, since its function is to set up the backward rotating component of the field magnetic flux Φ'. The forward rotating component of the field current, on the other hand, must not only have a no-load component $S'O$ sufficient to set up its component of the rotating flux Φ, but must also neutralize the mutual magnetic effect of the armature current component OF', which is the torque-producing component of the armature. Upon the same suppositions as used in the diagram of Fig. 466, where the armature current and voltage were considered to be in phase with each other, $F'H$ is proportional to the flux necessary to induce the voltage which drives the current $F'O$ through the armature. The diameter of the locus semicircle is found in the same way as for the semicircle representing a phase of a polyphase motor, as are the various elements of performance. KJ represents the slip line constructed as in Fig. 466; the loci of copper loss may also be shown as in that figure.

Impedances can be substituted for the single-phase motor circuits, for the purpose of making computations, as was done with the polyphase motor.* Referring to Fig. 464, the shunt circuit representing the no-load losses should be connected at cd instead of ab, in order that great accuracy may be obtained in representing the conditions of a polyphase motor. In the single-phase motor a similar arrangement is required for great accuracy, and allowance must be made for the extra reactive effects of the exciting currents; but for most purposes it is sufficiently accurate to have the no-load circuit connected from a to b, which will take a current equal in phase and quantity to the no-load current, and it is upon the basis of such an approximation that the locus diagram Fig. 486 is drawn.

201. Starting Single-phase Induction Motors. — Since single-phase induction motors are not *per se* self-starting, special starting devices must be included in their design and construction. This sometimes takes the form of what is called a **Phase splitter**. The field is wound with two coils similar to the windings of a

*Art. 192.

two-phaser, and at starting these are connected in parallel to the circuit, one directly, and the other through dead resistance or capacity. This throws the currents in the two coils into a difference of phase, which may be accentuated by winding one coil so that it has greater self-inductance than the other ; and the machine then starts as a two-phaser. After the machine is running, both coils are connected directly to the circuit, or one coil is cut out, and the motor operates as a single-phaser. This operation of "phase spliting," as applicable to such motors, cannot give a large difference of phase between the currents in the two motor circuits with a reasonably large power factor, and consequently single-phase induction motors started in this way must have either a very small starting torque or an unreasonably small power factor at starting. Another method frequently used in small motors, such as small fan motors, is to use **Shading coils.** That is, the motor field magnet may be constructed with salient pole pieces like an alternator field magnet, — the core of course being laminated. Around a similar tip of each pole piece is wound a short-circuited coil. The currents in the short-circuited coils tend to oppose the growth of magnetic flux under these pole tips, and hence it rises to a maximum first in the unwound tips, and then in the wound tips, causing the effect of a two-phase winding.

A more satisfactory method is to start the motor as a repulsion motor,* but this requires a commutator and more expensive mechanism. When this plan is adopted, the armature is wound with an ordinary reëntrant direct-current winding connected in the usual manner to a commutator, and it is self-starting with large torque when a short-circuiting connection joins the brushes and the brushes are displaced from the neutral plane. When the armature has come to full speed, an automatic device may be used to short-circuit the commutator, and the machine will then run as an induction motor.

202. Efficiency of Induction Motors and Methods of making Tests. — Polyphase induction motors can be built to give about the same efficiency as direct-current motors, and for somewhat less cost on account of the absence of a commutator and the low insulation required on the armature conductors, but with a counterbalancing extra cost on account of the high grade and

* Art. 205.

expensive sheet-iron stampings which are required for the field magnet. The most satisfactory design, as already explained, calls for a machine of short axial length and large diameter in order that sufficient teeth may be used to reduce magnetic leakage to small proportions. This is well illustrated in Fig. 487, which is from the photograph of the armature of a 300 horse-power, 25-cycle, 10-pole motor of standard design, having a speed of 300 revolutions per minute. The length of the core is quite small compared to its diameter.

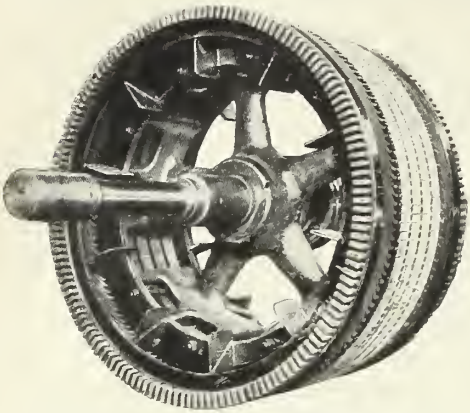


FIG. 487. — Illustration of a Coil-wound Induction Motor Armature, showing a Short Core of Large Diameter.

1. By far the quickest method of experimentally obtaining the operating characteristics of an induction motor, including efficiency, torque, regulation, and power factor, is by means of the circle or locus diagram.* The no-load exciting current can be obtained in phase position and scalar value by means of amperemeters, voltmeters, and wattmeters when the armature is running free. The current locus is fixed with approximate accuracy by using the same instruments to determine the phase position and quantity of the standstill current, using a reduced voltage for the measurements and considering the current to be proportional to voltage (or by obtaining it more accurately from the reactances at full load). The vector difference between the standstill and the no-load current is approximately the armature standstill current. The length and position of the vector of the armature standstill current determines the current locus. The resistance of the primary circuit may be measured, and that of the secondary circuit may be obtained by dividing the primary induced voltage (reduced to the secondary circuit by the ratio of transformation) by the product of the computed armature standstill current and the cosine of its angle of lag.

* Arts. 191, 192.

Correction can be made for variations in the exciting current if great accuracy is desired. Having thus laid out the locus, the desired operating quantities may be read off for any load.

2. *Direct Measurement.* In testing the efficiency of these motors, the output may be measured by a brake or transmission dynamometer, but the input must be measured by one of the wattmeter methods explained earlier. The two-wattmeter method is the best, but care must be taken to determine whether the readings of the two instruments are additive or subtractive, since the power factor of a partially loaded induction motor is likely to be quite low and at no load may be only a few per cent. The power factor is determined by taking simultaneous readings of amperemeter, voltmeter, and wattmeter in one circuit, if the machine is balanced; but if the circuits differ, readings for each circuit must be taken. The power factor is then the true watts divided by the apparent watts. This method requires that the motor shall be operated with its full load if the full load efficiency is to be obtained, and therefore may prove inconvenient, and it does not give any way of separating the losses.

3. *Stray Power Method.* — A method similar to that described for testing transformers (Art. 147) is often more convenient and satisfactory. By this plan the core losses, friction, and windage losses are determined by measuring by wattmeter the power which is absorbed by the motor when running light under normal voltage and frequency. The losses may be of such value that the field current flowing is considerable, especially as the power factor is likely to be rather low, and the I^2R loss cannot be neglected, but a correction can be made after the test for copper losses is completed. To measure the copper losses, the machine is locked so as to remain stationary, in which case the armature serves the purpose of a short-circuited secondary circuit, and such a reduced impressed voltage is applied as to cause any desired current to flow in the field winding. The wattmeter readings give the I^2R losses for the current flowing, and the losses for any other current may be at once calculated. A certain amount of core loss is included in this measurement, but an approximate correction may be made on account of it by considering its ratio to the total corrected core losses as the 1.6 power of the voltage applied in the copper loss test is to the

1.6 power of the normal voltage. From these results the total losses and the efficiency at any load may be calculated.

A motor running, without load, or with part load, on an unbalanced circuit, is likely to absorb widely different amounts of power in its coils; one coil may even return power to the circuit, while the others absorb the power required for operation plus that returned. In all such cases, the two-wattmeter method of measuring the power gives the net power absorbed by the machine.

4. *Power Factor.* — The power factor at any load may also be calculated from the circle diagram or it may be obtained for various loads by means of computations made from the readings of amperemeters, voltmeters, and wattmeters.

5. *Regulation and Torque.* — The exact regulation of a machine may be determined by actual running tests under load, in which the actual slip is measured, but the percentage slip may be taken to be approximately equal to the percentage drop of voltage in the armature windings. The starting torque can be measured by clamping a lever upon the pulley, and measuring the pull at the end of a fixed length of arm. For a machine having an improperly divided starting rheostat associated with the armature, this gives a value which is higher than the torque against which the motor will start and run up to speed. In such a machine, the *standing torque* and *starting torque* are different; but in a machine having a properly arranged starting rheostat associated with the armature, the maximum standing torque and the maximum starting torque are equal, and are practically equal to the maximum torque which the machine can exert. The torque and slip are, of course, obtainable from the circle diagram. All desired information in regard to the operation of induction motors may be determined by purely electrical measurements, and to a high degree of accuracy for commercial measurements. Using commercial amperemeters, voltmeters, and wattmeters which have been properly calibrated, the errors probably affecting the full load efficiency and power factor need not exceed one per cent.

The operating characteristics of a 50 horse power, four-pole, tri-phase, 60-cycle, 440-volt motor are given in Fig. 488. This shows a drop in speed of about 5 per cent from no load to full load, while the efficiency rises to about 88 per cent at slightly

over one half load. It is noted that the power factor drops rapidly for loads smaller than about one half load. In large machines the efficiency sometimes exceeds 95 per cent, while the

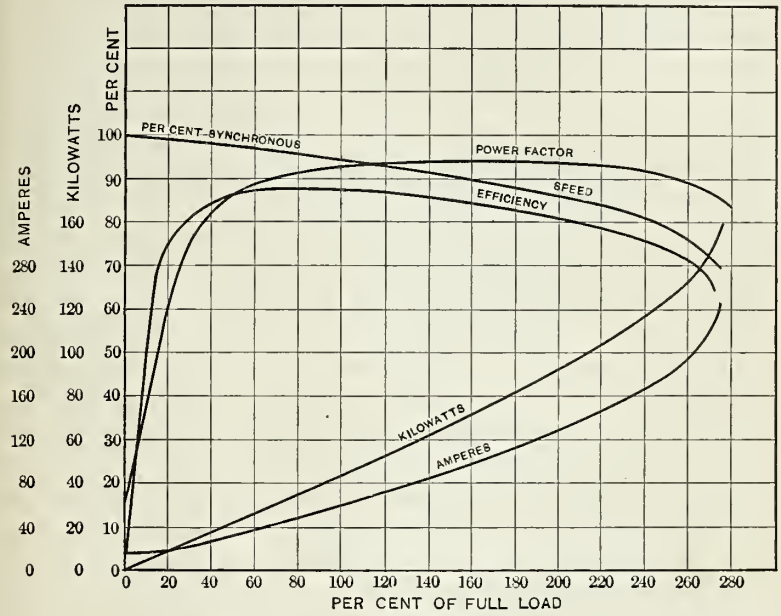


FIG. 488. — Operating Characteristics of a Tri-phase 50 H. P. Induction Motor.

slip is from 2 to 4 per cent at full load. Figure 489 shows the primary current per phase and the torque of a 5 horse power, four-pole, 60-cycle motor, in per cent of their normal values at full load, as functions of the armature speed. The figure therefore illustrates the conditions during starting, and especially the change in operating characteristics caused by change of voltage impressed upon the coils of a phase, obtained by changing the connections of the field coils from wye to delta.

Figure 490 shows the curves of torque as a function of speed and of current when a 25-cycle induction motor is operated from various taps of an autotransformer starter. It is to be noted that the starting current is reduced at serious expense of starting torque, but that a starting torque equal to full load running torque may be obtained with a current approximating to full load current. The various taps of the autotransformer give respectively 60, 70, 80, and 100 per cent of the full voltage, and

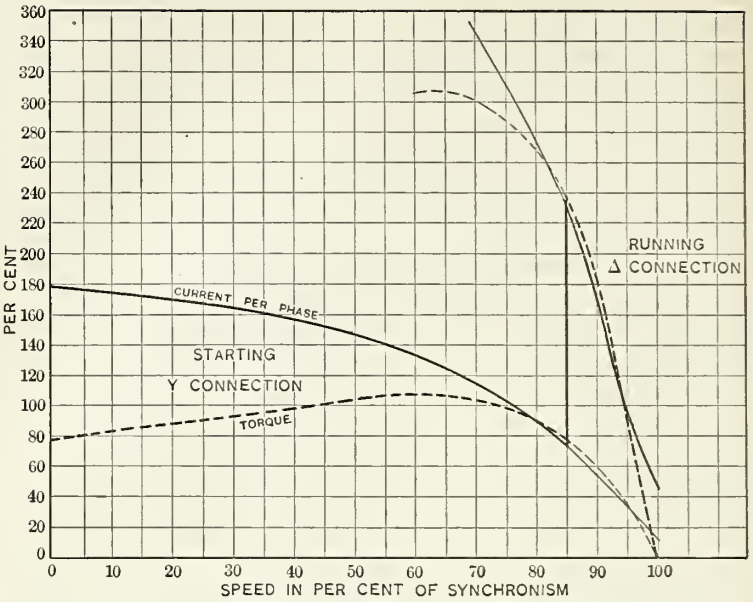


FIG. 489. — Curves of Primary Current and Torque of a Small Tri-phase Induction Motor, with the Primary Coils connected in Wye and in Delta.

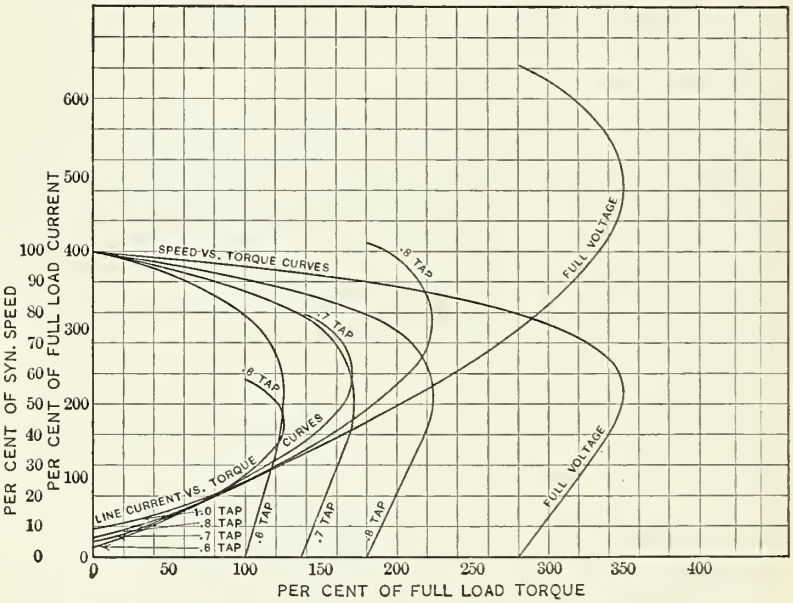


FIG. 490. — Torque Speed and Torque Current Curves of a 25-Cycle, Four-pole Induction Motor when started and run on various Taps of an Autotransformer.

the starting torques corresponding to the respective taps are therefore in ratio with each other as 36, 49, 64, and 100, that is, as the squares of the respective voltages. The maximum torques for the several voltages, which arise at 55 per cent of synchronous speed in this machine, are in the same ratio, as also are the torques corresponding to the several voltages at any given speed.

203. Effect of Frequency. — An examination of the formulas relating to the design of induction motors shows that the frequency of the current for which a machine is designed does not greatly affect its efficiency, slip, power factor, or starting torque, but that for a given speed the number of poles selected must be directly related to the frequency. Increasing the number of poles of a given machine reduces the cross section of each pole, but the number of lines of force at each pole is equally reduced, so that the magnetizing current is unaltered. Consequently, induction motors of equal merit may be designed for all reasonable frequencies, though magnetic leakage may interfere with the operation when the poles become too numerous.

On the other hand, when a machine which has been designed for a certain frequency is operated at another frequency, the synchronous speed is changed in direct proportion to the frequency, the percentage slip is practically unaltered, the starting torque varies inversely from the frequency, and the efficiency and power factor both vary in the same direction as the frequency because the magnetic density is inversely proportional to the frequency, as in transformers.

The frequencies which are commonly used with induction motors cover a wide range. In Europe, 50 periods per second is frequently adopted for tri-phase and single-phase motors; while in this country the frequencies of 60 periods per second and 25 periods per second have been generally adopted. The former is equally suitable for lighting and for stationary power purposes. The latter was early adopted at Niagara Falls and is used for plants where power service is of greater importance than the lighting service. The ratio of these two frequencies being 12 : 5 makes frequency changing inconvenient by limiting the numbers of pairs of poles practicable for use on the motor and generator of the frequency changer, within a reasonable range of speeds. The minimum number of poles available for use in

the frequency changing between these frequencies is 5 pairs on the 25-cycle machine and 12 pairs on the 60-cycle machine, giving a speed of 300 revolutions per minute. The next practicable combination is 10 pairs of poles and 24 pairs of poles, giving a speed of only 150 revolutions per minute, and therefore making an unduly expensive machine. If the lower frequency is reduced to 24 periods per second while the other is maintained at 60 periods, rather more flexibility is obtained, as speeds of 720, 360, 240, 180, etc., are then available for frequency changers. Nevertheless it would be of distinct advantage to electrical engineering in this country if the two commonly used frequencies were standardized at values more readily interconvertible. This would be accomplished if the lower frequency were standardized at 30 periods per second.*

The frequency of 40 periods per second was adopted in certain of the earlier power transmission plants in this country, and still persists in a few. In most instances the frequencies other than 60 periods per second and 25 periods per second have disappeared in this country.

204. Polyphase Induction Motor with Exciting Current supplied to the Armature. Motor with Unity Power Factor. — If a polyphase induction motor has its armature coils wound as a continuous reëntrant winding, and has direct current supplied to the coils through slip rings, so as to form as many magnetic poles as there are poles in the rotating field of the field magnet, the machine becomes a synchronous motor, and the lagging current of the primary windings may be neutralized when the machine is running in synchronism, or a leading current may be drawn from the alternating-current supply mains.†

Now assume the simple case of a two-pole tri-phase induction motor having a continuous reëntrant winding tapped to a commutator. Suppose that three brushes bear upon the commutator, spaced 120° apart, as in Fig. 491, where B is the armature winding, C the commutator, and d, d, d the brushes. Then, when the armature is at rest, a rotating field may be created by current in its windings by attaching the brushes to three-phase supply mains. If the connections are properly made and the brushes are set at the proper positions, this rotating field may be given the same direction of rotation and space phase as the rotating

* Art. 183. † Art. 168.

field set up by windings on the field magnet. If the two magnetic fields are superposed and the direction of their fluxes is the same, the exciting current in the field winding decreases, since the total flux required to link the field winding is limited by the amount necessary to set up a counter-voltage in the primary winding, which is equal to the vector difference of the impressed voltage and the *IR* drop in the winding. If the current introduced through the brushes is increased by raising the voltage impressed on the brush circuit, the exciting

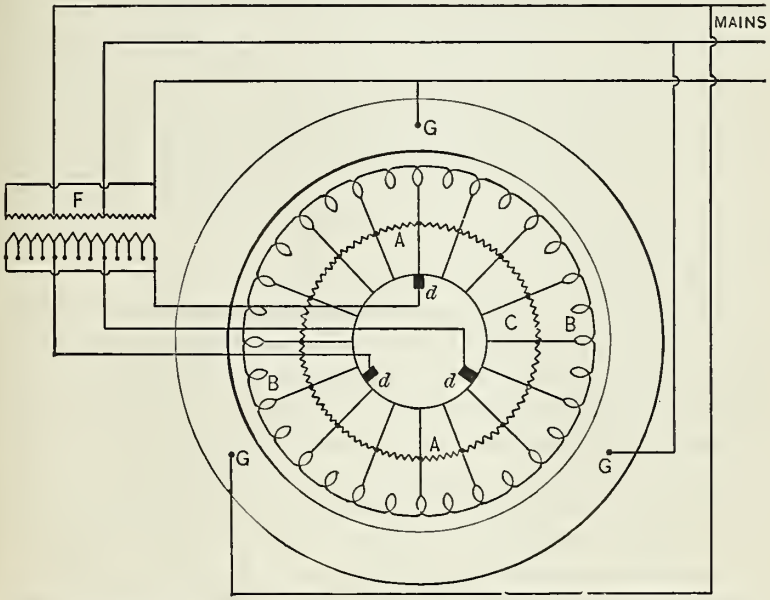


Fig. 491. — Diagram of a Polyphase Induction Motor capable of High Power Factor.

current in the primary winding can be entirely replaced, and further increases of current through the brushes tend to call for a neutralizing leading current in the primary winding.

Since rotation of the armature does not affect the rotation of the primary field, and the commutator maintains a uniform relation of the magnetic axes of the armature winding with respect to the brushes, the same time and space relations between the two magnetic fields exist, when the armature is running at any speed, as exist when the armature is standing still. That is, whatever may be the speed of the armature, its wind-

ing continues to be tapped through the medium of the brushes and commutator at three fixed points in space. Therefore, since the windings spanning between the brushes, in the three segments X, Y, and Z, are at all times unchanged, except directly under the brushes as commutation goes on, the strength and rotation of the field set up by the introduction of the triphase currents through the brushes is practically unchanged by changes in the speed of the armature. When the brushes are placed directly under the field taps G, G, G, as shown in Fig. 491, exciting current furnished through the armature will tend to lag, and no advantage will be obtained in increased power factor; but by shifting the brushes slightly, and overexciting through the armature brushes, the counter-voltage of the field windings may be made to take such a position with reference to the impressed voltage that a leading current will be drawn from the mains sufficient to neutralize the lagging current to the armature, or even cause a resultant leading current in the supply mains.

By joining the commutator bars by resistance A, the motor may be run as an ordinary short-circuited armature induction motor, while the exciting current is being supplied through the brushes d, d, d. Evidently in this case part of the current entering through the brushes is wasted in the parallel path through A, but by making the length of armature coil between the commutators sufficiently small, the resistance of A between brushes can be made great enough to reduce the current through it to a reasonably small proportion. This gives a motor with good starting torque and good regulation, and also affords opportunity for control of the power factor. Various modified arrangements for accomplishing the purpose may be used; thus, two separate windings may be placed in the armature slots, one to serve purely in the capacity of a commutated field magnetizing winding, and the other a short-circuited winding, which makes unnecessary the use of resistance A shown in the figure. The plan is equally applicable to single-phase and polyphase machines.

After the motor has been started in this way upon the high-starting torque and power factor, the brushes can be raised and the commutator can be short-circuited, thus reducing the machine to an ordinary induction motor having all of the advan-

tages of low armature resistance, or it may be allowed to run as a self-regulating motor of high power factor. In order to start the motor through the armature, the requisite exciting current can be obtained by connecting the brushes d, d, d to the supply mains through the medium of an autotransformer, or transformer having a suitable number of voltage regulation taps, as shown at F, Fig. 491. The effect of such lagging current as flows to the transformer F can be neutralized by properly placing the brushes.

205. Electromagnetic Repulsion and Repulsion Motors. — If a coil of wire is held in an alternating magnetic field in such a way that the lines of force pass through its turns, an alternating voltage is set up in it which has 90° difference of phase from the alternating magnetic flux. This in turn causes a current in the coil, and the coil experiences a force at each instant tending to move it in the magnetic field, which is proportional in magnitude and direction to the product of the corresponding instantaneous values of current and magnetic flux, paying due attention to their relative algebraic signs; and the average force for a cycle of flux is equal to the average of the instantaneous torques during the period. If the coil could have no self-inductance, and the phase of the current could therefore be in quadrature with that of the magnetic flux, the average forces during alternate quarter periods would be equal but in opposite directions (compare Fig. 199), and the average force during a whole period would be zero, so that the coil would have no tendency to move; but in all practical cases a coil, or even a flat disk, must have some self-inductance, so that the current lags behind the impressed voltage, and the current phase is therefore more than 90° behind the phase of the magnetism. In this case the instantaneous values of the force, when plotted in a curve, give a figure similar to the dotted curve in Fig. 200. The ordinates of the large loops represent a negative or repulsive force, and the ordinates of the small loops a positive or attractive force, and the summation of the instantaneous forces during a period has a finite negative value. This shows that the coil experiences a repulsive force which tends to move it out of the magnetic field. If the coil is pivoted, the force tends to turn it into such a position that the lines of force of the field do not thread through its turns. The conditions here set forth

were first fully explained and illustrated in a remarkable lecture by Professor Elihu Thomson.*

If an armature is wound with uniformly spaced short-circuited coils or conductors, the repulsive effects in the different coils will balance each other when the armature stands still ; but if the coils have their independent ends separately connected to the opposite bars of a commutator having as many bars as there are sets of conductors in the armature, brushes may be so arranged as to short-circuit each coil when it is in a position to give a force in one direction. This arrangement was suggested by Professor Thomson,† and is illustrated in Fig. 492, and represents the first type of repulsion motor. The motor is self-starting, and runs by virtue of the repulsion between the magnetic field and the coils, which, as they come into the active position, are short-circuited by the brush connections. Such a motor is bulky, inefficient, and expensive, since only a portion of the armature can be made continuously effective.

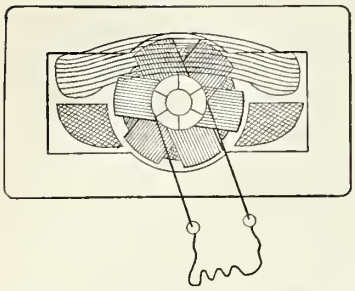


FIG. 492. — Diagram for Showing the Principle of a Repulsion Motor with Open Circuit Armature Coils.

A better method of arranging a repulsion motor is to use an armature with a re-entrant winding and commutator like a direct-current armature, the field magnet being laminated, as originally developed by Anthony, Jackson, and Ryan. The arrangement is as illustrated in Fig. 493. In this figure the lead wire *cc* makes a short-circuiting connection between the brushes *b*, *b'* which rest on the commutator to which the armature winding is attached. The brushes are usually set a little less than 45° from the neutral plane. Then the alternating poles *N*–*S* induce, by transformer action, currents in the armature winding which is short-circuited through the brushes and conductor *cc*. The voltage induced in the armature winding is zero when measured between commutator bars lying on a diameter perpendicular to the magnetic axis of the inducing flux, and is a maximum when measured between commutator bars on a diameter parallel

* Novel Phenomena of Alternating Currents, *Trans. Amer. Inst. E. E.*, Vol. 4, p. 160.

† *Trans. Amer. Inst. E. E.*, Vol. 4, p. 160.

to the flux. A large current flows through conductor *cc* when the brushes are placed in the latter position, but the currents flow in the armature conductors so that they neutralize each other's torque. When the brushes are placed in the position at right angles, no current flows and therefore no torque is produced. Consequently, the brushes are placed in a compromise position, giving a considerable current flow accompanied by the production of a considerable torque.

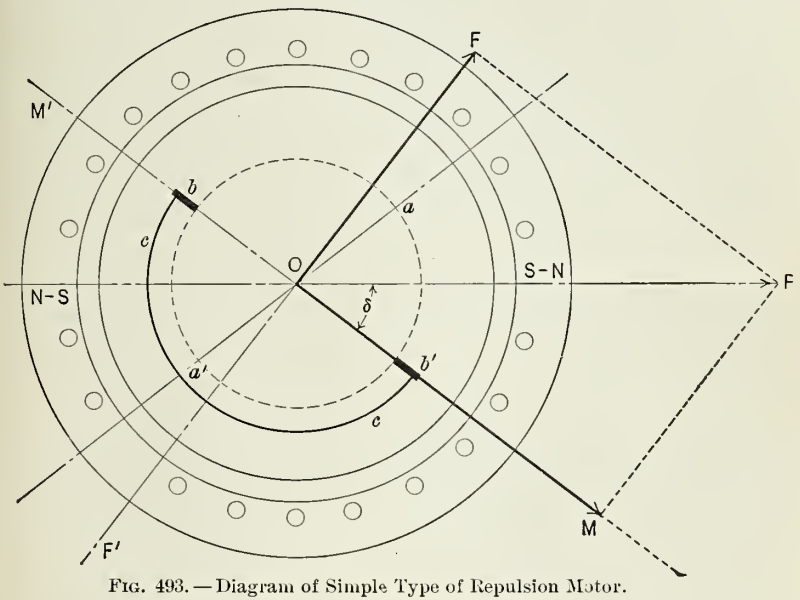


FIG. 493. — Diagram of Simple Type of Repulsion Motor.

A convenient method of determining the action in the repulsion motor is to resolve the magneto-motive force OP of the field magnet into two quadrature components, one of which, OM, is in line with the short-circuited brushes, and the other, OF, at right angles thereto. The field windings are wound through the holes or slots in the outer ring of Fig. 493, and produce the magneto-motive force OP, proportioned to their ampere-turns. Now, since the vector expression $\overline{OP} = \overline{OF} + \overline{OM}$ holds true, we can replace these field windings by two sets in space quadrature, but connected electrically in series, such that one set, which we shall call the *torque* or F coils, creates the magneto-motive force OF, and the other, which we shall call the *mutual* or M coils, creates the magneto-motive force OM.

Assume first an ideal motor in which the F and M coils and the armature winding are of negligible resistance. Remember that the armature winding is of the closed coil variety and is short-circuited between the brushes by wire cc. Assume also that the reluctance of the magnetic circuit is uniform and the magnetic leakage zero. If the armature is stationary and an alternating voltage E is impressed across the extremities of the F and M coil circuit, causing a current I to flow through the circuit, an alternating magneto-motive force is set up along the line MM', and another along the line FF' by reason of the M and F windings respectively. The flux from the M windings or poles induces in the armature windings bands of currents which complete their circuit from b through c to b'. The armature windings having negligible resistance and leakage inductance by assumption, this armature current I_M flows with negligible voltage drop. The value of this current is $I_M = \frac{1}{s''} I$, where s'' is the ratio of transformation of the transformer thus formed between the M coils and the armature windings or A coils, or $s'' = \frac{n_A}{n_M}$, where n_A and n_M are the equivalent numbers of turns in the A and M coils respectively. The resistance of the M coils being also assumed to be negligible, I_M in the A coils acts to destroy the mutual reactance of the M coils, like the current in the secondary circuit of any transformer; then, there being no induced voltage drop in the A coils, the induced voltage in the M coils is zero, which means that the magneto-motive force of the M coils has been counterbalanced and neutralized by that of the A coils. On the other hand, the magneto-motive force caused by the current I in the F coils does not set up current in the armature or A coils, because the brushes b and b' rest on neutral points with reference to the voltages induced by reason of the magnetic flux along FF' caused by this F magneto-motive force. Hence, there must be an induced counter-voltage in the F coils. This is $E_F = -IX_F$, where I is the line current and X_F the reactance of the F coils. The impressed voltage E is thus all used in drop through the F coils, in the ideal motor, there being no drop in the M coils, the other part of the complete field circuit, since their reactance is destroyed by the mutual effect of the A coils, and all resistance

is considered negligible. The voltage E_M, across the M coils, cannot be absolutely equal to zero, or E_F be equal to $-IX_F$, in an actual machine, since neither resistance nor leakage reactance can be entirely eliminated, but for our purposes of discussion the assumption of an ideal motor is warranted.

The foregoing two effects are combined in the armature and an alternating magnetic flux Φ_F, emanates from the field magnet on the line FF', and heavy sheets of alternating currents I_M flow through the armature coils and the conductor cc between b and b'. Moreover, the time phases of Φ_F and I_M are in opposition, since I_M must be in phase opposition to the current I in the inducing coils to accord with the laws of the transformer if losses and magnetic leakage are negligible.* Therefore, the armature current I_M and field or F flux Φ_F are in phase relation and space position to exert a torque upon one another. Now suppose that the armature rotates under the influence of this torque. Then the two sheets of conductors lying on the armature between b and b' will generate a voltage E_c between those points, due to cutting the flux Φ_F, so that $E_c = kS\Phi_F$, where k is a constant, and S is the speed of rotation in per cent of the frequency f. But Φ_F is in phase with I. Hence E_c and I are in the same time phase and E_c is in phase opposition with I_M. The voltage E_c is alternating, because its instantaneous values are proportional to ϕ_F at each instant, but it is also proportional to the speed of the armature. The current component I_c flowing through the armature under the influence of E_c creates a magnetic flux Φ_c which passes through the M coils and sets up in them a voltage E_M lagging 90° behind it and proportional to its rate of change. But $I_c = \frac{E_c}{X_M}$, when X_M is the reactance (mutual) offered on account of the flux Φ_c, and, therefore, since the circuit is assumed to be purely reactive, I_c lags 90° behind E_c, and hence is 270° behind or 90° ahead of I_M. This makes E_M ninety degrees behind $E_F = -IX_F$ in time phase. But since E_F is 90° behind I, E_M must be in phase opposition to I and the vector product of the two must represent real power.

We now have two induced voltages E_M and E_F in quadrature which cause the drop E in the field circuit or

$$E = -E' = -\sqrt{E_M^2 + E_F^2}.$$

* Chap. X.

Likewise, we have two quadrature components of current in the armature I_M and I_c, which are flowing in what is the equivalent of a transformer secondary to the M windings, and the current through the brushes must be equal to their resultant, or,

$$I_A = \sqrt{I_M^2 + I_c^2}.$$

The stator current is

$$I = -\frac{E_F}{X_F} = \frac{E \cos \lambda}{X_F},$$

where λ is of such value that $\tan \lambda = \frac{E_M}{E_F}$. Evidently E_F, E_M, and I can be expressed graphically by means of locus diagrams. Thus, let OE, Fig. 494, be the impressed voltage. Then since

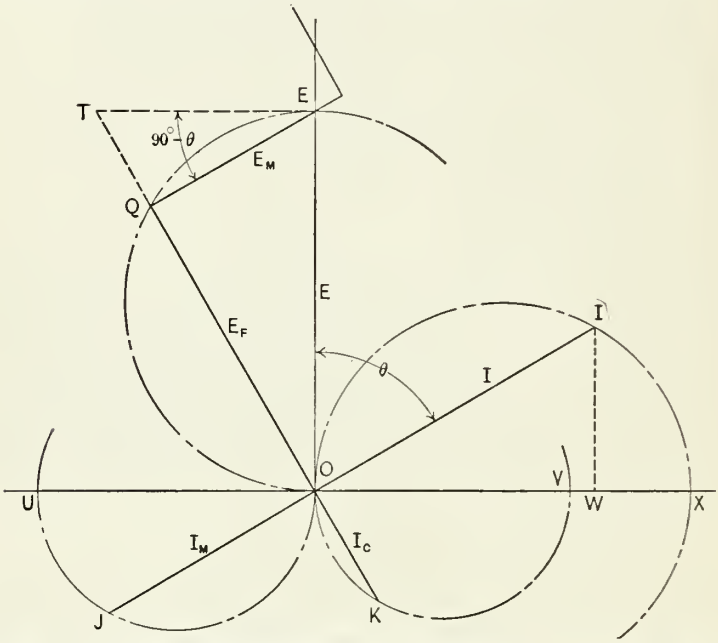


FIG. 494. — Vector Diagram of Relations in a Repulsion Motor having Negligible Resistance and Leakage Reactance.

the induced voltages E_M and E_F are in quadrature and their vector sum equals $-OE$, they may be represented respectively by the vectors EQ and QO. The point Q travels from E at standstill, when E_M is zero, to O, when E_F is zero, along the semicircle OQE The current, $\bar{I}$, in the stator is at right angles to

$OQ = -\bar{E}_F$, under the conditions assumed. It may therefore be represented by the line OI for the particular value of $\bar{E}_F$ shown in the figure. The point I travels from X when the armature is at rest and the current $\bar{I}$ has the value $\frac{E}{X_F}$, to O when infinite speed has been reached and $\bar{E}_F$ is zero. The secondary current is composed of the two components I_M and I_c. As has been explained, $\bar{I}_M$ sets up a magneto-motive force equal and opposite to that of $\bar{I}$ in the M coils. It may therefore be represented by the line OJ. The point J travels along the circular locus OJU with change in load, but always takes such a position that JOI is a straight line. The scalar value of I_M is equal to $I\frac{s}{s'}$, where s equals the ratio between the turns in the M field coils and the F field coils, or $\frac{n_M}{n_F}$, and s' is the ratio between the effective armature turns and the F turns, or $\frac{n_A}{n_F}$. This is evident, since $\bar{I}$ and the mutually induced armature current I_M must have the ordinary transformer relation $\frac{I_M}{I} = \frac{n_M}{n_A} = \frac{s}{s'} = \frac{1}{s''}$. This ratio is constant so long as the brushes are not moved, so that if I traces a semicircle, J must follow a similar locus. The diameter OU equals $OX\frac{s}{s'}$. The second component of armature current $\bar{I}_c$, as can be gleaned from the preceding discussion, is always in phase with $\bar{E}_F$ and at right angles to E_c. It is also proportional to $\bar{I}$ and the armature speed. Since $I_c = \frac{E_c}{X_A}$, $I = -\frac{E_F}{X_F}$, and the reactances are proportional to the square of the numbers of turns in which the voltages E_c and E_F are induced, then $\frac{I_c}{I} = \frac{E_c n_F^2}{E_F n_A^2}$, and, as will be shown later, $E_c = E_F s' S$; hence $I_c = I\frac{S}{s'}$, S being the armature speed in per cent of the impressed frequency. Being proportional to and set up by $\bar{E}_c$, I_c may be represented by the line OK. The point K travels the semicircle OKV, which has a diameter OV equal to $\frac{E_c}{X_A}$, and reaches V at infinite speed of the armature. The resultant of OJ and

OK is the total armature current, and it also, as seen by the geometrical construction, traces a circular locus having a diameter UV.

The power input is $OI \times OE \cos EOI = IE \cos\theta = E_M I$ and may be represented by IW in the figure. The torque, under the assumption made, is $\frac{E_M I}{S}$. Since the generated voltage E_M is proportional to the product $S\Phi_F$, and Φ_F is the maximum value of the flux from the F coils, and since Φ_F is proportional to E_F, we have $S \propto \frac{E_M}{E_F} \propto \tan(90°-\theta) \propto \frac{\text{Power factor}}{\cos(90°-\theta)}$. It may, therefore, be represented by the line TE. The output under the assumption of negligible losses is equal to the input. The angle of lag θ equals $\cos^{-1}\frac{E_M}{E_F} = \tan^{-1}\frac{E_F}{E_M}$.

The relation of the voltages in the two sets of imaginary field coils $E_M = E_F Ss$ may be found in this manner: $\frac{E_c}{E_F} = \frac{n_A}{n_F} S = s'S$, and $\frac{E_c}{E_M} = \frac{n_A}{n_M} = \frac{s'}{s} = s''$, from which $E_M = E_F Ss$. Substituting this in the formula $E = -\sqrt{E_M^2 + E_F^2}$ gives $E_F = -\frac{E}{\sqrt{1+S^2s^2}}$. Also $I = -\frac{E_F}{X_F} = \frac{E}{X_F\sqrt{1+S^2s^2}}$. Substituting the values of I_c and I_M, obtained in terms of I, S, s, and s', in the formula $I_A = \sqrt{I_c^2 + I_M^2}$ gives $I_A = \frac{I}{s'}\sqrt{s^2+S^2}$, and therefore

$$I_A = \frac{E_F}{X_F s'}\sqrt{s^2+S^2} = \frac{E\sqrt{s^2+S^2}}{X_F s'\sqrt{1+S^2s^2}}.$$

From the above formulas and the diagram much information concerning the operation of the repulsion motor may be obtained. If the speed is such that $S = 1$, which is sometimes called synchronous speed, since then 360 electrical degrees of the polar pitch is turned through by the motor armature during the time of one cycle of the impressed voltage, I_A reduces to $\frac{E}{s'X_F}$. When s is also made unity by placing the brushes 45° from the neutral plane, the primary current becomes $I = \frac{E}{\sqrt{2}X_F}$.

By completing the circular arcs in the locus diagram the loci are obtained for the machine driven backward, in which case it runs as a generator. The conditions are illustrated by the dotted halves of the circular loci of Fig. 494.

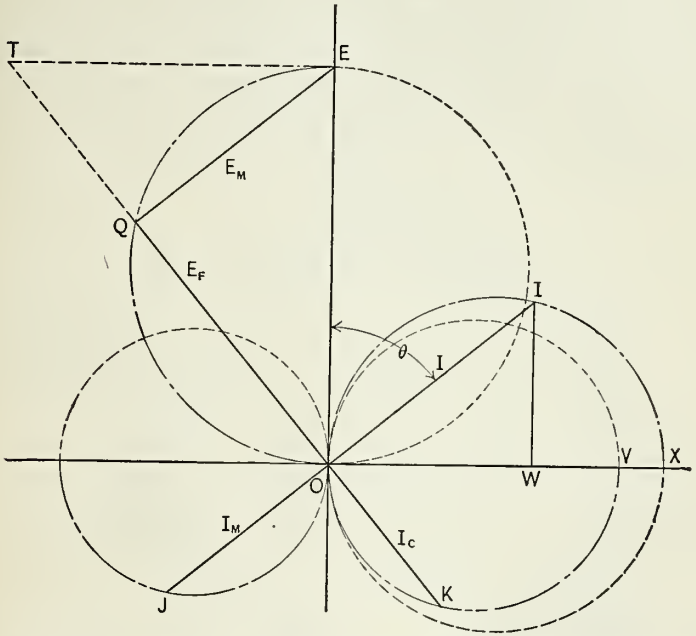


FIG. 495. — Vector Diagram of a Repulsion Machine as Generator and Motor when Winding Resistance and Reactance are Neglected.

Figure 496 shows diagrammatically the windings of a repulsion motor when the stator coils are actually divided into parts. In this case the brushes are directly under the windings marked "mutual coils." At standstill the magneto-motive forces of the armature and mutual field currents neutralize each other except for the drops required to overcome resistance and leakage reactance. The sheets of current in the armature react upon the unneutralized flux which is set up by the current in the "torque coils," causing the torque which results in rotation. As in the discussion of the simple repulsion motor, the voltage set up in the armature conductors by the rotation of the armature is (neglecting the iron loss angle) in phase with the torque flux and hence with the field current. The component of armature current caused by this flux lags nearly 90° behind this voltage,

and hence its flux entering the mutual coils sets up a voltage in those coils in quadrature with the voltage in the torque coils. An extra current must flow through the stator circuit to oppose the magneto-motive force of this new component of the armature current. At synchronous speed the two stator fields become equal, and beyond that speed the mutual field becomes the stronger, for which reason the repulsion motor commutates better below synchronous speed than above. The formulas and diagram given for the simple repulsion motor apply also to this modified type.

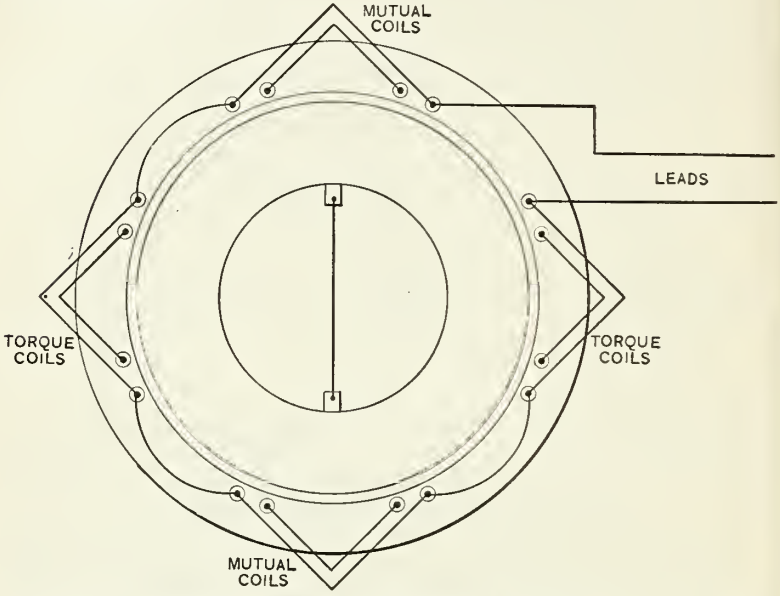


FIG. 496. — Conventional Diagram of a Repulsion Motor having the Stator Coils divided into Two Parts.

As explained, the usual forms of commercial repulsion motor have field cores like those of induction motors. In order then to prevent undue reactance in the armature circuit, caused by the low reluctance of the iron path, special arrangements may be used as shown in Fig. 497. In this case, in a two-pole machine, the pair of mutually induced poles are at M and M', while the torque poles excited by current in the armature are at F and F'. The two short-circuited brushes are directly under the points M and M' and collect the mutually induced currents in the armature, neutralizing the reactance in the field. The

two brushes under F and F' are connected in series with the field coils and the mains. The armature currents in the short circuit paths react on the torque poles F and F' and cause the armature to rotate. The series current through the rotor sets up the motor flux, which produces a torque by its action on the current in the stator coils. The short-circuited induced rotor

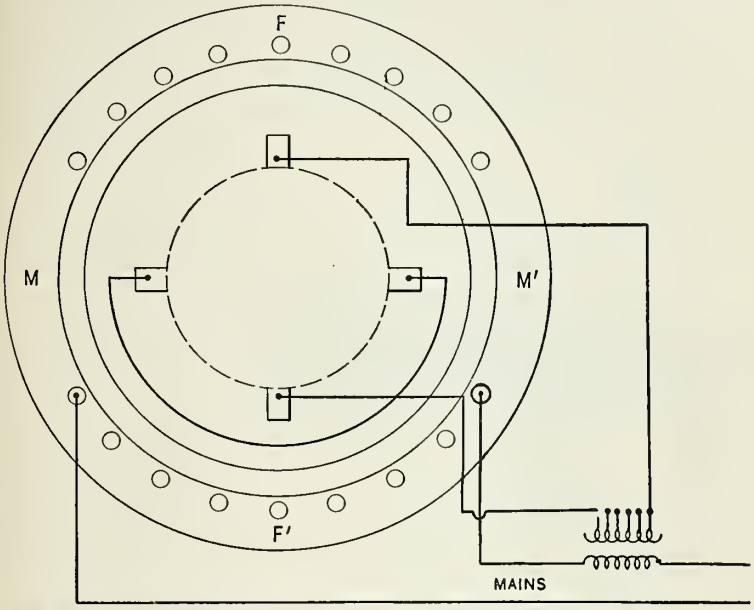


FIG. 497. — Compensated Repulsion Motor, Compensating Current in Armature.

current opposes its magneto-motive force to that of the current in the stator coils, reducing the reactance of the latter to a reasonable value. At starting, the stator circuit has very low impedance. When the machine is running, however, the top and bottom bands of the rotor conductors cut the flux set up vertically by the line current in the rotor. This creates a voltage in phase with the line current, which in turn sends, through the short-circuited rotor circuits, a current lagging nearly 90° behind the line current. The flux from this current passing into the stator at M or M' creates in the stator coils an induced voltage lagging approximately 90° behind its own phase, and hence in approximate opposition to the line current. Therefore, the stator coils have set up in them a counter-vol-

tage proportional to the speed of the rotor, which is similar in effect to the counter-voltage in the armature of a direct or alternating-current series motor. In this motor the excitation of the field magnet is furnished by currents in the armature conductors, which are at full line frequency when the rotor is standing still; at synchronism the frequency in each conductor is zero; and above line synchronism the conductor frequency again gains a value proportionate to the relative speed above synchronism of the rotor with respect to the frequency of the line current. As a result, at synchronism the power factor approaches unity, and above synchronism the motor may draw leading current from the line.

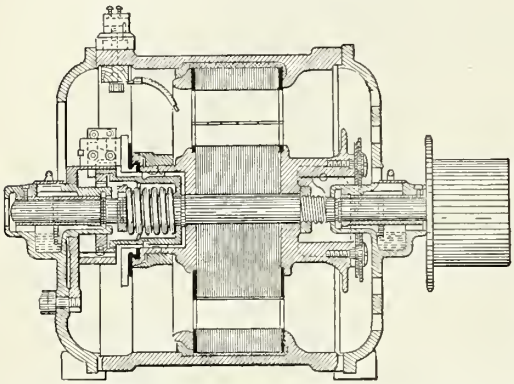


FIG. 498.— Cross Section of a Single-phase Motor constructed to start as a Repulsion Motor and run as an Induction Motor.

Figure 498 shows a cross section of a single-phase motor which starts as a repulsion motor and runs as an induction motor. In this case the brushes are movable, and when the motor comes to speed the commutator is short-circuited by a ring of copper and the brushes are raised from contact. The movements are accomplished by means of a centrifugal governor on the shaft within the armature spider.

Figures 499 and 500 show the normal performance of a 20 horse power, 60-cycle, six-pole motor of the type illustrated in Fig. 498. As the repulsion motor commutates best at speeds below the theoretical synchronous speed for its armature as an induction machine, this combination of repulsion motor starting and induction motor running gives an excellent form of single-phase machine.

206. Series Alternating Current Motors. — Since a direct-current series motor does not reverse its direction of rotation when the current is simultaneously reversed in its field and armature windings, it might be expected to run when supplied

with an alternating current. This is the case when the field and armature cores are sufficiently well laminated to avoid ex-

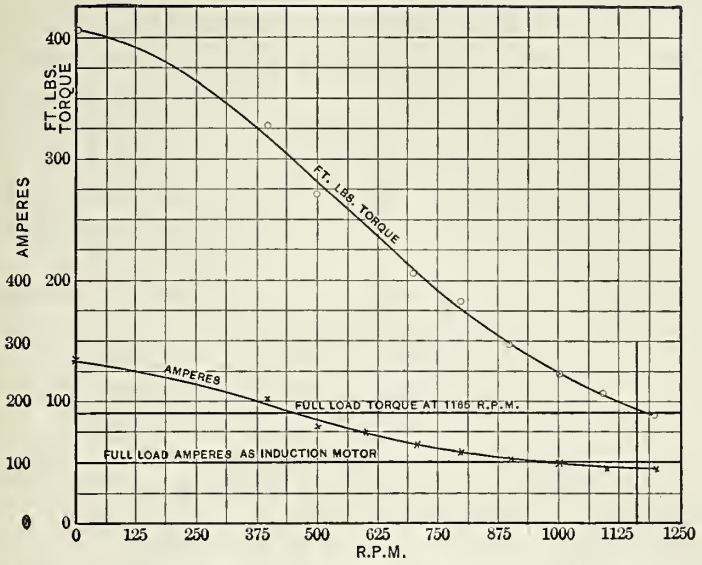


FIG. 499. — Torque and Current Curves of Single-phase Induction Motor arranged to start as a Repulsion Motor, such as is shown in Fig. 498.

cessive eddy currents and the electric and magnetic circuits are so designed as to keep self-inductive reactance down to reasonable limits.

In the case of the direct-current motor it is common to make the fields strong to prevent skewing of the flux, with resultant

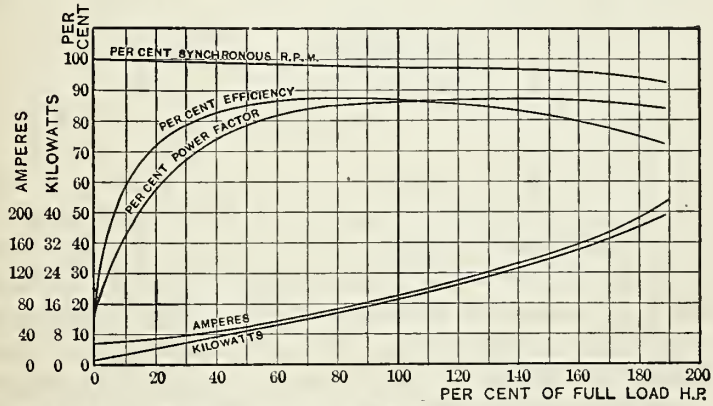


FIG. 500. — Performance Curves of Motor such as is shown in Fig. 498.

poor commutation, while the armature is made magnetically weak for the same purpose, and in order to reduce the self-inductance of the coils under commutation to a minimum. In the alternating-current series motor the field magnet is made magnetically weak to reduce its self-inductance to a minimum, while the armature is made relatively strong as a magnet. The self-inductance of the latter is reduced or overcome by means of **Compensating coils** as described below.

The general construction of the motor, as frequently used, is seen in Fig. 501. In this figure the field coils A are undistrib-

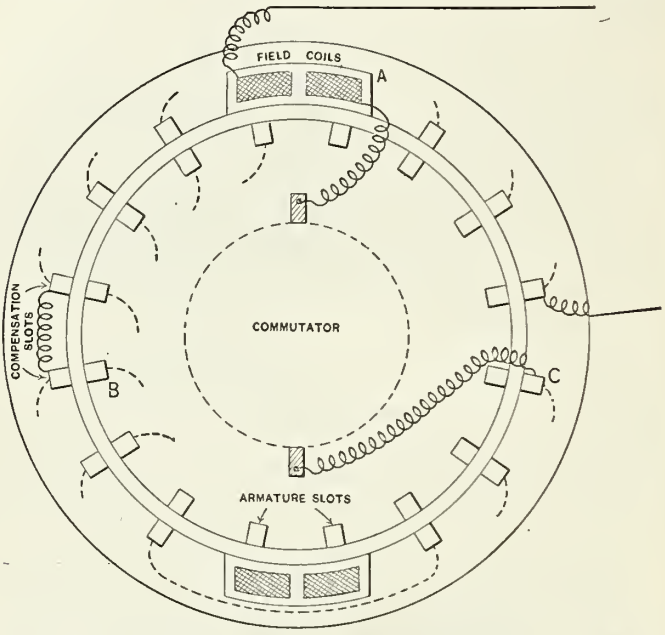


Fig. 501. — Diagrammatic Sketch of a Series Alternating Current Motor.

uted and surround the armature, the armature winding B is of the continuous closed circuit type, and the compensating coils C are distributed along the long pole faces and carry a current which opposes the magneto-motive force of the armature current. Sometimes the field windings are distributed, but this is more or less disadvantageous, as it interferes with the best use of the compensating windings. More often the ordinary polar projections are used with the compensating windings distributed over

the pole shoes. For mechanical convenience every other pole only need bear a field winding, the alternate ones merely carrying the magnetic flux.

The methods of connecting up the windings are shown diagrammatically in Figs. 502 to 507. In Fig. 502 is shown an **Inductively compensated** series motor. In this case the compensation winding distributed over the pole faces is short-circuited upon itself, and by the effect of mutual induction it has generated within it a short-circuited current having approximately the same magneto-motive force as that in the armature. In this way the armature self-inductance is in large part destroyed.

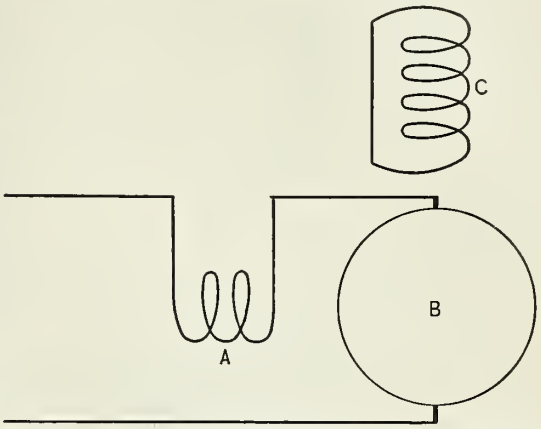


Fig. 502. — Diagram of an Inductively Compensated Series Motor.

The armature may be considered equivalent to the primary winding and the compensation winding to the secondary winding of a transformer, and the transformer diagrams previously given apply to this case.*

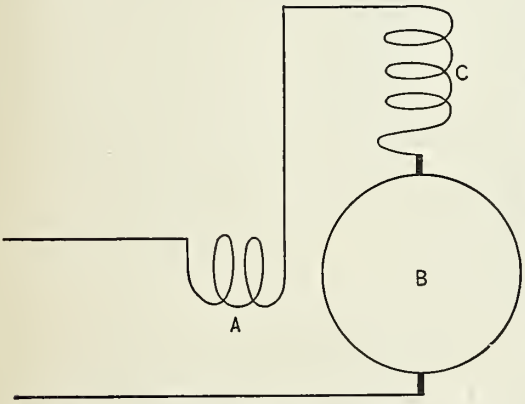


Fig. 503. — Conductively Compensated Series Motor.

Figure 503 shows the connections of a **Conductively compensated** series motor. Here the compensation winding C is in series with the field winding A and the armature B. By properly proportioning the turns in winding C the

* Art. 124.

motor may be **Overcompensated** or **Undercompensated**, *i.e.* C may be made magnetically stronger or weaker than B. It will be noted that less voltage will be available for the armature with this connection than when the inductively compensated arrangement is used.

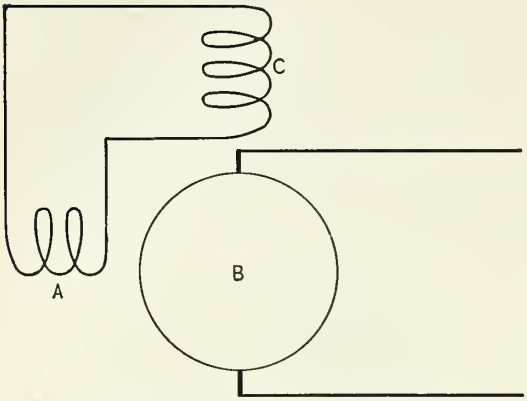


FIG. 504. — Series Motor with Inductively Connected Field and Compensation Windings.

Figure 504 shows the connections when the field and compensation windings are connected in series with each other and the requisite current is mutually induced in them by the armature acting as a transformer primary winding. In this case full line voltage is available across the armature brushes.

Figure 505 shows the field windings A in series with the compensation windings C arranged for electrical connection

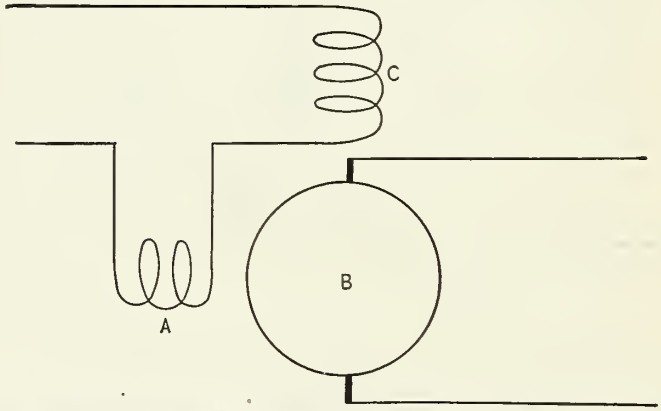


FIG. 505. — Series Repulsion Motor. Field and Compensation Windings Independent of the Armature Winding.

with the supply circuit independently of the armature windings B. With this arrangement the stator and rotor circuit voltages can be controlled independently of each other.

A motor so connected is sometimes called a **Series repulsion motor.**

An arrangement somewhat similar to the last is shown in Fig. 506, where the compensating coils alone are connected independently to the line, and hence the value of the compensation can be varied at will without interfering with the voltage applied to the armature and field windings, which are in series.

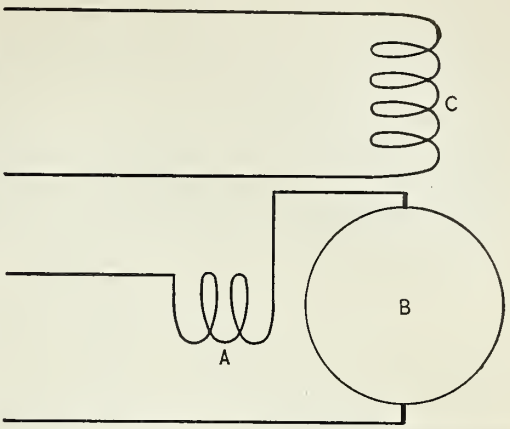


Fig. 506. — Series Repulsion Motor. The Compensation Winding is connected to the Line independently.

Figure 507 diagrams a true repulsion motor, similar to that shown in Fig. 496, except that the current of the stator coils setting up the torque flux is obtained directly from the armature.

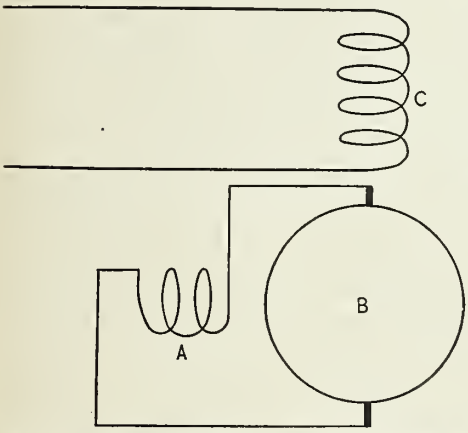


Fig. 507. — Repulsion Motor with Torque Coils in Series with Armature Circuit.

Figure 508 shows a conductively compensated series motor such as is illustrated in Fig. 503, except that a non-reactive shunt *D* is added around the field windings for the purpose of reducing the impedance of the field circuit.

It is evident that by bringing the terminals of the field winding *A*, the armature terminals *B*, and the terminals of the compensation winding *C* to a proper switching device a motor can readily be given any of the connections shown in Figs. 496, and 502 to 507. In this way a motor may be started with the connection that gives best commuta-

tion, power factor, and torque, and may then be converted to such other connections as will give the proper speed and best power factor and commutation for the speed and load which are to be maintained.

The compensation winding cannot absolutely neutralize the effect of the armature magneto-motive force because of the magnetic leakage around the conductors of both the armature and compensation coils, which is not mutual, but bridges across the core teeth. Nor will it fully oppose the armature magneto-motive force in the mutual circuit unless distributed entirely around the armature when the armature winding pitch is 180 electrical degrees. Thus, take the inductively connected compensation when the compensating current times the ratio of the compensating turns to the armature turns is approximately equal to the armature current, which gives equal and opposite magneto-motive forces for the two. Then the distribution of magneto-motive force in the compensating coils will be different from that in the armature if the former is wound upon an arc equal to the pole width while the latter covers 180 electrical degrees. If the armature winding has a pitch equal to the width of the pole face, the two magneto-motive forces can be made practically equal and opposite in their effects, and they therefore practically neutralize each other. Under ordinary construction, then, the armature and compensation winding self-inductance is not absolutely overcome, but it is reduced to reasonable values. When the motor is to be used alternatively with direct currents and alternating currents, conductive compensation should be used, since otherwise with the magnetically strong armature and weak field, excessive skewing of the field flux would occur with the direct currents, with resultant bad commutation and low maximum power.

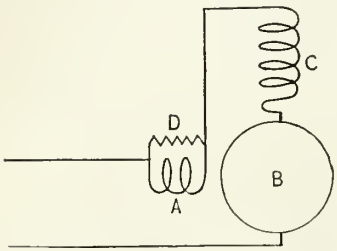


FIG. 508. — Compensated Series Motor with Non-reactive Shunt around Field Windings.

The alternating-current commutator motor if sufficiently small may be started by throwing it directly upon the line. In the case of heavy motors, such as are used in railway work, it is desirable to start by varying the supply voltage by means of

transformers or autotransformers having variable voltage taps brought out from the secondary windings. When necessary to improve the power factor, the field coil may be shunted by a non-inductive resistance as shown at D in Fig. 508.

207. Vector Diagrams of Alternating Current Series Motors, and Expressions for Voltage. — The vector diagram of the series motor may be constructed as in Fig. 509. Consider the conductively compensated type first. Here the current flowing is

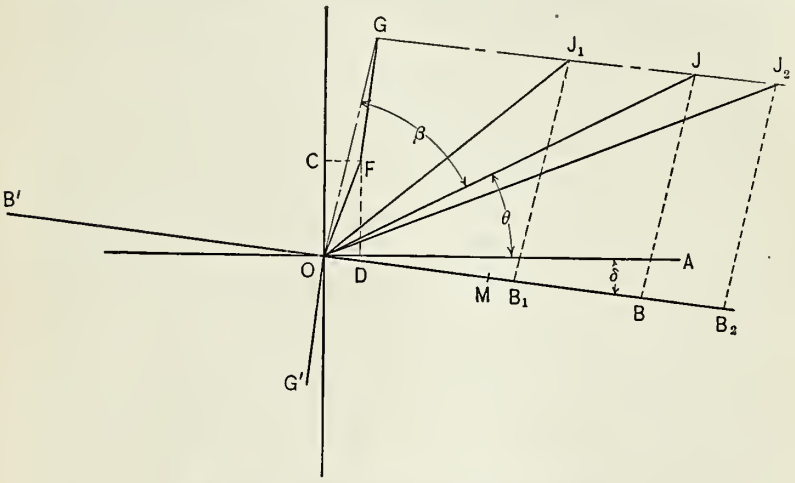


FIG. 509. — Vector Diagram of a Series Motor.

represented by OA. The magnetic field flux, represented by OM, lags behind this because of the iron loss angle and by reason of the fact that there is a considerable amount of true power consumed in the short-circuited armature coils lying under the brushes during commutation. The voltage drop in the leakage reactance of all the windings is represented by OC, which is in quadrature with OA, and in their combined resistance by OD which is in phase with OA. Thus OF may be considered as the voltage drop in the local impedance. The line FG represents the voltage drop equal and opposite to OG', the voltage generated in the field windings by the alternation of the field flux, $OM = \Phi$, through the field windings. The line OB is the drop of impressed voltage equal and opposite to the voltage OB' created by the armature conductors cutting the field flux Φ as the armature rotates. Voltage OB' must be in phase opposition

to the flux OM, as it tends to drive a current in opposition to the current caused to flow by the impressed voltage. Combining these three voltage drops OF, FG, and OB, gives OJ as the impressed voltage for the current OA when the speed is proportional to the voltage OB. The angle of lag is JOA, the power input is $OJ \times OA \times \cos\theta$, and the torque is proportional to $OM \times OA \times \cos\delta$.

Suppose now that the current OA, Fig. 509, is kept constant in value as the speed varies, a condition entirely possible in traction work. Then $\Phi = OM$ is constant; and hence the voltage OG is constant. Now, suppose the impressed voltage is reduced by means of a controller until the speed has lowered to such a value that the induced voltage caused by the armature rotation is $-OB_1$; then the impressed voltage is OJ_1, the angle of lag is increased, and the power input and output have decreased. The torque, however, which is proportional to the current OA times the flux $\Phi (= OM)$ times cos δ, has remained constant. At standstill the impressed voltage that maintains a current OA is voltage OG. By the construction it is seen that the straight line GJ_1JJ_2 extended to the right is the locus of the impressed voltage vector for constant current and torque but variable speed. If the speed increases beyond the original value, considered, for instance, so that the armature counter-voltage becomes $-OB_2$, the impressed voltage must become OJ_2 to maintain the same current, and the angle of lag is decreased and the input and output are increased. It will be observed that the speed is proportional to the voltage on the line OB.

By considering the vertical component of OG constant and the angle δ negligible in Fig. 509, a simple circular locus diagram can be constructed for the compensated series motor when operating at various loads (variable currents) under constant voltage. The counter-voltage E_c generated by the armature conductors cutting the field flux may be considered as replaceable by a variable resistance such that $R_c = \frac{E_c}{I}$. Then the current flowing through the motor is $I = \frac{E}{\sqrt{X^2 + (R + R_c)^2}}$, where X is the uncompensated reactance, R is the true resistance of the windings plus such additional amount as is required to be substituted for iron and commutator losses to give an equiv-

alent power loss, and R_e is the apparent resistance due to the counter-voltage. But the reactance is constant, while the resistance $(R + R_e)$ varies when the current, and hence the speed, and with it E_c, varies. Therefore, we have a case of a circuit containing a constant reactance and variable resistance.* Then lay off OE, the voltage E, Fig. 510, and at right angles with this lay off OA such that $OA = \frac{E}{X}$. Then the semicircle OQA represents the required locus. At standstill R_e is zero and the total

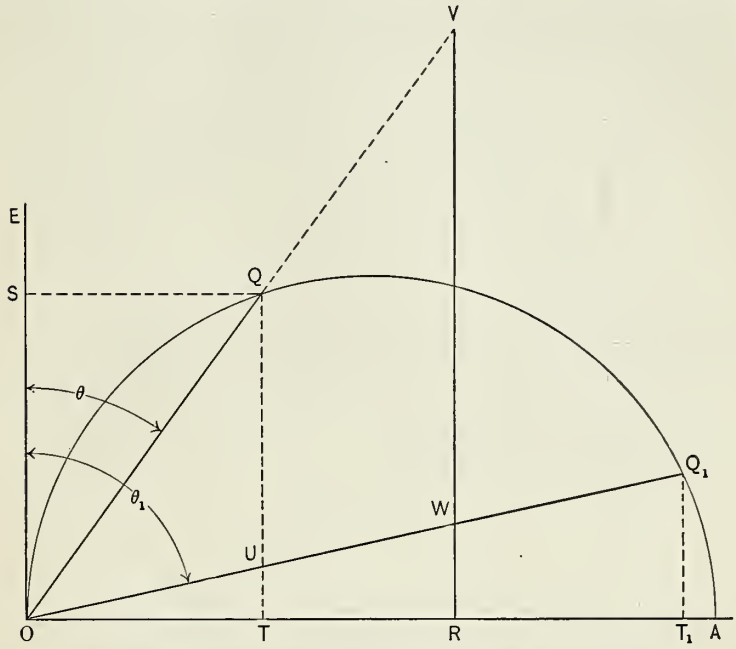


FIG. 510. — Current Locus of a Series Alternating Current Motor.

resistance is R. If, then, OQ_1 is the starting current, Q_1T_1 represents the standstill losses. The power input for any current OQ is $P_i = EI \cos\theta = OE \times QT$, which varies with QT. The angle of lag is $EOQ = \theta$; and $\tan\theta = \frac{QS}{QT} = \frac{E_x}{E_a}$, where E_a is the active and E_x the quadrature component of the impressed voltage. Therefore $\tan(90° - \theta) = \frac{E_a}{E_x} = \frac{R + R_e}{X}$. But X is con-

* Art. 70 (*a*).

stant and R is constant, but R_c varies as the speed and the counter-voltage E_c. Therefore, a vertical line such as RV, Fig. 510, can be erected and the speed and counter-voltage will be approximately proportional to the intercept WV when current OQ flows. WV at proper scale times the current then equals the motor output. The torque is approximately proportional to the current squared, under the conditions assumed. By mean proportionals, $OQ_1^2 : OQ^2 : : OT_1 : OT$; hence the torque can be measured off directly on the line OA if the proper scale is used. As the copper losses vary as I^2, it is sometimes assumed that all losses vary as I^2. In this case TU would be proportional to the losses for current OQ, and UQ be proportional to the output.

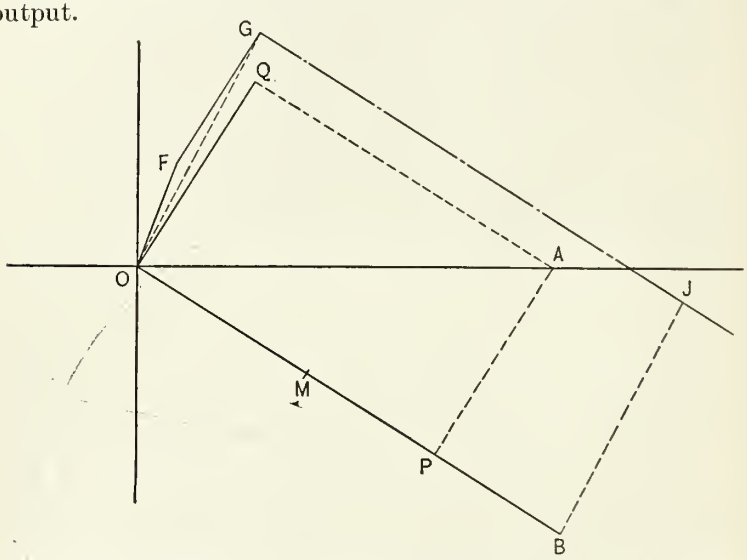


FIG. 511. — Vector Diagram of a Series Alternating Current Motor with Shunted Field Circuit.

The power factor of the motor may be increased if desired by shunting the field circuit as shown in Fig. 508 at D. In this case, the diagram of voltages becomes as shown in Fig. 511. The current entering the motor terminals is OA. It divides into two components assumed to be at right angles, through the non-inductive shunt and the inductive field coils ; let the former component be OQ and the latter OP. The magnetic field flux $OM = \Phi$ is (neglecting the iron loss angle) in phase with OP, the current in the field coil ; and the counter-voltage OB' is in oppo-

sition to Φ ; while the field-induced voltage GF is at right angles with Φ. Joining the local impedance drop OF with FG and OB gives an impressed voltage OJ. The voltage locus is GJ for constant current OA and variable speed. This arrangement makes it possible by using a variable shunt resistance to neutralize the lagging currrent and even draw leading current from the line. Various other methods of controlling the phase displacement between the impressed voltage and current are omitted here for lack of space.

The diagrams just discussed (Figs. 509 to 511) apply equally to motors having inductive compensation, since the local drop in the impedance of the compensation windings is furnished by the impressed voltage as in any transformer.

The vector diagram of an inductively excited motor such as diagrammed in Fig. 504 is practically similar to that shown in

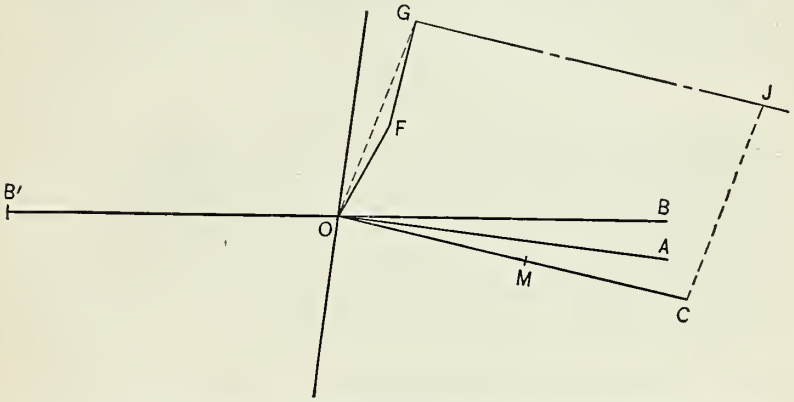


Fig. 512. — Vector Diagram of a Repulsion Motor or an Inverted Inductive Series Motor.

Fig. 509. The only difference in the diagrams lies in the fact that the secondary induced current is slightly more than 180° behind the primary current on account of the losses in the winding circuits ; hence its opposite component in the primary current is slightly ahead of the total primary current (OA, Fig. 509). As the field induced voltage (OG', Fig. 509) is at an angle to its current equal to quadrature plus the iron loss angle, it is thrown slightly to the left. The result is a slightly lower power factor. With A and C of Fig. 504 connected to the line and the brushes of B short-circuited there results a split coil repulsion motor.

The vector diagram for such a motor may be drawn quite similarly to that of Fig. 509, since the speed-induced counter-voltage is transferred directly to the compensating or mutual inducing coil C. Thus, assume that current OA of Fig. 512 is flowing through the torque and mutual coils of Fig. 496. Current flows in the armature, lagging slightly less than 180° from OA, supposing the reluctance of the path of the flux which links the mutual coils and armature coils is small. The voltage set up by the armature rotation is in phase opposition to OM. This induces in the mutual coil C a practically equal and opposite voltage OC. The self-inductive voltage of the field GF is at right angles to OM. Then if OF is the total local drop of voltage due to magnetic leakage and conductor resistance, the impressed voltage is OJ, and the construction is, as said, similar to that of Fig. 509.

The voltage induced in the field coils of a series motor by the torque flux is

$$E_f = \frac{\sqrt{2}\,\pi f n \Phi_f}{10^8} = IX_f, \tag{1}$$

where n is the number of field turns, f the frequency of the supply current, X_f the reactance of the winding, and I the motor current.

If the magnetic flux were constant instead of alternating, the counter-voltage induced in the armature by its rotation, with the brushes in the neutral plane, would be

$$E = \frac{a\Phi v}{10^8} \times \frac{p}{p'},$$

in which a is the number of conductors on the armature, Φ is the total flux per field pole, v is the number of revolutions of the armature per second, p is the number of pairs of poles in the magnetic field, and p' is the number of paths in parallel for current to flow through the armature winding. But the flux is of sinusoidal alternations instead of being constant, and, Φ being the maximum flux per pole, the counter-voltage set up in the armature by its rotation is

$$E_c = \frac{a\Phi v}{\sqrt{2} \times 10^8} \times \frac{p}{p'}. \tag{2}$$

The third voltage used in the diagrams represents that absorbed by resistance and leakage reactance and is

$$E_z = IZ_l, \tag{3}$$

in which Z_l is the impedance from resistance and leakage reactance.

If R_c is a resistance such that $IR_c = E_c$, then

$$R_c = \frac{E_c}{I} = \frac{a\Phi v}{\sqrt{2} \times 10^8 \times I} \times \frac{p}{p'}. \tag{4}$$

Putting $2w$ for $\frac{a}{n}$ and S for $\frac{v}{f}$, gives from Equations (1) and (4)

$$R_c = \frac{p}{\pi p'} wSX_f. \tag{5}$$

The motor input is

$$P_i = EI \cos \theta,$$

the output is $P_o = I^2 R_c$,

and the torque is $T = \frac{R_c I^2}{2\pi v}$.

208. Commutation and Other Characteristics of Commutating Alternating-current Motors. — The process of commutation in alternating-current motors differs from that of direct-current machines by reason of the voltages induced in the short-circuited coils under the brushes, caused by the rapidly alternating character of the field flux passing through them. Thus, to prevent sparking, it is necessary to neutralize these induced voltages to reasonable values or reduce the resultant currents and yet perform the ordinary function of reversing the current in the coils while short-circuited under the brushes. It will also be noted that the value of the current in the short-circuited coils is dependent upon two conditions at each instant of the primary current cycle. Namely, the actual currents in successively short-circuited coils depend upon the load and the instantaneous value of the field flux. Thus, at the instant when the field flux is a maximum, no current is induced in the short-circuited coils due to its change, but the coils enter the condition of short circuit bearing maximum line current; and when the field flux is zero, a maximum current is induced in the short-circuited coil, but the line current is zero.

The short-circuited coils being, with respect to the field windings, in effect transformer secondaries of very low resistance, the short circuit currents become so great, if not reduced by some special construction, as to seriously demagnetize the fields and interfere with the operation of the motor by destroying its torque. This, taken in connection with the liability to serious sparking, makes the question of commutation of very serious importance in designing alternating-current commutator motors.

One of the methods of reducing the short-circuit currents is to introduce high resistance strips into the leads between the commutator bars and the armature windings. This is useful; but leads of sufficient resistance, having large enough radiating surfaces to dissipate the heat generated in them, are difficult to insert in the confined space available. Therefore, though suitable when the armature is moving and the resistance strips are in use only a small part of a revolution, they are liable to be destroyed if the motor is stalled or fails to start promptly. Nevertheless, this method has proved valuable.

Another method is to insert reactance coils in the commutator leads and thus avoid the heating caused by the previous method, but with the disadvantage of adding to the self-inductance of the short circuit and thus making commutation difficult. Various arrangements have been proposed of this nature. One of them uses a differentially wound coil so that the line current largely neutralizes the reactance in its path while the short-circuited current sets up full reactive flux in the coil.

Another method uses a wide space between commutator bars and a split brush between the halves of which is a reactance coil. The two halves of the brush are so spaced that they touch two adjacent bars, and hence the short-circuit current must flow through the reactance. Special devices may be arranged so that the line current is not affected by the reactance.

The use of overcompensation with connections of the motor windings in the relations of a series repulsion motor has been found desirable for reducing the transformer voltage in the short-circuited coils. The use of commutating interpoles placed over the brushes has also been found advantageous. These poles can have windings which are shunt-connected across the brushes and through a variable autotransformer so that their strength may be changed with the speed and load. A method

shown in Fig. 513 of connecting the commutating poles gives one of the many possible arrangements. Thus, part A of the commutating pole winding is shunted in series with resistance B across the field coil, which gives a current strength in A proportional to the field strength. The part F of the same winding is connected in magnetic opposition to A and receives current from the secondary circuit of a small transformer G, the primary winding of which is connected across the armature brushes and which receives current in proportion to the armature speed. Then at low speeds, when the current to be reversed in the armature conductors and the field flux tending to induce heavy secondary currents in the short-circuited coils are large, the commutating poles are strong. When the speed is high and the armature current and field flux are low, the commutating pole flux is low. The reactance H indicated in series with the small transformer is required to adjust the value and phase of the current.*

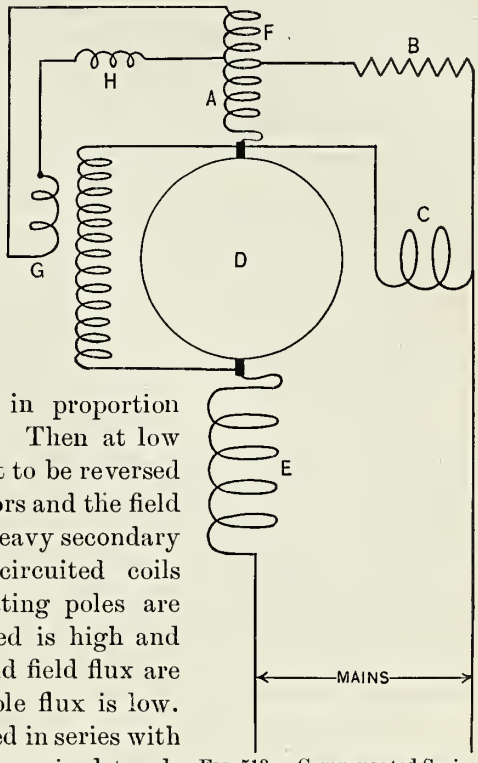


FIG. 513.—Compensated Series Alternating-current Motor having Commutating Poles connected for Automatic Strength Variation.

However, since commutating poles or overcompensation are effective to any extent only when the armature is rotating and hence its conductors are cutting the commutating flux, such schemes require also the use of resistance, or an equivalent device, in the leads between armature coils and commutator.

The commutation of repulsion motors is better when the speeds are below synchronism than when the speeds exceed synchronism, while the reverse is true of plain series motors whether compensated or uncompensated.

* McAllister, *Alternating Current Motors*, p. 302.

The operating characteristics of a series alternating-current motor are shown in Fig. 514. As in a direct-current series-motor, the torque starts at a high value and decreases as the speed increases and the current and field flux decrease. The power rises rapidly to a maximum when the product of torque and speed is a maximum and then decreases. The power factor rises rapidly to nearly unity.

From the formula and diagram already developed (page 887) it may be shown that $\tan \theta = \frac{X_f}{R_c} = \frac{\pi p'}{wSp}$, approximately, where θ is the angle of lag in a series-motor. Then to reduce θ to a

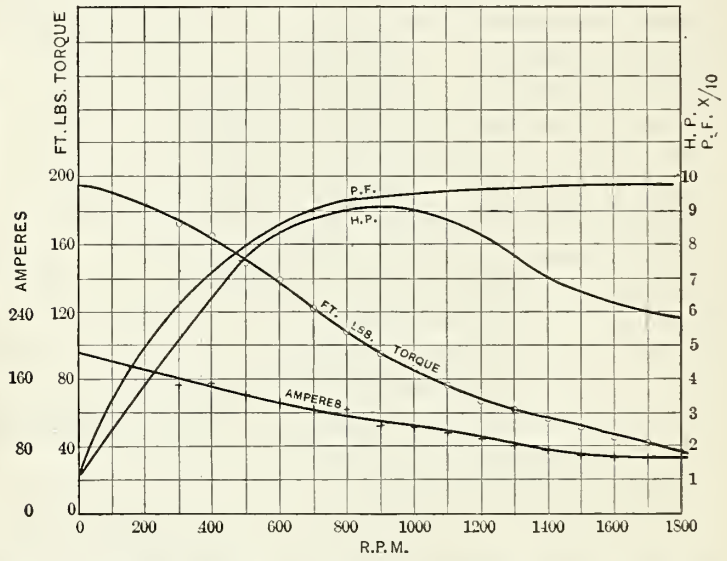


FIG. 514. — Operating Characteristics of a Series Alternating-current Motor, Compensated.

minimum, w, S and p should be as great as possible — a fact which has been found true in practice. To keep the power factor at the highest possible value, it is evidently desirable to reduce the air gap to the minimum allowable by mechanical considerations.

The frequency for which large series-motors are built in this country is usually 25 cycles per second, though many smaller motors of the series and repulsion types are used on circuits having a frequency of 60 cycles per second. Some European traction

circuits, on which commutating alternating-current motors are used, have frequencies of 15 periods per second or even less.

Alternating-current motors having their field windings connected in shunt with the armature can be built, though difficulty is experienced in keeping the field flux and armature current in phase; and further the place for motors of this type is occupied satisfactorily by the single-phase induction motor or repulsion motor.

The efficiency of alternating-current commutator motors can best be obtained by measuring the input by a wattmeter and the output by a rated generator or other mechanical absorbing device. However, the losses can be approximately determined if desired. The copper losses can be calculated. The field core losses can be measured by a wattmeter when the field is excited, the armature being removed. And the armature core loss and other rotating losses can be approximately obtained by driving the armature by a rated motor when the field is fully excited for the load and speed desired.

209. Frequency Changers and Motor Converters. — Frequency changers are much used in this country for transforming low frequency currents to higher frequency where electric power transmitted from a distance is needed for lighting.

One form of frequency changer, called a **Motor-generator**, consists of a motor operated from the circuit of one frequency and driving a generator connected with the circuit of the other frequency.* The two machines are usually mounted on one bed plate and have a common shaft, thereby making a motor-generator set, and a starting motor is sometimes also mounted on one end of the shaft. The numbers of magnet poles in the field magnets of the two machines must be in the ratio of the two frequencies. When two such machines are expected to operate in parallel at both ends of the sets, it is necessary that the alignment of the field magnets and armature coils shall be exactly corresponding in the two, or excessive cross currents will flow. It is also necessary to synchronize at both ends of the set, since the set may parallel correctly at one end and yet be out of correct phase for paralleling at the other end. In this case it is necessary to slip one or more poles at the motor end by opening the field switch, or in some other

* Art. 183.

manner bring the machine into paralleling phase at both ends. The number of points in one revolution of the shaft of such a machine at which paralleling can be accomplished at both ends of the set is equal to the greatest common factor of the numbers of pairs of poles respectively in the field magnets of the motor and the generator. The generator terminal voltage of an unloaded set which is to be put in parallel with a loaded set will also not be in exactly the same phase as the terminal voltage of the loaded set, on account of the mechanical lag of the loaded motor and the electrical lag of the loaded generator, even though the motors have been brought into the correct relations for paralleling the sets.

A Frequency converter as distinguished from a motor-generator consists, in its ordinary form, of a synchronous motor driving the armature of an induction motor backwards, *i.e.* against its direction of rotation as a motor, the armature winding of the synchronous machine and the field winding of the induction machine being connected to the supply mains of lower frequency, while the armature winding of the induction machine is connected to the higher frequency circuit. The connections are shown in Fig. 515. The power which must be supplied by the synchronous motor in driving the induction motor armature backward is equal to the torque exerted between the induction motor field flux and armature current times the speed of the armature, while that transferred by the induction motor itself is that which would be transferred if the armature were stationary and the same current flowed. Therefore, if a is the frequency of the supply mains and b is the frequency of the secondary mains, the induction motor must supply $\frac{a}{b}$ times the total power transformed and the synchronous motor must supply $\frac{b-a}{b}$ times the total power. Then, if it is required to convert 200 kilowatts from 25 to 60 cycles, the induction motor field must transform $\frac{25}{60} \times 200$, or $83\frac{1}{3}$ kilowatts, while the synchronous motor must supply $116\frac{2}{3}$ kilowatts to the armature shaft. The armature losses must be supplied in equal ratio. The voltage given out will be equal to $\frac{b}{a}$ times the voltage that would be generated

in the armature windings if they were stationary. The capacity of synchronous motor and induction motor field winding must be then in the proportion given, but the capacity of the induction motor armature must be that of the entire output. The losses of the armature tend to be high on account of the high frequency of the magnetic cycles to which its core is subjected.

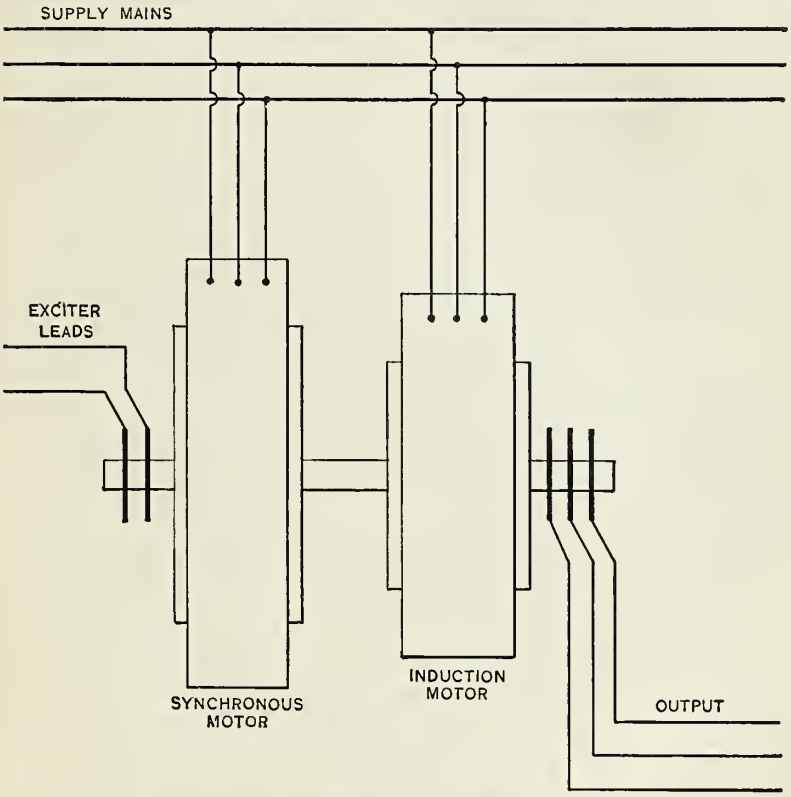


FIG. 515. — Connections of a Frequency Converter.

A **Motor-converter** is a machine for converting comparatively high voltage alternating currents to comparatively low voltage direct currents without the intervention of a transformer. It consists of the combination of an induction motor and a rotary converter as shown in Fig. 516. The armature of the motor sends polyphase currents directly into the windings of the converter armature, but slip rings are not needed, as the machines are connected together mechanically. Suppose that the two

machines have equal numbers of poles, then when the induction motor armature reaches one half synchronous speed, the current it sends to the synchronous converter armature has a frequency equal to the frequency of the counter-voltage induced in the converter armature. The frequency of the voltage induced in

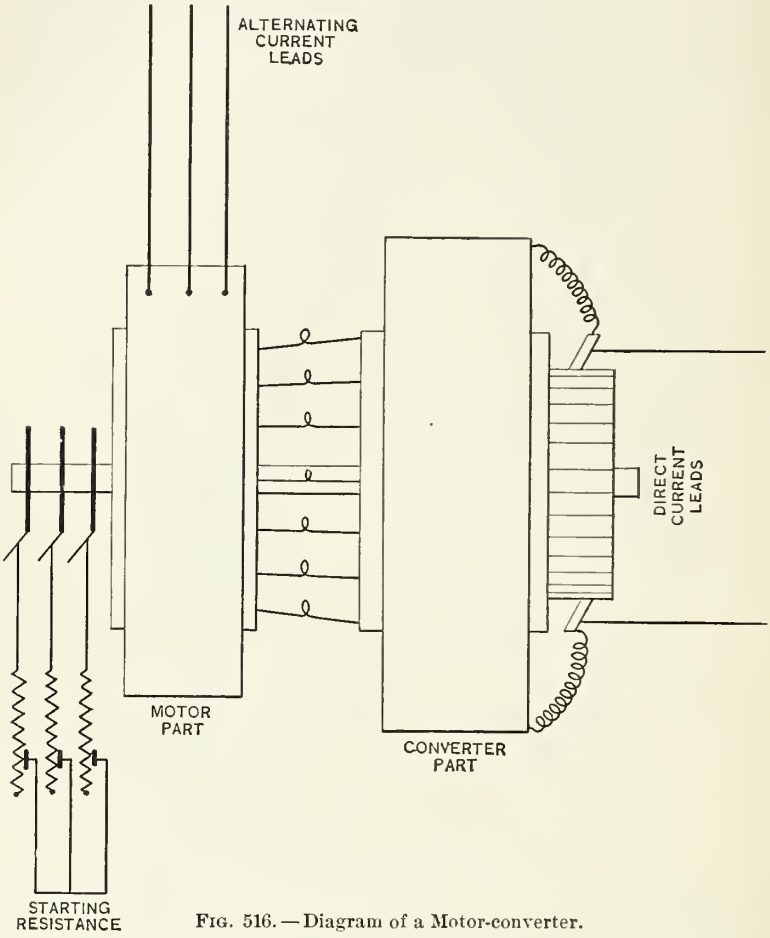


FIG. 516. — Diagram of a Motor-converter.

the armature conductors of the induction motor falls proportionally with the armature speed, from the frequency of the supply circuit to zero, as the armature increases in speed from standstill to synchronism. The voltage likewise falls off during this change of speed. At the same time the frequency of the voltage induced in the armature conductors of the synchro-

nous converter armature rises proportionally with the armature speed, from zero to the frequency of the supply circuit, in the same range of speed, and the voltage itself increases. At half synchronous speed the induced voltages are of the same frequency. If the speed tends to rise, the counter-voltage of the synchronous converter armature tends to reverse the resultant voltage in the circuit of the armatures. Therefore, if the induction motor and synchronous converter are of the proper relative proportions and have equal numbers of field poles, the speed will be maintained equal to one half the speed corresponding for the induction motor to synchronism with the supply current. Part of the power converted by the synchronous converter is received mechanically through its shaft from the induction motor and part electrically from the induction motor armature windings. If the machines have unequal numbers of field poles, the speed of the set is proportional to the ratio of the number of poles on the induction motor to the sum of the numbers of poles on the two machines.

In addition to permitting the use of high alternating voltages without the use of transformers, the motor-converter has the advantage of furnishing a low frequency to the rotary portion of the apparatus.

210. The Mercury Vapor Rectifier. — The mercury vapor rectifier or converter has come much into use for operating series direct-current arc lamp circuits, and charging storage battery circuits. It consists of an exhausted globe as shown in Fig. 517. A small amount of mercury is placed in the tube or globe, and when the latter is shaken, causing the mercury to bridge across the terminals A and B, a current flows from the alternating-current mains between the terminals. This frees mercury vapor, and current flows alternately into the terminals C and C' and out of the terminal A. By virtue of the peculiar property of the vapor in connection with the terminal electrode, current apparently cannot flow out of the terminals C and C', so that it flows in on alternate half cycles. Consider an instant when it is flowing into C: It then passes from the iron or carbon electrode, through the vapor, into the mercury at the bottom, out of terminal A, through the storage battery E, through the reactance coil F, and thence to the alternating-current return main. Upon the next half cycle, the current

flows into the rectifier at C', out at A, and thence through the battery in the same direction as before, then through F' and out to the other main.

A drop of voltage of about 15 volts occurs when current passes through the tube, which makes it quite efficient for supplying series arc lighting circuits, in which the current is low and the voltage high. For such circuits the alternating current is usually supplied to the rectifier by a constant-current transformer.* Polyphase tubes are also used for charging storage batteries and similar purposes. In a tri-phase tube three positive electrodes similar to C and C' are used. The secondary windings of the supply transformers are connected in wye. The free battery terminal is connected to the neutral point and the three electrodes are connected to the three wye terminals.

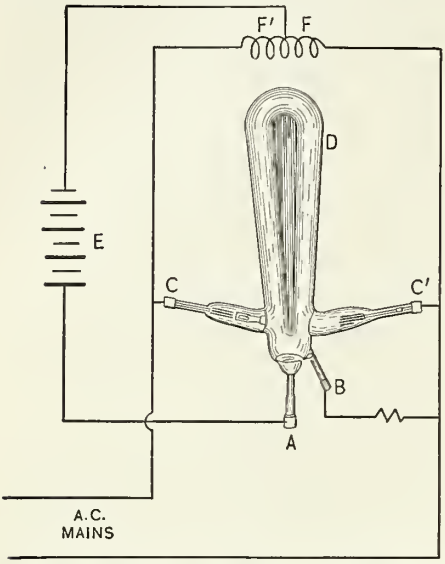


FIG. 517.—Single-phase Mercury Vapor Rectifier.

Electrolytic rectifiers are also used to some extent. They are based on the fact that an aluminum electrode in a cell will pass a current only in one direction (*i.e.* that which makes the aluminum the anode) unless the voltage exceeds a certain critical figure. This peculiarity of aluminum is also made use of in the electrolytic lightning arresters, now largely used on high voltage transmission circuits. For this purpose what are equivalent to long strings of aluminum cells are joined in series from a line wire to ground or from one line wire to another. When a current is passed through them, they quickly take a property of very high resistance. This property gradually disappears with time, when the arrester must be "recharged," by having a suitable voltage impressed upon it for a short time.

* Art. 142.

CHAPTER XIII

SELF-INDUCTANCE, MUTUAL INDUCTANCE, AND ELECTROSTATIC CAPACITY OF PARALLEL WIRES. SKIN EFFECT

211. The Self-inductance of Parallel Wires. — The self-inductance of two parallel wires hanging upon a pole line, contained in a cable, or otherwise, has much influence on the processes of long distance power transmission, long distance telephony, and long distance telegraphy. In the ordinary alternating-current systems for electric lighting and for transmission of power over relatively short distances, the effects of the self-inductance of the transmission lines are not particularly serious, but even in those instances it is important to be able to predetermine the nature and magnitude of the effects.

An expression for the self-inductance of two parallel wires may be developed thus: Suppose that two parallel cylindrical conductors A and A', Fig. 518, form a circuit of indefinitely great length. Let I be the amperes of current flowing through the conductors, r the radius of each conductor measured in centimeters, and ∂ the distance between the axes of the conductors also measured in centimeters, and assume the current to be uniformly distributed over the cross section of each of the conductors.

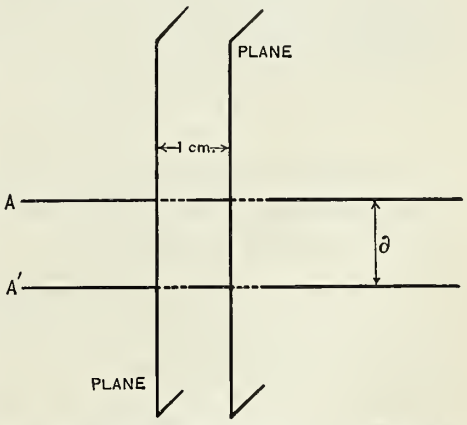


FIG. 518. — Diagram for studying Self-inductance of Parallel Wires, consisting of Two Conductors A, A' of Indefinite Length cut at Right Angles by Two Imaginary Planes.

Also let μ and μ' be respectively the magnetic permeability of the medium surrounding the conductors and of the material composing the conductors. The intensity of the magnetic field

(H_a) measured in dynes at a point outside of the conductor A, at a distance a centimeters from its center and due to the current in A, is

$$H_a = \frac{2\,I}{10\,a}. \tag{1}$$

This may be readily understood from the fact that the lines of force due to the current in the straight conductor are circles with their planes perpendicular to the conductor, and if the magnetic force at any point in space at a distance a centimeters from the conductor is F_a dynes, the work done against this force in moving a unit magnet pole around the conductor is $W = 2\,\pi a F_a$ ergs. In this case the electromagnetic intensity, and therefore the force exerted on a unit pole, is perpendicular to the conductor, since the lines of force are circles with their planes perpendicular to the conductor. Also, the work done in carrying a unit pole around the conductor is equal to $W = 4\,\pi n \frac{I}{10}$ in any case in which n is the number of times the current is circled and the current is measured in amperes. The symbol n stands for the number of turns in the coil if the conductor concerned were a coil; but in this case $n = 1$, and therefore

$$2\,\pi a F_a = \frac{4\,\pi I}{10},$$

or

$$F_a = H_a = \frac{2\,I}{10\,a}.$$

The magnetic density (B_a) in lines of force per square centimeter at the point a is therefore

$$B_a = \mu H_a = \frac{2\,\mu I}{10\,a}. \tag{2}$$

Now, considering a space cut out by two planes perpendicular to the axes of the conductors and one centimeter apart (see Fig. 518), within this space at a distance a centimeters from conductor A a number of lines of force equal numerically to

$$B_a da = d\Phi_a = \frac{2\,\mu I da}{10\,a}$$

will pass through a radial width da centimeters; and the total number of lines of force set up by the current in A which pass

through the area bounded on two sides by the surface of A and the center of A' and on the other two sides by the planes, is

$$\Phi_a = \int_r^{\partial} \frac{2\,\mu I da}{10\,a} = \frac{2}{10}\,\mu I \log_e \frac{\partial}{r}. \tag{3}$$

At any point p within conductor A and at a distance of b centimeters from the center of A, the magnetic intensity will be the same as though the current within a circle of radius b centimeters were condensed at the center of the conductor, as far as the magnetic effect of the current in conductor A is concerned. Since the magnetic effect at the point p of the uniform layer of the current in the conductor outside of radius b centimeters will be zero, the strength of field at p may be written

$$H_b = \frac{2\,I}{10\,b} \times \frac{\pi b^2}{\pi r^2} = \frac{2\,bI}{10\,r^2}, \tag{4}$$

and the magnetic density at p is

$$B_b = \frac{2\,\mu' bI}{10\,r^2}. \tag{5}$$

Hence within the conductor and between the two planes one centimeter apart, a number of lines of force numerically equal to

$$B_b\,db = d\Phi_b = \frac{2\,\mu' bI\,db}{10\,r^2}$$

will pass through a radial width db. But this flux does not link with the whole current in the conductor A, but only with the current equal to $\frac{\pi b^2}{\pi r^2} I$ amperes which is within the circle having a radius of b centimeters. The product of the current with the number of lines of force inclosed by it, giving the magnetic linkages for current within the whole conductor, therefore is

$$\int_0^r \frac{\pi b^2}{\pi r^2} \times \frac{2\,\mu' Ibdb}{10\,r^2} = \frac{1}{2}\,\frac{\mu' I}{10}. \tag{6}$$

The self-inductance of conductor A given in henrys for a centimeter of length is equal to 10^{-8} times the number of lines of force linking the current when it has a value of one ampere; and, summing the effects of the magnetic field without and

within the conductor by adding (3) to (6) when I is one ampere, and multiplying by 10^{-8} gives

$$L = \left(2\,\mu \log_e \frac{\partial}{r} + \frac{1}{2}\,\mu'\right) 10^{-9}, \qquad (7)$$

which is the self-inductance in henrys per centimeter of conductor length.

The effect of the return conductor A' in a single-phase circuit is to exactly double the flux which is linked or inclosed by the current, and therefore the self-inductance per centimeter of length of circuit is the same as the self-inductance for two centimeters length of conductor.

Given in terms of common logarithms the self-inductance per centimeter of length of conductor is

$$L = \left(4.605\,\mu \log \frac{\partial}{r} + \frac{1}{2}\,\mu'\right) 10^{-9};$$

and in terms of 1000 feet of conductor the self-inductance is

$$L_1 = \left(.1404\,\mu \log \frac{\partial}{r} + .01524\,\mu'\right) 10^{-3} \text{ henrys}$$

or

$$.1404\,\mu \log \frac{\partial}{r} + .01524\,\mu' \text{ millihenry.}$$

When the conductors are of copper suspended in the air or embedded in the usual insulating materials, $\mu = \mu' = 1$ and the self-inductance per 1000 feet of conductor is

$$L_2 = .1404 \log \frac{\partial}{r} + .01524 \text{ millihenry.}$$

As a rule the value of $2 \log \frac{\partial}{r}$ derived from the distances apart of diameters of wires in ordinary overhead electric circuits is quite large compared with the term $\frac{1}{2}$, but in the case of conductors in underground cables the ratio $\frac{\partial}{r}$ may be so small that the constant term $\frac{1}{2}$ comprises a large part of the self-inductance. In underground cables, however, the conductors are so close together that the uniformity of distribution of the currents over the cross section of each conductor cannot be relied upon, as the conductors will influence each other and the results may be modified.

The following table gives the resistances and self-inductances of wires of different sizes at a number of distances apart.

TABLE SHOWING SELF-INDUCTANCES OF SOLID CYLINDRICAL COPPER CONDUCTORS AT VARIOUS DISTANCES APART, MEASURED FROM CENTER TO CENTER

B. & S. Gauge No.	Diameter in Fractions of Inch	Resistance in Ohms per 1000 Ft. of Wire at 20°C.	Self-inductances in Millihenrys per 1000 Ft. of Conductor									
			1 Inch C. to C.	3 Inches C. to C.	6 Inches C. to C.	12 Inches C. to C.	18 Inches C. to C.	24 Inches C. to C.	36 Inches C. to C.	60 Inches C. to C.	72 Inches C. to C.	120 Inches C. to C.
0000	.4600	.04893	.1049	.1718	.2141	.2564	.2811	.2986	.3233	.3545	.3656	.3968
000	.4096	.06170	.1119	.1789	.2212	.2635	.2882	.3057	.3304	.3616	.3727	.4039
00	.3648	.07780	.1190	.1860	.2282	.2705	.2952	.3127	.3374	.3686	.3797	.4109
0	.3249	.09811	.1261	.1931	.2353	.2776	.3023	.3198	.3445	.3757	.3868	.4180
1	.2893	.1237	.1331	.2001	.2424	.2847	.3094	.3269	.3516	.3828	.3939	.4251
2	.2576	.1560	.1402	.2072	.2495	.2918	.3165	.3340	.3587	.3899	.4010	.4322
3	.2294	.1967	.1473	.2143	.2565	.2988	.3235	.3410	.3657	.3969	.4080	.4392
4	.2043	.2480	.1543	.2213	.2636	.3059	.3306	.3481	.3728	.4040	.4151	.4463
5	.1819	.3128	.1614	.2284	.2707	.3130	.3377	.3552	.3799	.4111	.4222	.4534
6	.1620	.3944	.1685	.2355	.2777	.3200	.3447	.3623	.3870	.4181	.4293	.4604
7	.1443	.4973	.1755	.2425	.2848	.3271	.3518					
8	.1285	.6271	.1826	.2496	.2918	.3341	.3588					
9	.1144	.7908	.1897	.2567	.2989	.3412	.3659					
10	.1019	.9972	.1968	.2638	.3060	.3483	.3730					
12	.0808	1.586	.2109	.2779	.3202	.3624	.3871					
14	.0641	2.521	.2250	.2920								
16	.0508	4.009	.2392	.3062								
18	.0403	6.374	.2533	.3203								
20	.0320	10.14	.2675	.3345								
22	.0253	16.12	.2817	.3487								

In the foregoing discussion of the self-inductance of two parallel conductors the upper limit of integration in Equation (3) is taken as ∂, which carries the integration from the edge of conductor A to the center of conductor A' for the magnetic linkages resulting from flux outside of the conductor A. This is obviously correct for each thin cylindrical layer of current in conductor A', and is therefore correct for the whole.

In case the solid cylindrical conductors are of different diameters, the formula for the self-inductance per centimeter of conductor length becomes

$$L = \left(\mu \log_e \frac{\partial^2}{r_1 r_2} + \frac{1}{2}\mu'\right) 10^{-9}, \tag{8}$$

in which r_1 and r_2 are the radii of the two conductors.

The equation for parallel hollow cylindrical conductors is, per centimeter of conductor length,

$$L = \left[\mu \log_e \frac{\partial^2}{r_1 r_2} + \frac{1}{4}\mu_1 \left\{ \frac{r_1^2 - 3(r'_1)^2}{r_1^2 - (r'_1)^2} + \frac{4(r'_1)^4}{[r_1^2 - (r'_1)^2]^2} \log_e \frac{r_1}{r'_1} \right\} + \frac{1}{4}\mu_2 \left\{ \frac{r_2^2 - 3(r'_2)^2}{r_2^2 - (r'_2)^2} + \frac{4(r'_2)^4}{[r_2^2 - (r'_2)^2]^2} \log_e \frac{r_2}{r'_2} \right\} \right] 10^{-9}, \tag{9}$$

in which μ, μ_1, μ_2 represent the magnetic permeabilities respectively of the medium surrounding the conductors and the two conductors, r_1, r_2 represent the external radii of the two conductors, r'_1, r'_2 represent the internal radii of the hollow cylinders.*

The foregoing formulas relate only to cylindrical wires, but apply with an ample accuracy to conductors of other forms of cross section, such as square or oval, provided the distance between the conductors is large compared with their thickness through. In the case of flattened conductors in underground cables where the conductors are near together, these formulas can be applied approximately in lieu of satisfactory exact formulas which have not been developed on account of the complexity introduced in the integrations when the conductors deviate far from cylindrical form.

If the currents are not uniformly distributed over the cross section of the conductors, as may be the case if conductors A

* See Maxwell, *Electricity and Magnetism*, Art. 685.

and A' lie close together and are so thick that the magnetic flux from each one sets up an appreciable redistribution of the flux in the metal of the other, the inductive reactance computed from the formulas may go quite wide of the truth if the reactance is taken as $2\pi fL$ and L computed from the formulas.

In case the alternating currents are of rather high frequency and the diameters of the conductors are considerable, the currents tend to forsake the central part of each conductor and become of greater density toward its surface on account of the so-called **Skin effect**,* and an approximation of the self-inductance may then be made by means of formula (9). When the frequency is sufficiently great to cause the currents to concentrate substantially in a thin cylindrical film of the metal at the surface, the inner and outer radii of the cylindrical conductor become equal to each other.

Formula (7) may be written in the following form in case the conductors are of the same radius and the same material and are sufficiently far apart compared with their thickness so that they do not affect the current distribution in each other,

$$L = (2\mu \log_e \frac{\partial}{r} + \frac{v}{2}\mu')\, 10^{-9},$$

in which v is a variable coefficient with limits from unity when the currents are uniformly distributed over the cross section to zero when the frequency is high enough to bring the currents to the surface.

The minimum value for the self-inductance of a circuit composed of two indefinitely long cylindrical conductors obviously occurs when the conductors are separated by the least practicable distance. If the current can be assumed to be uniformly distributed over their cross sections, Equation (7) reduces for the minimum value to

$$L = (2\mu \log_e 2 + \tfrac{1}{2}\mu')10^{-9} = (1.386\,\mu + .5\,\mu')\, 10^{-9},$$

and when μ and μ' are equal to unity this becomes $L = 1.886 \times 10^{-9}$ henrys per centimeter of conductor length. This reduces to $L = 0$ in the limiting case of the two conductors very close together and carrying currents of such high frequency that the currents concentrate wholly on the surface,

* Art. 212.

since in that case (the conductors being so close together) the currents in the conductors react on each other so that the currents in the tube conductors come to occupy positions in narrow strips of the surfaces facing each other.

When the conductors of equal cross section are concentric with respect to each other and the inner conductor is solid, a similar integration shows that the self-inductance of the pair of conductors per centimeter of length of the cable is

$$L = \left[2 \log_e \frac{R}{r} + \left(\frac{R^2 + r^2}{r^2}\right)^2 \log_e \left(\frac{R^2 + r^2}{R^2}\right) - \left(\frac{R^2 + r^2}{r^2}\right) \right] 10^{-9},*$$

where r is the radius of the inner conductor and R is the radius of the inner cylindrical surface of the external conductor.

In case the inner conductor is also tubular, the distribution of the current is limited to the area of a hollow cylinder and the self-inductance of the concentric cable per centimeter of length of the cable then becomes

$$L = \left[2 \log_e \frac{R_1}{r} + \frac{2}{(r^2 - r_1^2)^2} \left\{ r_1^4 \log_e \frac{r}{r_1} + R^4 \log \frac{R}{R_1} - \frac{1}{2}(r_1^2 + R^2)(r^2 - r_1^2) \right\} \right] 10^{-9},$$

in which r_1 and r are respectively the inner and outer radii of the inner conductor and R_1 and R are the inner and outer radii of the outer conductor.

Prob. 1. A pair of solid electric light wires are erected on a pole line 15 inches apart center to center for a distance of 12 miles. The wires are of 0000 B. & S. gauge. What is the self-inductance of the loop composed of these two conductors? What is the inductive reactance of the loop at a frequency of 25 cycles per second?

Prob. 2. A concentric underground cable 3000 feet long is composed of a solid No. 6 B. & S. gauge wire surrounded by an external conductor composed of a spiral wrapping of fine wire but giving the effect of a hollow cylinder of 300 mils internal radius with the same conducting cross section as the inner wire. The insulation between these two conductors is composed of

* Russell, *Alternating Currents*, Vol. 1, pp. 53–54.

oiled paper of which the permeability may be considered unity. What is the self-inductance of the loop composed of these two conductors?

It must be remembered that the foregoing formulas for the self-inductance of parallel wires are applicable only to lengths which are parallel for such large distances compared with the distances apart of the conductors that the ends of the loops need not be considered, and they are not applicable to coils or to wires with frequent turns in them, inasmuch as the magnetic force at the turns does not respond to the condition $H_a = \frac{2\,I}{10\,a}$. The equations are sufficiently accurate in their representation of the facts to meet the requirements of most computations with respect to overhead and underground electric light and power circuits, telephone circuits and telegraph circuits; but may prove seriously in error if applied to wiring on switchboards or inside of buildings on account of the many turns which may exist in such wiring, and also may prove seriously in error if applied to circuits composed of conductors deviating far from cylindrical form. Flattening the conductors tends to decrease the self-inductance. Also dividing the total current between two circuits, each having cylindrical conductors of half the total cross section, reduces the reactive voltage in the circuit because the current per circuit is reduced to a greater degree than the self-inductance is increased.

Overhead wires of considerable thickness and the conductors of underground cables of greater cross section than No. 6 B. & S. gauge wire are usually made by stranding a sufficient number of smaller wires to afford flexibility while maintaining the desired aggregate cross section. In applying the formulas to such conductors it is usual to assume that they give the same magnetic effects as smooth, cylindrical conductors of equal cross section. The error introduced by this assumption is difficult to determine for any particular instance, as it depends upon the various proportions entering into the strand. It is usually in the direction of a positive error, that is, it makes the computed self-inductance somewhat too large, but the error is usually quite small.

The foregoing formulas all deal with circuits composed of parallel conductors completing a loop. When a conductor is

vertical with respect to the earth and removed from parallel conducting bodies, a flow of current may occur through the effects of radiation or of electrostatic capacity. In this case the self-inductance per centimeter of length of a cylindrical conductor is of the form of

$$L = \left(2 \log_e \frac{A}{R} - B\right) 10^{-9},*$$

in which R is the geometric mean distance from each other of the elements of current in the conductor and is equal to $re^{-\frac{1}{4}}$ or r (r being the radius of the wire), depending upon whether the current is uniformly distributed over the cross section of the conductor or is concentrated on its surface.

When the loop composed of two conductors carries currents which are in different phases in the conductors, as is the case with a loop composed of either pair of wires of a three-phase circuit, the self-inductive voltage introduced into the loop is not equal to the reactance of the loop multiplied by the current in one of the wires. The currents in the two conductors being not in the same phase, the maximum magnetic flux set up in the loop is proportional to their vector sum instead of their algebraic sum. The self-inductive voltage introduced in the loop in such cases is therefore equal for a balanced circuit to the loop reactance per unit length of conductor (as given above) times $I \cos\left(\pi - \frac{\pi}{n}\right)$, where I is the current per conductor and n is the number of phases. This voltage is in quadrature with the vector representing the sum of the two currents. In the case of a symmetrically placed three-phase circuit, a convenient manner of computing the effect of self-induced voltage on the regulation of the circuit is to consider the voltage per conductor, which is equal to the conductor reactance multiplied by the conductor current and acts in quadrature to the conductor current. This voltage is opposite to the reactive drop along the wire which is measured between the conductor and the neutral point.

Prob. 1. The parallel conductors of a balanced three-phase circuit are each composed of a No. 0 B. & S. gauge wire and are located at the corners of an equilateral triangle so that they are

* Maxwell, *Electricity and Magnetism*, Arts. 685–693. Steinmetz, *Transient Electric Phenomena and Oscillations*, Chap. VIII, Sec. III.

36 inches apart, center to center. The length of each conductor is 10 miles, and each carries a current of 50 amperes. What is the value of the self-inductive voltage induced per conductor? What is the value of the self-inductive voltage induced in a loop composed of two of the conductors?

Prob. 2. In Prob. 1, suppose the voltage between conductors at the receiving end of the line is 10,000 volts, what is the delta voltage and also what is the wye voltage at the generator, assuming that the only causes contributing to line drop are resistance and self-inductance?

212. The Distribution of Current in a Wire. — The distribution of current over the normal cross section of a homogeneous conductor along which it flows is uniform, provided the current is steady, as this is the distribution that requires the smallest voltage to cause a given current to flow. The proof of this theorem is as follows: The total power lost in heating the conductor is, according to Joule's law, I^2R, where R is equal to $\frac{l\rho}{A}$, l, ρ, and A being respectively the length in centimeters, the specific resistance in ohms per centimeter cube, and the area in square centimeters of the conductor. Considering the conductor to be divided into elementary filaments of equal area and of resistance r, the current flowing in one of these is i and the power lost in it is i^2r. The total power lost in the conductor is Σi^2r, and the conditions afford the two simultaneous equations

$$\Sigma i^2r = I^2R,$$
$$\Sigma i = I.$$

These conditions can be simultaneously fulfilled only when the currents in the filaments are all equal to each other, and the distribution of the current over the cross section of the conductor is therefore uniform. If the filamentary currents were not uniform, the values of ir for the different filaments would differ from each other and from IR, which would cause a diversion of current from one filament to another until the uniform distribution was again reached.

The theorem of uniform distribution over the cross section applies to steady currents as just shown, but it does not apply to alternating currents, as the effect of mutual induction between filaments comes in to disturb the distribution. Suppose

a homogeneous cylindrical conductor divided into concentric cylindrical elements of equal cross sections, then by the formulas developed in the preceding article,

$$\Phi = \frac{2\mu I}{10} \log \frac{\partial}{r} + \frac{\mu' I}{10},$$

which exhibits the flux within a radius ∂ which is set up by a steady current I flowing in the conductor, and shows that a greater number of lines of force surround the central element of the conductor than surround the elements nearer the surface, when the current is uniformly distributed over the cross section of the conductor. In fact, when the current is uniformly distributed over the cross section of the conductor, the element composing the outside of the cylindrical conductor is surrounded by $\frac{\mu' I}{10}$ less lines of force than the central element. When the current flowing through the conductor is alternating, however, the uniformity of distribution is disturbed because a counter-voltage is set up in each element which is equal to $-\frac{d\phi_f}{dt}$, where ϕ_f is the flux which is set up by the current and surrounds the filament under consideration.

Since ϕ_f increases from the outside of the wire towards the central elements, unless the current is all concentrated on the outer surface of the conductor, the counter-voltage is greatest at the center and least at the surface of the conductor. Consequently there is a tendency for the current to forsake the center of the conductor and to seek a place nearer the surface. This tendency is directly proportional to the frequency of the cycles of flux when the current is sinusoidal. This tendency is opposed by the tendency of the current to a distribution which will give the least loss of energy, and alternating current therefore distributes itself in a conductor so that these two tendencies are balanced against each other, for which reason the current becomes distributed over the cross section so that its density increases from the center to the surface of the conductor. *This makes an increase in the actual resistance to the flow of alternating current, and in the loss of energy caused by the current flowing through the conductor.*

The resistance R_a, which a straight conductor of length l exposes to an alternating current, in ratio to the true resistance

(or resistance which the conductor exposes to a steady current) R, may be calculated by the following formula in which R_a and R are given in ohms, f in periods per second, and l in centimeters,*

$$\frac{R_a}{R} = 1 + \frac{\mu^2}{12}\left(\frac{2\pi fl}{10^9 R}\right)^2 - \frac{\mu^4}{180}\left(\frac{2\pi fl}{10^9 R}\right)^4 + \frac{\mu^6}{2440}\left(\frac{2\pi fl}{10^9 R}\right)^6 - \text{etc.}$$

$$= 1 + \frac{\mu^2}{12}\left(\frac{2\pi fA}{10^9 \rho}\right)^2 - \frac{\mu^4}{180}\left(\frac{2\pi fA}{10^9 \rho}\right)^4 + \frac{\mu^6}{2440}\left(\frac{2\pi fA}{10^9 \rho}\right)^6 - \text{etc.}$$

In the last expression A is the area in square centimeters of the cross section of the conductor and ρ is the specific resistance of the material measured in ohms per centimeter of length and square centimeter of cross section (per centimeter cube).

When the material of the conductor is copper, $\mu = 1$, and the formula may be written approximately for copper conductors, in which $\rho = 16 \times 10^{-7}$ ohms for the centimeter cube,

$$\frac{R_a}{R} = 1 + 30(d^2 f\, 10^{-9})^2,$$

where d^2 is the number of circular mils in the cross section. For other conductors the approximate formula is

$$\frac{R_a}{R} = 1 + 30\left(\frac{\rho}{\rho'} d^2 f \mu\, 10^{-9}\right)^2,$$

in which ρ is the specific resistance of copper, ρ' the specific resistance of the material composing the conductor concerned, and μ is the permeability of the conductor. When $d^2 f\mu$ in these approximate formulas exceeds 5×10^9, the resistance which a cylindrical conductor offers to alternating currents is approximately equal to the true resistance of a tube of the same material with its outside diameter equal to that of the conductor and with the thickness of its wall equal to $2500/\sqrt{f}$ mils.

The following table gives the ratio of R_a to R for various values of the product $d^2 f$ for cylindrical conductors, d being the diameter in mils.

* Maxwell, *Electricity and Magnetism*, Vol. II, Chap. 13; Rayleigh, On Self-Induction and Resistance of Straight Conductors, *Philosophical Magazine*, May, 1886, p. 381.

Product of Circular Mils and Frequency	$\frac{R_a}{R}$	Product of Circular Mils and Frequency	$\frac{R_a}{R}$
10,000,000	1.003	70,000,000	1.15
15,000,000	1.007	80,000,000	1.19
20,000,000	1.012	90,000,000	1.24
30,000,000	1.03	100,000,000	1.30
40,000,000	1.05	125,000,000	1.47
50,000,000	1.08	150,000,000	1.67
60,000,000	1.11	Infinite	∞

This table shows that with an increase of the frequency of the circuit or of the diameter of the conductor, or of both, the energy loss per ampere flowing in the conductor increases, and that the energy loss may become very great (as though the current wholly forsook the middle parts of the conductor and confined itself to a thin exterior layer) when the product of the frequency of the current and the square of the conductor diameter becomes very great. On account of this apparent concentration of the current in a superficial skin of the wire the phenomenon is commonly called **Skin effect.**

The current density at any point within a conductor is shown by Thomson to fall off in going from the surface inwards, in the ratio of

$$1 : \epsilon^{-x\left(\frac{4\pi^2\mu f}{\rho}\right)^{\frac{1}{2}}},$$

where x is the distance of the point from the surface.* Gray † points out that, however great the diameter of a wire may be, its resistance to alternating currents of different frequencies will never be less than the true resistance of a wire of the diameter in centimeters given in the following table:

Frequency	Copper	Lead	Iron, $\mu = 300$
80	1.43	4.98	.195
120	1.17	4.08	.159
160	1.02	3.52	.138
200	.91	3.16	.123

* J. J. Thomson, *Recent Researches in Electricity and Magnetism*, pp. 260 and 281.

† *Absolute Measurements in Electricity and Magnetism*, Vol. II, p. 338.

The specific resistance of lead is not far from that of average German silver. It is about twice that of iron and about twelve times that of copper. The remarkably large skin effect indicated for iron by this table is due to its relatively large magnetic permeability. The tabular values should be proportional to the square roots of the specific resistances of the different metals and inversely proportional to the square roots of their magnetic permeabilities. For metals of a given specific resistance, the tabular values should be inversely proportional to the square roots of the frequencies. In view of the high frequency harmonics that it is desirable to preserve in telephone currents, the table indicates a reason why iron should be rejected as a material for telephone conductors. It must be understood that the permeability of iron wires depends upon the quality of the iron and the magnetic density set up in its mass, and that it may vary from a few tens of units to many hundreds of units. The tabular value of 300 is taken as a median or representative figure for iron subjected to rather low magnetizing forces.

The formula for the self-inductance of a cylindrical wire was developed in the preceding article on the assumption of a uniform distribution of current over the cross section of the conductor. The disturbance of this distribution by skin effect reduces the self-inductance, the formula for self-inductance in henrys per centimeter of length becoming

$$L_a = \left\{2\mu \log \frac{\partial}{r} + \mu' \left[\frac{1}{2} - \frac{1}{48}\left(\frac{2\pi f \mu' A}{10^9 \rho}\right)^2 + \frac{13}{8640}\left(\frac{2\pi f \mu' A}{10^9 \rho}\right)^4 - \text{etc.}\right]\right\} 10^{-9},$$

$$\text{or} \qquad L_a = L - \mu' \left[\frac{1}{48}\left(\frac{2\pi f \mu' A}{10^9 \rho}\right)^2 - \frac{13}{8640}\left(\frac{2\pi f \mu' A}{10^9 \rho}\right)^4 + \text{etc.}\right] 10^{-9}.$$

The limiting values of R_a and L_a are $R_a = R$ and $L_a = L$ for steady currents or alternating currents of low frequency in conductors of moderate thickness; and $R_a = \infty$ and $L_a = 2 \log \frac{\partial}{r}$ for very high frequencies.

213. Mutual Induction of Parallel Distributing Circuits. — Where two or more electric light or power circuits carrying alternating currents run parallel to each other, they act inductively upon each other, and in some cases the mutual induc-

tion may cause considerable interference with the uniformity of the voltage on the lines. The mutual inductance of any two parallel circuits of indefinitely great length may be easily calculated, provided the distances apart of the different wires composing the circuits are known. The mutual inductance of the two circuits given in henrys is equal to 10^{-8} times the number of lines of force which pass through or link with one of the circuits due to one ampere flowing in the other circuit, provided there is no iron in the magnetic path; and this number of lines of force is equal to the algebraic sum of the number of lines of force embraced by the first circuit which would be set up by the current in the individual conductors of the second circuit taken separately. The method of Art. 211 is therefore directly applicable to the calculation of the mutual inductance of two long and parallel, narrow circuits. The following examples represent the commonest arrangements of circuits on pole lines.

Suppose that a, a' and b, b' represent the conductors of two circuits, and that the order of the wires is $a - a' - b - b'$, the distance apart center to center of the wires of circuit A is x, of circuit B is y, and of the adjacent wires of the two circuits $(a' - b)$ is z; then if we consider the currents as concentrated at the centers of the wires, which makes but an insignificant error with the ordinary dimensions of conductors and circuits, and consider the space between two planes perpendicular to the circuits and one centimeter apart, the number of lines of force due to a current of one ampere in a', which pass through the circuit B between the planes, is (Art. 211)

$$\Phi_{a'} = \int_z^{y+z} \frac{2\,da}{10\,a} = \frac{2}{10} \log_\epsilon \frac{y+z}{z},$$

and the number of lines of force due to a current of one ampere in a which pass through the circuit B between the planes, is

$$\Phi_a = -\int_{x+z}^{x+y+z} \frac{2\,da}{10\,a} = -\frac{2}{10} \log_\epsilon \frac{x+y+z}{x+z}.$$

The total number of lines of force set up by the current of one ampere in circuit A, which pass through the circuit B between

the planes, is $\Phi_a + \Phi_{a'}$, the number which link through the B circuit in a length of l centimeters is

$$l(\Phi_a + \Phi_{a'}),$$

and the mutual inductance of the parallel circuits of length l centimeters is

$$M = \frac{l(\Phi_a + \Phi_{a'})}{10^8} = \frac{2\,l}{10^9}\left(\log_\epsilon \frac{y+z}{z} - \log_\epsilon \frac{x+y+z}{x+z}\right)$$

$$= \frac{2\,l}{10^9} \log_\epsilon \frac{(x+z)(y+z)}{z(x+y+z)} = \frac{4.60}{10^9} \log_{10} \frac{(x+z)(y+z)}{z(x+y+z)}.$$

If $x = y$, this becomes

$$M' = \frac{2\,l}{10^9} \log_\epsilon \frac{(x+z)^2}{z(2\,x+z)} = \frac{4.60\,l}{10^9} \log_{10} \frac{(x+z)^2}{z(2\,x+z)};$$

and if $x = y = z$, it becomes

$$M'' = \frac{4.60\,l}{10^9} \log_{10} \frac{4}{3} = \frac{.575\,l}{10^9},$$

where l is the length of the parallel circuits in centimeters.

The mutual inductance in millihenrys per 1000 feet of distance in which the circuits are parallel is, using common logarithms,

$$M = .1404 \log \frac{(x+z)(y+z)}{z(x+y+z)};\quad M' = .1404 \log \frac{(x+z)^2}{z(2\,x+z)};$$
$$M'' = .01752.$$

Exchanging the order of the wires so that circuit A is between the conductors of circuit B, thus $b - a - a' - b'$, changes the formulas. Here the algebraic sum of the number of lines of force set up by the circuit A which link with circuit B is equal to the total number of lines of force set up by circuit A minus the number passing backwards through $b - a$ and $a' - b'$. Suppose that distance $a - a'$ is equal to x and distances $b - a$ and $a' - b'$ are each equal to y, then the total number of lines of force set up by one ampere in a length of one centimeter of circuit A, is

$$\Phi_a = .4 \log \frac{x}{r} + .2,$$

(r being the radius of the conductor), and the number of lines due to one ampere in circuit A, which pass between the planes through the space $b - a$, is

$$\Phi_{ab} = .2 \log_\epsilon \frac{y}{r} + .1 - .2 \log_\epsilon \frac{x+y}{x} = .2 \log_\epsilon \frac{xy}{r(x+y)} + .1,$$

and $$M' = \frac{l}{10^8}(\Phi_a - 2\,\Phi_{ab}) = \frac{4\,l}{10^9}\left(\log_\epsilon \frac{x}{r} - \log_\epsilon \frac{xy}{r(x+y)}\right)$$

$$= \frac{4\,l}{10^9} \log_\epsilon \frac{x+y}{y} = \frac{9.20\,l}{10^9} \log_{10} \frac{x+y}{y}.$$

If $x = y$, $M'' = \frac{9.20\,l}{10^9} \log_{10} 2 = \frac{2.77\,l}{10^9}$.

The mutual inductance in millihenrys per 1000 feet of distance in which the circuits are parallel is, using common logarithms,

$$M' = .2807 \log_{10} \frac{x+y}{y}; \quad M'' = .08451.$$

If the circuits are not in the same plane, as, for instance, they are arranged thus,

$$a - a',$$
$$b - b',$$

and the distance $a - a'$ is x, the distance $b - b'$ is y, the distance $a' - b'$ is z, the distance $a' - b$ is w, $a - b$ is v, and $a - b'$ is u; then the formulas are

$$\Phi_a = .2\left(\log_\epsilon \frac{u}{r} - \log_\epsilon \frac{v}{r}\right) = .2 \log_\epsilon \frac{u}{v},$$

$$\Phi_{a'} = .2\left(\log_\epsilon \frac{w}{r} - \log_\epsilon \frac{z}{r}\right) = .2 \log_\epsilon \frac{w}{z},$$

and

$$M = \frac{l}{10^8}(\Phi_a + \Phi_{a'}) = \frac{2\,l}{10^9} \log_\epsilon \frac{uw}{vz} = \frac{4.60\,l}{10^9} \log_{10} \frac{uw}{vz}.$$

If one circuit is directly beneath the other and $x = y$, $v = z$, then $w = u = \sqrt{x^2 + z^2}$, and the formula becomes

$$M' = \frac{2\,l}{10^9} \log_\epsilon \frac{x^2 + z^2}{z^2} = \frac{4.60\,l}{10^9} \log_{10} \frac{x^2 + z^2}{z^2}.$$

If $x = y = v = z$,

$$M'' = \frac{4.60\,l}{10^9} \log_{10} 2 = \frac{1.38\,l}{10^9}.$$

The mutual inductance in millihenrys per 1000 feet of distance in which the circuits are parallel is, using common logarithms,

$$M = .1403 \log \frac{uw}{vz}; \quad M' = .1403 \log \frac{x^2 + z^2}{z^2}; \quad M'' = .04225.$$

In case the arrangement is $\begin{matrix} a - b \\ b' - a' \end{matrix}$, one circuit being directly above the other and the distance $a - b$ equal to the distance $b' - a$, that is, the wires are on the corners of a square with the plane of each circuit tracing the diagonal, then the flux set up by either circuit is tangential with the other and the mutual inductance is zero, there being no linkages.

These results plainly show that the mutual inductance of two circuits is entirely independent of the actual distances apart of the conductors composing the circuits, but depends wholly upon the relative values of the distances. The mutual inductance of two circuits is a maximum when the circuits are exactly superposed, in which case $M = \sqrt{L'L''} = L$, and decreases as the distance between the circuits is increased in comparison with the distance apart of the conductors of each circuit; consequently, mutual inductance between circuits on the same pole line may be reduced by decreasing the distance apart of the conductors of each circuit and increasing the distance apart of the circuits. A better way to avoid annoyance from the effects of mutual inductance in most cases is to transpose the positions of the circuits with reference to each other at fixed intervals of distance on the pole line, so that the inductive effects of the circuits on each other are in opposition in different equal parts of the line, and neutralize each other for the line as a whole. If there are more than two circuits on a pole line and transpositions are needed, it is important to make the transpositions so that the magnetic mutual reactions which occur between any of the circuits all balance off and neutralize. For this purpose it is necessary that the transpositions shall be made properly. In any transposition interval, one line may be run straight through, one should be transposed at the middle, one at a quarter way from each end of the interval, one at the quarters and the middle, etc.

The effect of mutual induction between two circuits is to set up a voltage in one when the current in the other varies. If

the current is a sinusoidal alternating one, this voltage is (Arts. 119 and 120) $2\pi fMI$, and the effect of an alternating current in one circuit upon another circuit is easily determined if M is known. When the two circuits are fed from the same single-phase alternator, the induction of one upon the other is in quadrature with the current in the first, and the phase relation of the voltage induced in the second relative to the impressed voltage therein depends on the current lag in the first. If this is zero, the induced and impressed voltages are in quadrature, while they are in opposition if the lag is 90°. The result is a displacement of the voltage waves and a drop of voltage along the lines. The effect of reversing the connection of one of the circuits to the alternator causes an interchange of the voltage relations of the two circuits, but nothing else.

If the circuits are fed from different alternators the frequencies of which are slightly different, the inductive voltage and impressed voltage interfere in each circuit so as to form pulsations or beats, the frequency of which is equal to the difference of the two alternator frequencies, and the amplitude of which is the sum of the impressed voltage and the induced voltage. This may cause a perceptible winking of incandescent lamps connected to mutually inductive circuits carrying heavy currents of nearly the same frequency.

214. Effects of Self and Mutual Induction in Polyphase Circuits. — The effects of self and mutual induction in polyphase circuits may be determined by using the principles set forth in the preceding articles, provided the resultant effect of the differing phases is always considered. In unbalanced circuits, if mutual inductive effects are allowed to occur, they may increase the defect in balance. Mutual induction between the phases of one polyphase feeder may be avoided by suitably placing the conductors. Thus, in the case of a four-wire two-phase feeder, the wires may be located on two cross arms so that the planes of the conductors of the two phases form the diagonals of a square, thus $\begin{matrix} a & b \\ b & a \end{matrix}$, where a, a are the conductors of one phase and b, b are the conductors of the other. In this case there is no mutual induction between the circuits. When this arrangement cannot be utilized with quarter phase lines, transpositions must be resorted to.

Three-phase circuits may be erected on insulators located so that the three wires occupy the corners of an equilateral triangle, thus $\begin{matrix} & a & \\ c & & b \end{matrix}$. The lines of force set up by the current in any one of the conductors are circles surrounding the conductor and are therefore tangent to the plane of the loop composed of the other two conductors. Hence they do not link the loop and no mutual induction occurs. When this arrangement cannot be utilized, transpositions may be used to neutralize mutual inductance.

When two or more feeders are near each other, as on the same line of poles or towers, mutual induction between feeders may be neutralized by transposing the wires of each feeder spirally in such a way that the inductive actions in successive lengths balance each other. With the equilateral triangle arrangement of three-phase conductors, the spiraling of each feeder composed of three conductors may be carried out with regard only to the effects of the feeders on each other and on other neighboring circuits; but when the equilateral triangle arrangement is not used, the spiraling of each feeder should be accompanied by suitable transpositions to neutralize the mutual induction occurring between the phases of the feeder.

The vector relations of the voltages in four-wire two-phase circuits when the conductors lie in a plane (as when they are mounted on the pins of the same cross arms on a pole line or are supported one above another by suspension insulators) are illustrated in Fig. 519. The assumed relative positions of the wires are illustrated in the upper part of the figure, at AB, and the vector relations in the left hand and lower parts. The vector diagram is drawn for conditions of balanced load at the receiver end of the line.

In the vector diagram of Fig. 519, OE_A and OE_B are the voltages at the receiver in the respective phases 90° apart, and OI_A and OI_B are the corresponding currents. The voltage at the generator $OE°$ in each circuit is equal to the vector sum of the voltage at the receiver and components numerically equal and opposite respectively to the IR drop in the circuit $E'E$, the IX drop in the circuit $E''E'$, and the voltage $E°E''$ introduced in the circuit by the inductive influence of the current in the other branch of the two-phase circuit. The generator voltages

in the two circuits are therefore $OE_A°$ and $OE_B°$. The effect of the mutually induced voltage is therefore to increase the numerical difference and decrease the angular difference between the generator voltage and the voltage delivered to the receiver in

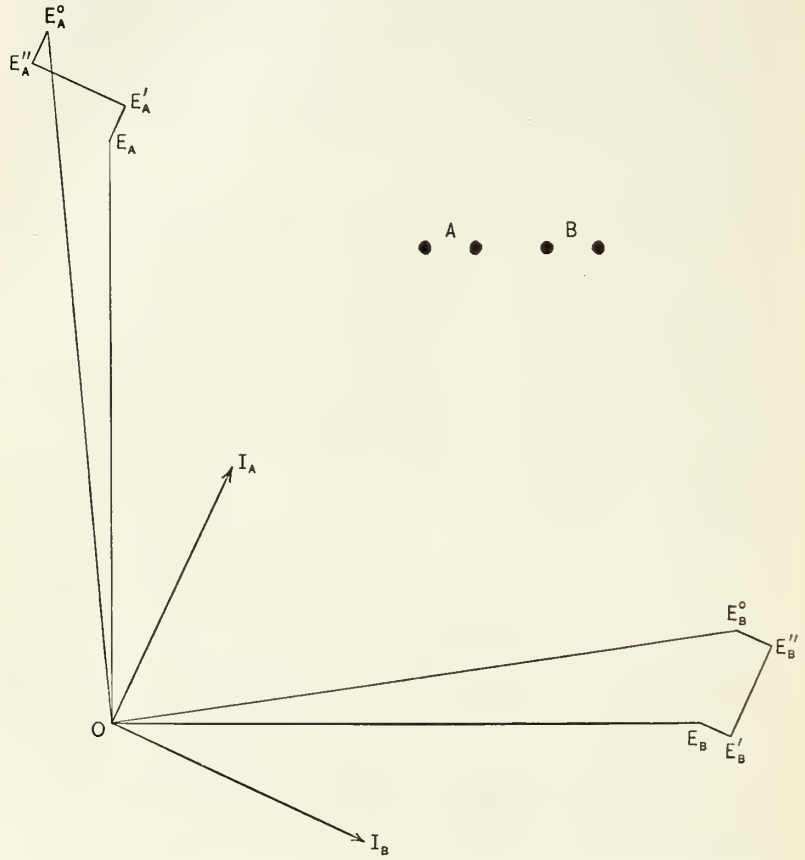


FIG. 519. — Diagram of Vector Voltage Relations in Four-wire, Four-phase Circuits when the Four Wires lie in a Plane, as at AB.

the leading phase, and to decrease the numerical difference and increase the angular difference between the generator voltage and the voltage delivered to the receiver in the lagging phase, compared with the relations when mutual induction is absent.

If $E_A°$ is the line voltage at the generator for the leading phase, E_A the line voltage at the receiver, R_A, L, and M the line resistance, self-inductance, and mutual inductance, I_A the cur-

rent, and Z, R, and X the impedance, resistance, and reactance of the load in the same phase, then

$$\bar{I}_A = \frac{\bar{E}_A}{Z} = \frac{\bar{E}_A}{R + jX} \text{ and } \bar{I}_B = -jI_A = -j\frac{\bar{E}_A}{R + jX}.$$

Also

$$\bar{E}_A = \bar{E}_A{}^\circ - \bar{I}_A R_A - j\omega L\bar{I}_A - j\omega M\bar{I}_B$$

$$= E_A{}^\circ - \frac{R_A\bar{E}_A(R - jX)}{R^2 + X^2} - j\frac{\omega L\bar{E}_A(R - jX) - \omega M\bar{E}_A(X + jR)}{R^2 + X^2}$$

$$= \bar{E}_A{}^\circ - \bar{E}_A\left[\frac{RR_A + XX_A + \omega MR}{R^2 + X^2} + j\frac{RX_A - XR_A - \omega MX}{R^2 + X^2}\right].$$

Hence

$$\bar{E}_A{}^\circ = \bar{E}_A\left[1 + \frac{RR_A + XX_A + \omega MR}{R^2 + X^2} + j\frac{RX_A - XR_A - \omega MX}{R^2 + X^2}\right].$$

By the same process

$$\bar{E}_B{}^\circ = \bar{E}_B\left[1 + \frac{RR_A + XX_A - \omega MR}{R^2 + X^2} + j\frac{RX_A - XR_A + \omega MX}{R^2 + X^2}\right].$$

When the load is non-reactive, $X = 0$ and the equations for $E_A{}^\circ$ and $E_B{}^\circ$ become

$$\bar{E}_A{}^\circ = \bar{E}_A\left(1 + \frac{R_A + \omega M}{R} + j\frac{X_A}{R}\right),$$

$$\bar{E}_B{}^\circ = \bar{E}_B\left(1 + \frac{R_A - \omega M}{R} + j\frac{X_A}{R}\right).$$

It may be observed from the equations as well as the diagram that, when mutual induction exists between the phases, the generator voltage is not balanced in case the voltage and current at the load are balanced; and conversely, if the generator voltage is balanced, it is necessary to abate mutual induction or else introduce special phase regulators in order to obtain balanced voltage conditions at the load. Inasmuch as the IZ drop of a distribution line is ordinarily much greater than the mutually induced voltage between circuits on the same poles, the unbalancing by mutual induction may not be very serious.

Figure 520 shows the effect of mutual induction in a three-phase circuit with three wires in one plane, as when they are erected on the same cross arm on each pole of a pole line, or are suspended in a vertical plane by suspension insulators. The lines OE_{ab}, OE_{bc}, and OE_{ca} are balanced delta line voltages

at the receiver, OE_a, OE_b, and OE_c representing the corresponding wye voltages from each line conductor to the neutral point, and OI_a, OI_b, and OI_c are the line currents. In this case the mutual inductance of the middle conductor b on the loop composed of conductors a and c is zero, provided the distance between a and b is equal to the distance between b and c, since

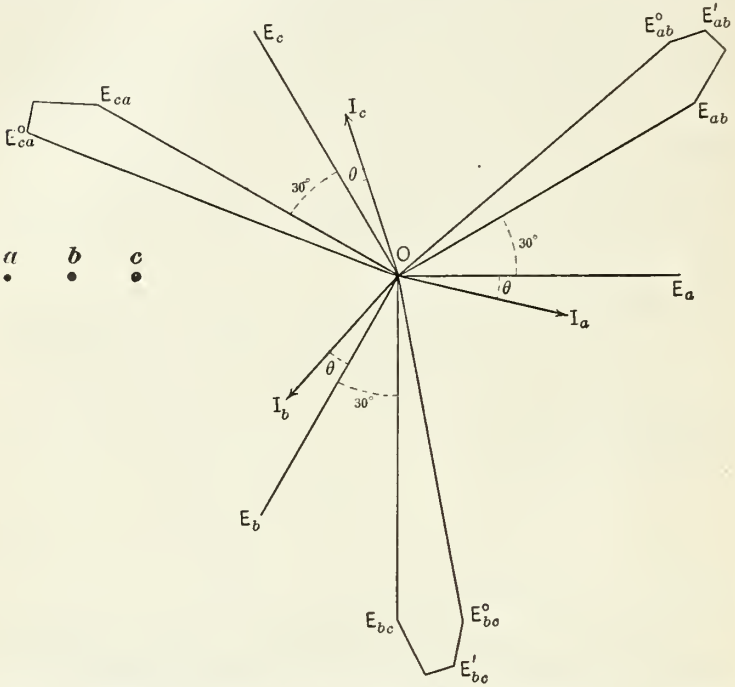


FIG. 520. — Diagram for showing the Effect of Mutual Induction in a Three-phase Circuit when the Wires are in One Plane, as at abc.

under those circumstances the flux set up through the loop $a - c$ by current in b is half upward and half downward. Conductor c, however, possesses mutual inductance with respect to the loop $a - b$, and conductor a has mutual inductance with respect to loop $b - c$. In case conductor b is not exactly halfway between a and c, mutual inductance also exists between conductor b and loop $a - c$.

In a balanced three-phase circuit the mutual induced voltage set up in a loop of two conductors, such as a and b, by current in the third conductor is

$$-j\sqrt{2}\,2\pi f M I_L \sin\,[\alpha + 120° - (30° + \theta)],$$

when the instantaneous delta voltage on the loop is proportional to $\sin \alpha$. Therefore, if $\theta = 0$, that is, the receiver current is in phase with the wye voltage at the receiver, the mutually induced voltage is in the same phase as the delta voltage at the receiver. If θ is not equal to zero, the mutually induced voltage differs in phase from the delta voltage at the receiver by the angle θ.

If, in the arrangement of conductors illustrated in the left-hand part of Fig. 520, the conductor b is moved perpendicular with respect to the plane of loop $a - c$, the mutual inductance of a with respect to $b - c$ and of c with respect to $a - b$ decreases until b has reached a point at which distances $a - b = b - c = c - a$. Under the latter circumstances, the conductors occupy positions corresponding to the corners of an equilateral triangle, the lines of force due to any one conductor are tangent to the plane of the loop composed of the other two, and the mutual inductance is zero. If conductor b is moved farther from the $a - c$ plane, the mutual inductance again increases, but the direction of the fluxes of a and c through the loops $b - c$ and $a - b$ are reversed, so that the mutually induced voltages are reversed. This is obvious from the following considerations: If distance $a - c$ is x, distance $a - b$ is y, and distance $b - c$ is z, then the mutual inductance of conductor a on loop $b - c$ is

$$M_a = \frac{2\,l}{10^9} \log_\epsilon \frac{x}{y},$$

and the mutual inductance of conductor c on loop $a - b$ is

$$M_c = \frac{2\,l}{10^9} \log_\epsilon \frac{x}{z}.$$

When b is nearer the plane of $a - c$ than (distance $a - c$) $\times \sin 60°$, $\log_\epsilon \frac{x}{y}$ and $\log_\epsilon \frac{x}{z}$ are positive and therefore M_a and M_c are positive; but when b is farther from the plane, $\log_\epsilon \frac{x}{y}$ and $\log_\epsilon \frac{x}{z}$ are negative and M_a and M_c therefore become negative.

In Fig. 520, EE' for each branch of the diagram represents the line resistance and reactance drop and $E'E_0$ represents the mutually induced voltage, when the receiver voltages and currents are as shown. The vectors $OE_{ab}°$, $OE_{bc}°$, and $OE_{ca}°$,

therefore, represent the delta voltages at the generator. In this illustration the distance between conductors a and b is taken equal to the distance between conductors b and c, the three conductors being in a plane, and there is no mutual voltage in the $c-a$ loop. It will thus be observed that the voltages at the generator end of the line must be unbalanced in case the voltages at the receiver end are to remain balanced, unless the conductors can be located on the apices of an equilateral triangle or transpositions are resorted to.

215. Electrostatic Capacity of Parallel Conductors. — When the voltage used for electrical transmission of power is high, the amount of transmitted current is relatively small, especially when the load is light, and the reactance drop in the line may be largely affected by the electrostatic capacity between the conductors. It is, therefore, necessary to investigate the magnitude of the electrostatic capacity between parallel conductors.

Electrostatic capacity of two neighboring conductors with respect to each other may be defined as the ratio of the electrostatic charge on each to the difference of potential between the conductors which is set up by the charges, one of which charges is positive and the other of which is equal numerically but of opposite kind. The capacity in farads is the ratio between the charge measured in coulombs and the difference of potential measured in volts. When the conductors are large parallel plates whose distance apart is small compared with their superficial dimensions, the capacity may be readily found on the assumption that a charge distributes uniformly over them, which is true except for portions near the edges. These conductors being charged, and away from the influence of other conductors, one carries a positive charge and the other carries an equal negative charge. The electrostatic field between them is uniform except near the edges, and the difference of potential measured from one to the other is equal to the intensity of the field multiplied by the centimeters of distance between the plates. The measure of the intensity of the field is the force in dynes which would be experienced by a C. G. S. electrostatic unit of charge moving from the plate of lower potential to the plate of higher potential.

The force which measures the field intensity is numerically equal to $1/K$ times the electrostatic lines of force per square

centimeter in the dielectric (measured in a plane perpendicular to the flux); and this flux is equal to $4\pi\sigma$, in which σ is the electrostatic charge per square centimeter on each plate, measured in electrostatic units, and K is the dielectric constant of the insulator lying between the plates. Also $Q = \sigma A$ where A is the area of each plate in square centimeters. Therefore, by definition, the capacity of the condenser composed of the parallel plates is, in electrostatic units of measure,

$$C_a = \frac{Q_a}{E_a} = \frac{Q_a}{F_a d} = \frac{KQ_a}{4\pi d\sigma_a} = \frac{KA}{4\pi d}.$$

To transform from C. G. S. electrostatic units to C. G. S. electro-magnetic units, with which the practical units conform, it is necessary to divide by v^2 where v is the velocity of light, and to transform from C. G. S. electro-magnetic units to farads it is necessary to multiply by 10^9 since a C. G. S. electro-magnetic unit of capacity is 10^9 times as large as a farad. The value of v has been experimentally proved to be substantially 3×10^{10}. Consequently, the capacity of the parallel plates in farads is

$$C = \frac{KA\,10^{-9}}{1.13\,d},$$

A being the area of each plate in square centimeters and d the distance between them measured in centimeters. For inch measure the formula is $C = 2.25\dfrac{KA\,10^{-9}}{d}$.

In the case of concentric cylinders of great length compared with the radial clearance between them, one of which carries a positive charge and the other an equal negative charge, the surface density of the charge is uniform on each cylinder except close to the ends. Considering the end effects negligible in long cylinders, the lines of force in the electrostatic field are radial. Inasmuch as the radial force exerted on an electrostatic unit charge at any point P outside of a uniformly charged cylinder of indefinitely great length is

$$\frac{2q}{Kr} \text{ dynes},$$

where q is the charge in C. G. S. electrostatic units per centimeter of length of the cylinder and r is the radial distance in

centimeters from the axis of the cylinder to the point P; * and also, inasmuch as the force inside of a uniformly charged cylinder is zero, when the influence of the ends may be neglected; therefore the capacity in C. G. S. electrostatic units of the concentric cylinders is

$$C_a = \frac{Q_a}{E_a} = \frac{Q_a}{\int_{r_1}^{r} F_a dr} = \frac{Klq}{2q\int_{r}^{} \frac{dr}{r}} = \frac{Kl}{2\log_\epsilon \frac{r_2}{r_1}},$$

in which r_1 and r_2 are respectively the outer radius of the inner cylinder and the inner radius of the outer cylinder, and l is the length of each cylinder in centimeters. Reducing the equation to represent capacity in farads gives

$$C = \frac{Kl\,10^{-9}}{1800\log_\epsilon \frac{r_2}{r_1}};$$

and, finally, reducing the equation to represent farads per 1000 feet of each cylinder and using common logarithms gives

$$C_1 = \frac{7.35\,K10^{-9}}{\log_{10}\frac{r_2}{r_1}}.$$

When the cylinders are not concentric but the distance between centers is a centimeters, one cylinder being entirely surrounded by the other, the last equation becomes

$$C_1 = \frac{7.35\,K10^{-9}}{\log_{10}\left[\frac{r_2^2 + r_1^2 - a^2}{2\,r_2 r_1} + \sqrt{\left(\frac{r_2^2 + r_1^2 - a^2}{2\,r_2 r_1}\right) - 1}\right]}.$$

These formulas apply to an underground insulated conductor surrounded by a lead sheath, in the usual construction of high voltage cables for underground use.

* The force at P due to the charge on any element of length of the cylinder is $\frac{qdx}{K(x^2 + r^2)}$. Only the radial component of this is effective, as components parallel to the cylinder neutralize each other, and the radial component is $\frac{qdx}{K(x^2 + r^2)} \sin\theta = \frac{qdx}{K(x^2 + r^2)} \times \frac{r}{\sqrt{x^2 + r^2}} = \frac{qrdx}{K(x^2 + r^2)^{\frac{3}{2}}}$. The force due to the entire cylinder of indefinitely great length is therefore

$$\int_{-\infty}^{\infty} \frac{qrdx}{K(x^2 + r^2)^{\frac{3}{2}}} = \frac{2q}{Kr} \text{ dynes.}$$

In case the cylinders are side by side instead of one inside the other, as in the case of two overhead line wires or two separately insulated conductors lying side by side in the same underground cable, the equation takes a different form. The following development of the equation is simple.

Considering two indefinitely long parallel conducting filaments A and A' (perpendicular to the plane of the page), Fig. 521, charged respectively with $+q$ and $-q$ electrostatic units of charge per centimeter of length, the force experienced by a unit charge at any point P in the electrostatic field of filament A and distant r centimeters from the filament would be $\frac{2q}{Kr}$ if filament A' were removed, and the potential at the point due to A is

$$V = \int_r^\infty \frac{2q}{Kr}\, dr.$$

Also, the potential at P due to A' alone is

$$V' = -\int_{r'}^\infty \frac{2q}{Kr'} dr,$$

if r' is the radial distance of the point from the filament A'. The total potential at P due to the charges on A and A' is therefore

$$V_P = V + V' = \frac{2q}{K}\left[\int_r^\infty \frac{dr}{r} + \int_\infty^r \frac{dr}{r'}\right] = \frac{2q}{K}\log_\epsilon \frac{r'}{r}.$$

Therefore, if the point P moves, the potential V_P at the point P does not change provided the locus of P maintains the ratio $\frac{r'}{r}$ constant; and this locus is a circle with its center O (Fig. 521) located on the line joining A and A' extended and with radius of length a which is a mean proportional between the lengths OA and OA'. If P_1 is the intersection of the circle with the line OA', these conditions arise,

$$\frac{A'P}{AP} = m,$$

$$\frac{OP_1}{OA} = \frac{OA'}{OP_1},$$

in which P is any point on the circle, P_1 is the point where the circle intersects the line OA', and m is a constant. It therefore follows that

$$\frac{OP}{OA} = \frac{OP_1}{OA} = \frac{OA'}{OP_1} = \frac{A'P_1}{AP_1} = \frac{A'P}{AP} = m,$$

and by division in the third and fourth ratios,

$$OP_1 = \frac{A'P_1 \times AP_1}{A'P_1 - AP_1}.$$

The equipotential surfaces around each one of the charged filaments are cylinders of radius increasing from zero to infinity

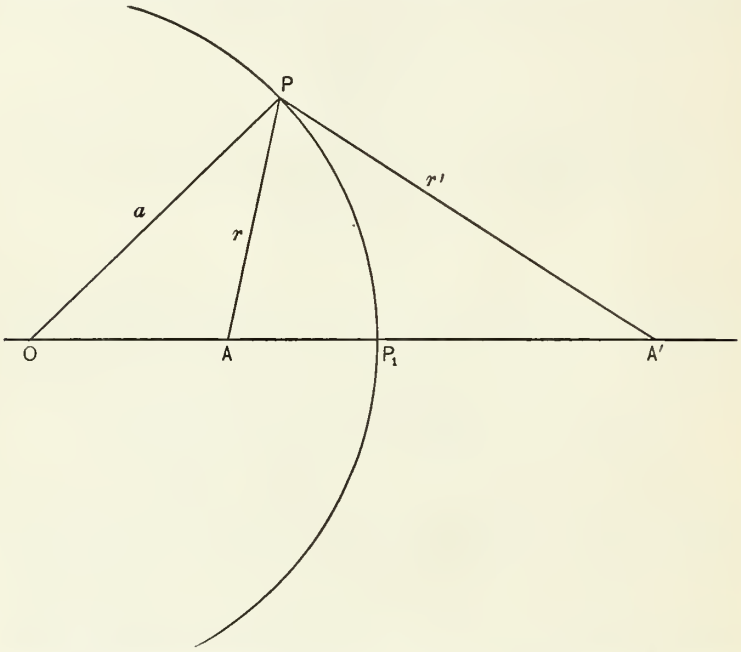


Fig. 521. — Diagram showing a Locus of Equal Potential between Two Parallel Charged Filaments A and A'.

as the point P is taken at increasing distances from the filament. These merge into a plane of zero potential when $A'P_1 = AP_1$, P_1 being halfway between the two equally but oppositely charged filaments. This plane touches infinity and has the same potential (namely, zero) as space at infinite distance from the charges.

Introducing a cylinder of conducting material with its outer surface coincident with the position of any one of the equipotential surfaces will not alter the electrostatic field; and the electrostatic field of two charged cylindrical parallel metal conductors may be computed by considering the surfaces of each as an equipotential surface surrounding a charged filament, the

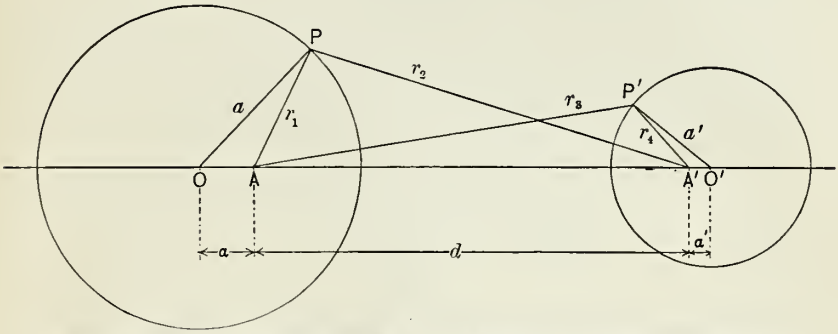


FIG. 522. — Diagram for illustrating Method of finding the Mutual Potentials of Two Charged, Cylindrical, Parallel Conductors.

center to center distance between the cylinders being the center to center distance OO' between the corresponding equipotential circles (Fig. 522). Then, according to the preceding paragraphs, the mutual potential at the surface of cylinder A is

$$V_A = \frac{2q}{K} \log_\epsilon \frac{r_2}{r_1},$$

and the mutual potential at the surface of the cylinder A' is

$$V_{A'} = \frac{2q}{K} \log \frac{r_4}{r_3}.$$

Therefore, the difference of potential in C. G. S. electrostatic units is

$$E_a = V_A - V_{A'} = \frac{2q}{K} \log \frac{r_2 r_3}{r_1 r_4} = \frac{2q}{K} \log_\epsilon \frac{aa'}{\alpha\alpha'},$$

since the relations proved by Fig. 521 are in Fig. 522,

$$\frac{r_2}{r_1} = \frac{A'P}{AP} = \frac{OP}{OA} = \frac{a}{\alpha} \text{ and } \frac{r_3}{r_4} = \frac{AP'}{A'P'} = \frac{O'P'}{O'A'} = \frac{a'}{\alpha'}.$$

The electrostatic capacity with respect to each other of the two

parallel cylindrical conductors given in C. G. S. electrostatic units is therefore

$$C_a = \frac{Q_a}{E_a} = \frac{Kl}{2 \log_\epsilon \frac{aa'}{\alpha\alpha'}},$$

where l is the length of each cylinder in centimeters.

It is desirable to convert this formula into one involving only the radii a and a' and the center to center distance ∂ of the cylinders, by eliminating the distances α and α'. This may be easily done, since $\alpha + \alpha' + d = \partial$. Then putting

$$\frac{\partial^2 - a^2 - (a')^2}{2\,aa'} = b$$

gives

$$\frac{aa'}{\alpha\alpha'} = b \pm \sqrt{b^2 - 1}.$$

Whence

$$C_a = \frac{Kl}{2 \log_\epsilon (b \pm \sqrt{b^2 - 1})}.$$

When the cylinders are eccentric, as in the case under consideration, the positive sign applies before the radical in the denominator of the last expression.

To bring this to farads per 1000 feet of conductor in the circuit and using common logarithms, this must be multiplied by 7.35×10^{-9}, so that

$$C_1 = \frac{7.35\,K10^{-9}}{2 \log_{10} (b + \sqrt{b^2 - 1})} = \frac{3.68\,K10^{-9}}{\log_{10} (b + \sqrt{b^2 - 1})}.$$

When the two cylinders are of the same diameter, $a = a'$, whence

$$b = \frac{1}{2}\left(\frac{\partial}{a}\right)^2 - 1,$$

and

$$b + \sqrt{b^2 - 1} = \left[\frac{\partial}{2\,a} + \sqrt{\left(\frac{\partial}{2\,a}\right)^2 - 1}\right]^2.$$

Therefore

$$C_2 = \frac{3.68\,K10^{-9}}{\log_{10}\left(\frac{\partial}{2\,a} + \sqrt{\left(\frac{\partial}{2\,a}\right)^2 - 1}\right)^2} = \frac{1.84\,K10^{-9}}{\log_{10}\left(\frac{\partial}{2\,a} + \sqrt{\left(\frac{\partial}{2\,a}\right)^2 - 1}\right)}.$$

When the distance apart (center to center) of the conductors is large compared with the diameter of each conductor, as for

instance if this ratio is as great as is usual with overhead line wires, $\sqrt{\left(\frac{\partial}{2a}\right)^2 - 1}$ approximates to $\frac{\partial}{2a}$, and a closely approximate formula for the capacity in farads per 1000 feet of conductor may be written

$$C_2 = \frac{1.84\,K\,10^{-9}}{\log_{10}\frac{\partial}{a}}.$$

When the conductors are in air, the dielectric constant is unity; therefore for overhead line wires the capacity in farads, measured from conductor to conductor, per thousand feet of conductor, is usually obtained from the expression

$$C_2 = \frac{1.84 \times 10^{-9}}{\log_{10}\frac{\partial}{a}}.$$

Since $\log_e(b + \sqrt{b^2 - 1}) = \cosh^{-1} b$,* the exact formula for the capacity in farads per thousand feet of conductor may be written

$$C_1 = \frac{16.94\,K10^{-9}}{\cosh^{-1} b},$$

which is easily computed with the aid of a table of the functions of hyperbolic trigonometry, such as are contained in the Smithsonian Mathematical Tables. In this formula $b = \frac{\partial^2}{2a^2} - 1$ when the conductors are of equal diameters, in which case, putting c for the ratio of the center to center distance to the diameter of a conductor, $c = \frac{\partial}{2a}$ and $b = 2c^2 - 1$.

Therefore, $b + \sqrt{b^2 - 1} = 2c^2 + 2c(c^2 - 1)^{\frac{1}{2}} - 1 = (c + \sqrt{c^2 - 1})^2$, and the capacity in farads per thousand feet of conductor is

$$C_2 = \frac{8.47\,K10^{-9}}{\cosh^{-1} c}.$$

Pender and Osborne have shown that for parallel cylindrical conductors removed from the influence of other conductors, the

* McMahon, *Hyperbolic Functions*, Art. 19.

distribution of the charge around the cross-section which is required to keep the surfaces equipotential surfaces, gives a surface density of the charge on each conductor which conforms to an ellipse with its major axis in the line joining the centers of the two conductors.*

The capacity with respect to ground of an overhead conductor may be deduced from the reasoning applied to two parallel conductors. Suppose the conductor of diameter $2a$ is erected at a distance h above the surface of the earth and substantially parallel therewith, the distance h being many times larger than a. The capacity of 1000 feet of this conductor, with respect to a hypothetical equal conductor or image at a distance h below the surface of the earth, would be

$$C = \frac{3.68 \times 10^{-9}}{\log_{10} \frac{2h}{a}},$$

in which K becomes unity because the dielectric is air. But the surface of the earth corresponds with the plane of zero potential which is halfway between two conductors, and the difference of potential between the conductor concerned and the earth with a given charge Q on the conductor is one half as great as the difference of potential between the conductor and its image. Therefore the capacity of the conductor with respect to earth in farads per thousand feet of length is twice the foregoing, or

$$C = \frac{7.35 \times 10^{-9}}{\log_{10} \frac{2h}{a}}.$$

The foregoing expressions for the capacity of cylindrical conductors in concentric and excentric relations are directly applicable to the conditions of underground and overhead wires. The formulas applying to cylinders one inside the other apply not only to concentric cables, but are also useful for computing the capacity between the conductor or conductors of a lead-sheathed underground cable and the sheath. The formulas are developed for the relations of two conductors with respect

* Pender and Osborne, The Electrostatic Capacity between Equal Parallel Wires, *Electrical World*, 1910, Vol. 56, p. 667.

to each other, but the same reasoning may be extended to multiple conductor cables.*

The capacity current set up by each harmonic of voltage impressed between the conductors is

$$I = 2\pi f C E l,$$

where E is the effective value of the harmonic and l is the length of the wire in thousands of feet. The total charging current is the square root of the sum of the squares of all the harmonic currents. The minimum charging current therefore occurs when the impressed voltage is sinusoidal.

When dealing with polyphase circuits, the capacity between conductors is computed by the foregoing formulas, just as for single-phase circuits; but the disturbing effect on the electrostatic field of the additional conductors and the relation of the mesh current to the star current must be borne in mind when computing the charging current flowing in a conductor. Considering a three-phase circuit with the conductors at the corners of an equilateral triangle and balanced voltage, the electrostatic capacity of one of the conductors with respect to the other two is four thirds the capacity with respect to only one of the others; but the voltage impressed on this capacity is the voltage reaction from the first conductor to the average or mid-voltage of the other two, which is equal to $\frac{\sqrt{3}}{2}E$ if E is the line voltage. Therefore, in a balanced three-phase circuit with equal capacity for each pair of conductors, the charging current flowing in a conductor is equal to $\frac{2}{\sqrt{3}}(2\pi f C E l_1)$, when C is the capacity per pair of conductors, E is the delta line voltage, and l_1 is the length in thousands of feet of each conductor of the three-phase circuit. This gives the same charging circuit per conductor of the three-phase circuit as would be obtained by connecting, from each conductor to the neutral point, a condenser of capacity $2\,C$ microfarads per 1000 feet of the line (not per 1000 feet of wire), in which case the voltage impressed on each condenser would be $\frac{E}{\sqrt{3}}$.

* Russell, *Alternating Currents*, Vol. I. Heaviside, *Electrical Papers*, Vol. I, p. 42.

216. Distributed Resistance, Self-inductance, Leakage Conductance, and Electrostatic Capacity. — The relations of resistance, self-inductance, and capacity which have heretofore been considered in this book have associated the quantities as though they could be segregated in a circuit, and this is a sufficiently accurate hypothesis for most conditions of low voltage commercial circuits. In long distance transmission lines of very high voltage, and in some high voltage machines, however, the distributed quantities bring in additional effects which will be briefly discussed here. The complete discussion of these phenomena belongs more particularly to the special field of power transmission, and not within the scope of this work, and the following discussion therefore only gives the fundamental relations.

In the case of two parallel uniform conductors, it is obvious that each element of length of each conductor possesses a certain self-inductance, and likewise that each element of length of each conductor possesses a certain electrostatic capacity measured between the conductors. These elements of self-inductance and capacity being uniformly distributed in the case of the uniform conductors, the distributed self-inductance and capacity bear relations to each other in transmitting power which are analogous to the mass and compressibility of a vibrating uniform metal rod or string.

The drop of voltage along an element of length of a conductor when a variable current flows is

$$de = Ridx + Ldx\frac{di}{dt},$$

where R and L are the resistance in ohms and self-inductance in henrys of the circuit per unit (such as a centimeter, a thousand feet, or a mile) of length of conductor and dx is an element of the length. Therefore with equal uniform conductors composing the circuit, the rate of change of voltage between the conductors as a function of distance along the circuit, that is, the space-rate of change of voltage between the two conductors, is

$$\frac{de}{dx} = Ri + L\frac{di}{dt}. \qquad (1)$$

Also, the loss of current caused by the uniformly distributed capacity, and by leakage from conductor to conductor through

the insulation and by convection or corona effects, likewise assumed to be uniform along the circuit, is for each element

$$di = Gedx + C\,dx\frac{de}{dt},$$

where G and C are the conductance in mhos of the insulation* and the capacity in farads per unit of length of conductor. Therefore, the space-rate of change of current with distance along a conductor is

$$\frac{di}{dx} = Ge + C\frac{de}{dt}. \tag{2}$$

Differentiating equations (1) and (2) with respect to time and then with respect to distance gives the following four differential equations of the second order:

$$\frac{d^2e}{dx \cdot dt} = R\frac{di}{dt} + L\frac{d^2i}{dt^2},$$

$$\frac{d^2i}{dx \cdot dt} = G\frac{de}{dt} + C\frac{d^2e}{dt^2},$$

$$\frac{d^2e}{dx^2} = R\frac{di}{dx} + L\frac{d^2i}{dx \cdot dt},$$

$$\frac{d^2i}{dx^2} = G\frac{de}{dx} + C\frac{d^2e}{dx \cdot dt}.$$

By eliminating $\frac{d^2i}{dx \cdot dt}$ from the second and third of these and substituting the value of $\frac{di}{dx}$ given in equation (2), there results

$$LC\frac{d^2e}{dt^2} + (RC + LG)\frac{de}{dt} + RGe = \frac{d^2e}{dx^2}. \tag{3}$$

A similar process of elimination of $\frac{d^2e}{dx \cdot dt}$ from the first and fourth of the group of four equations and substituting the value of $\frac{de}{dx}$ from equation (1) gives

$$LC\frac{d^2i}{dt^2} + (RC + LG)\frac{di}{dt} + RGi = \frac{d^2i}{dx^2}. \tag{4}$$

If the impressed voltage is assumed to be sinusoidal at the generator with a frequency corresponding to $2\pi f = \omega$, equa-

* The insulation conductance in mhos is the reciprocal of the insulation resistance measured in ohms.

tion (3) may be solved in the following form, when neglecting the transient state,

$$e_x = M[\epsilon^{-ax}\{A \sin(\omega t - bx + \gamma)\} + \epsilon^{-a(l-x)}\{B \sin(\omega t - b(l-x) + \gamma)\}]. \quad (5)$$

This shows that the voltage measured between conductors at any point on the line is equal to the sum of a wave decreasing logarithmically as the length x of conductors between the generator and point is increased and a wave decreasing logarithmically as the length $l - x$ of conductors from the receiver at the far end of the line is increased. The former wave may be looked upon as a disturbance or wave sent out from the generator and the latter as the return wave produced by reflection of the original wave at the far end of the line. In this formula l represents the total length of conductor in the chosen units and M, A, B, a, b, and γ are quantities determined by the resistance, self-inductance, leakage conductance, and electrostatic capacity of the line and the frequency of the impressed alternating voltage. If the circuit is of infinite length, the second term of the right-hand member of equation (5) disappears for all finite values of x.

If all of the resistance, self-inductance, leakage conductance, and capacity in the circuit have constant values which are independent of the current or voltage, the generator current after the expiry of the transient state must as a consequence of the symmetry of the equations be sinusoidal when the generator voltage is sinusoidal, but the trigonometrical terms in the solution for current corresponding to equation (5) contain an additional term representing the angle of lag of the current.

Equation (5) shows that the voltage at any point is a sine function with respect to time, since at a fixed point, x is constant and t is the only variable. The period of this function is $\frac{2\pi}{\omega} = \frac{1}{f}$ seconds, which is the period of the generator voltage. On the other hand, the equation shows that the voltage at a given instant of time varies along the line as a sine function of x multiplied by a logarithmic decrement, and that the wave length in the units of length in which x is measured is

$$\lambda = \frac{2\pi}{b} \text{ units.}$$

That is, through the range of values of x extending from zero units of conductor length (kilometers, miles, or otherwise as the case may be) to $\frac{2\pi}{b}$ units of length, the voltage goes through a complete cycle of values which is of the form of a sinusoid multiplied by a logarithmic factor. The cycle is repeated along each additional $\frac{2\pi}{b}$ units in length of conductor.

Since the voltage and current at any fixed point go through sinusoidal cycles with a frequency $\frac{\omega}{2\pi}$ cycles per second, and points distant $\frac{2\pi}{b}$ units of length from each other go through their phases simultaneously, it is manifest that the voltage and current cycles are in the nature of disturbances which travel along the conductors at a rate which traverses $\frac{2\pi}{b}$ units of distance in $\frac{2\pi}{\omega}$ seconds. Hence, the velocity of propagation of the wave in distance traveled per second is

$$v = \frac{\omega}{b} = f\lambda.$$

The solution of equations (3) and (4) which are most convenient for purposes of computation are obtained in terms of the functions of hyperbolic trigonometry, as was early pointed out by Kennelly and McMahon.* When the voltage and current at any point on the line are sinusoidal functions of time, the vector relations of e and i with respect to their derivatives for a given value of t may be written

$$e = e_m \sin \omega t,$$

$$\frac{de}{dt} = e_m \omega \cos \omega t = j\omega e, \tag{6}$$

$$\frac{d^2e}{dt^2} = -e_m \omega^2 \sin \omega t = -\omega^2 e; \tag{7}$$

and correspondingly for the current,

$$i = i_m \sin (\omega t - \theta),$$

* McMahon, *Hyperbolic Functions*, Art. 37. Pender has worked out computing formulas, using the solution with circular trigonometry. See *Electrical World*, Vol. 56, p. 667.

$$\frac{di}{dt} = j\omega i, \tag{6'}$$

$$\frac{d^2i}{dt^2} = -\omega^2 i. \tag{7'}$$

Substituting these values of the derivatives in equations (3) and (4) gives a vector equation which is independent of t and in which the values of e and i may therefore be taken as either maximum or effective vector values,

$$\frac{d^2\bar{E}}{dx^2} = (-\omega^2 LC + j\omega(RC + LG) + RG)\bar{E}$$
$$= (R + j\omega L)(G + j\omega C)\bar{E};$$

and
$$\frac{d^2\bar{I}}{dx^2} = (R + j\omega L)(G + j\omega C)\bar{I}.$$

Writing P for $(R + j\omega L)^{\frac{1}{2}}(G + j\omega C)^{\frac{1}{2}}$ in the last two equations gives

$$\frac{d^2\bar{E}}{dx^2} = P^2\bar{E}$$

and
$$\frac{d^2\bar{I}}{dx^2} = P^2\bar{I}.$$

The solutions of these equations are

$$\bar{E} = A \cosh Px + B \sinh Px, \tag{8}$$

and
$$\bar{I} = A' \cosh Px + B' \sinh Px.* \tag{9}$$

To find the constants, put $x = 0$, when $\bar{E}$ and $\bar{I}$ become the generator voltage $\bar{E}_0$ and generator current $\bar{I}_0$. Therefore, since $\sinh Px = 0$ and $\cosh Px = 1$ when $x = 0$, $A = \bar{E}_0$ and $A' = \bar{I}_0$. Also, according to equations (1) and (6'),

$$\frac{d\bar{E}}{dx} = -(R + j\omega L)\bar{I} = -(R + j\omega L)(A' \cosh Px + B' \sinh Px),$$

the negative sign being required here because of the vector relations, $(R + j\omega L)\bar{I}$ being a counter voltage measured in the opposite direction from x; and by differentiation of equation (8),

$$\frac{d\bar{E}}{dx} = AP \sinh Px + BP \cosh Px.$$

Hence, $AP \sinh Px = -(R + j\omega L)B' \sinh Px,$

and $BP \cosh Px = -(R + j\omega L)A' \cosh Px.$

* McMahon, *Hyperbolic Functions*, Art. 14.

Therefore

$$B = -\frac{R+j\omega L}{P}A' = -\frac{\sqrt{R+j\omega L}}{\sqrt{G+j\omega C}}A' = -\bar{Z}_l A' = -\bar{Z}_l\bar{I}_0,$$

and $$B' = -\frac{P}{R+j\omega L}A = -\frac{\sqrt{G+j\omega C}}{\sqrt{R+j\omega L}}A = -\frac{A}{\bar{Z}_l} = -\frac{\bar{E}_0}{\bar{Z}_l},$$

$\bar{Z}_l$ being put for the impedance of the line per mile of conductor, $(R+j\omega L)^{\frac{1}{2}}(G+j\omega C)^{-\frac{1}{2}}$.
Consequently, equations (8) and (9) become

$$\bar{E} = \bar{E}_0 \cosh Px - \bar{Z}_l\bar{I}_0 \sinh Px,$$

and

$$\bar{I} = \bar{I}_0 \cosh Px - \frac{\bar{E}_0}{\bar{Z}_l}\sinh Px.$$

If x is measured from the receiver, when $x = 0$, $A = \bar{E}_l$ and $A' = \bar{I}_l$, in which $\bar{E}_l$ and $\bar{I}_l$ are the voltage and current at the receiver. Also in this case $\frac{d\bar{E}}{dx} = +(R+j\omega L)\bar{I}$, and therefore $B = \bar{Z}_l\bar{I}_l$ and $B' = \frac{\bar{E}_l}{\bar{Z}_l}$. Hence the equations representing the voltage and current at any distance x from the receiver, in terms of the voltage and current at the receiver, are like the foregoing except that $\bar{E}_l$ and $\bar{I}_l$ are substituted for $\bar{E}_0$ and $\bar{I}_0$ and the negative sign before the second term of the right-hand member changes to a positive sign.

Since

$$P = (R+j\omega L)^{\frac{1}{2}}(G+j\omega C)^{\frac{1}{2}} = \{RG - \omega^2 LC + j\omega(RC+LG)\}^{\frac{1}{2}},$$

the foregoing equations contain hyperbolic functions of complex arguments, which it is desirable to convert into functions of real arguments.* Putting

$$\{RG - \omega^2 LC + j\omega(RC+LG)\}^{\frac{1}{2}} = \alpha + j\beta,$$

gives

$$\bar{E} = \bar{E}_0 \cosh(\alpha+j\beta)x - \bar{Z}_l\bar{I}_0 \sinh(\alpha+j\beta)x,$$

$$\bar{I} = \bar{I}_0 \cosh(\alpha+j\beta)x - \frac{\bar{E}_0}{\bar{Z}_l}\sinh(\alpha+j\beta)x.$$

* This is for the purpose of convenience in computation with the mathematical tables that are ordinarily available, but tables are being developed by Kennelly and others which may make it more practicable to compute directly from the hyperbolic functions with complex arguments.

Since

$$\sinh(\alpha x + j\beta x) = \sinh \alpha x \cos \beta x + j \cosh \alpha x \sin \beta x^*$$

and

$$\cosh(\alpha x + j\beta x) = \cosh \alpha x \cos \beta x + j \sinh \alpha x \sin \beta x,$$

the foregoing convert to

$$\bar{E} = \bar{E}_0 \cosh \alpha x \cos \beta x - \bar{Z}_l \bar{I}_0 \sinh \alpha x \cos \beta x + j(\bar{E}_0 \sinh \alpha x \sin \beta x - \bar{Z}_l \bar{I}_0 \cosh \alpha x \sin \beta x), \quad (10)$$

$$\bar{I} = \bar{I}_0 \cosh \alpha x \cos \beta x - \frac{\bar{E}_0}{\bar{Z}_l} \sinh \alpha x \cos \beta x + j(\bar{I}_0 \sinh \alpha x \sin \beta x - \frac{\bar{E}_0}{\bar{Z}_l} \cosh \alpha x \sin \beta x). \quad (11)$$

These may be computed by the ordinary tables of functions of circular and hyperbolic trigonometry, such as the Smithsonian Mathematical Tables. A twenty-inch slide rule and a set of tables make computation by this formula relatively easy.

The values of α and of β may be computed from the original expression which it has been convenient to represent by $\alpha + j\beta$, that is,

$$\{RG - \omega^2 LC + j\omega(RC + LG)\}^{\frac{1}{2}} = \alpha + j\beta,$$

from which comes

$$RG - \omega^2 LC + j\omega(RC + LG) = (\alpha + j\beta)^2,$$

$$RG - \omega^2 LC + j\omega(RC + LG) = \alpha^2 - \beta^2 + 2j\alpha\beta,$$

and

$$\alpha^2 - \beta^2 = RG - \omega^2 LC,$$

$$2j\alpha\beta = j\omega(RC + LG).$$

Whence, taking positive and real values only, as being applicable to the premises here under consideration,

$$\alpha = \frac{1}{\sqrt{2}}[\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} + (RG - \omega^2 LC)]^{\frac{1}{2}}$$

and

$$\beta = \frac{1}{\sqrt{2}}[\sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} - (RG - \omega^2 LC)]^{\frac{1}{2}}.$$

Of these terms, α is the coefficient a associated with x in the exponential terms of equation (5). It is called the **Attenuation constant**, since it determines the rate of decrement with distance due to the exponential terms of the wave formula. The term

* McMahon, *Hyperbolic Functions*, Arts. 28-30.

β is the coefficient b associated with x in the sine terms of equation (5), and is called by Fleming the **Wave length constant** since the length of the space wave of voltage or current is equal to $\frac{2\pi}{\beta}$. The unit in which the wave length is measured depends on the unit of length corresponding to R, L, G, and C. If these are measured in ohms, henrys, mhos, and farads per mile of conductor, the last two being measured from conductor to neutral plane, then the space wave length is

$$\lambda = \frac{2\pi}{\beta} \text{ miles of conductor.}$$

The velocity of propagation of the space wave is then

$$v = \frac{\omega}{\beta} = f\lambda \text{ miles of conductor per second.}$$

The quantity $P = \alpha + j\beta$ has been called the **Propagation constant** of a circuit, since it is a complex quantity made up of the attenuation constant and the wave length constant.

The imaginary roots of the equations on the preceding page lead to solutions representing oscillatory phenomena.

To obtain the voltage and current at the receiving end of a transmission line when the current and voltage at the sending end are known, it is only necessary to substitute the total length of a line conductor in the place of x in equations (10) and (11), giving,

$$\bar{E}_1 = \bar{E}_0 \cosh \alpha l \cos \beta l - \bar{Z}_l \bar{I}_0 \sinh \alpha l \cos \beta l + j(\bar{E}_0 \sinh \alpha l \sin \beta l - \bar{Z}_l \bar{I}_0 \cosh \alpha l \sin \beta l), \tag{12}$$

$$\bar{I}_1 = \frac{\bar{E}_1}{\bar{Z}_1} = \bar{I}_0 \cosh \alpha l \cos \beta l - \frac{\bar{E}_0}{\bar{Z}_l} \sinh \alpha l \cos \beta l + j\left(\bar{I}_0 \sinh \alpha l \sin \beta l - \frac{\bar{E}_0}{\bar{Z}_l} \sinh \alpha l \cos \beta l\right), \tag{13}$$

$\bar{E}_1$, $\bar{I}_1$, and $\bar{Z}_1$ being the voltage, current, and impedance at the receiver.

Eliminating $\bar{I}_0$ from (12) and (13) gives

$$\bar{I}_1 = \frac{\bar{E}_0}{\bar{Z}_l \sinh \alpha l \cos \beta l + \bar{Z}_1 \cosh \alpha l \cos \beta l + j(\bar{Z}_l \cosh \alpha l \sin \beta l + \bar{Z}_1 \sinh \alpha l \sin \beta l)}, \tag{14}$$

and the ratio of $\bar{I}_1$ to $\bar{I}_0$ is

$$\frac{\bar{I}_1}{\bar{I}_0} = \frac{Z_l}{\bar{Z}_l \cosh \alpha l \cos \beta l + \bar{Z}_1 \sinh \alpha l \cos \beta l + j(\bar{Z}_l \sinh \alpha l \sin \beta l + \bar{Z}_1 \cosh \alpha l \sin \beta l)}, \tag{15}$$

from which the generator voltage and current may be computed if the characteristics of the line and the voltage, current, and impedance of the receiver are given.

It is to be observed that $\bar{Z}_l$ is quite different from $\bar{Z}_0 = \frac{\bar{E}_0}{\bar{I}_0}$. The former depends only on the frequency of the current and the electrical constants of the line conductors per unit length. It is independent of the length of the line and of the character of receiver into which the line delivers power. The impedance $\bar{Z}_0$, on the other hand, depends on the length of the line and the character of the receiver in addition to being dependent on the constants of the line per unit length. For a line of infinite length $Z_l = Z_0$, but for an ordinary line they bear a complex relation to each other. A relation that is sometimes useful to know is the ratio of the generator current when a short circuit occurs at the receiver to the generator current in normal working. The ratio is

$$\frac{\bar{I}'_0}{\bar{I}_0} = \frac{\tanh(Pl + \theta)}{\tanh Pl}$$

in which I'_0 and I_0 are the generator current respectively with the receiver end of the line short-circuited and in normal working, and $\theta = \tanh^{-1}\frac{Z_1}{Z_l}$.

The effect of either a short circuit or an open circuit at any point is readily observed from these equations. In the case of a short circuit occurring with a length x of conductor between it and the generator, $\bar{E}$ becomes zero at that point, and

$$\bar{E}_0 \cosh \alpha x \cos \beta x - \bar{Z}_l \bar{I}_0 \sinh \alpha x \cos \beta x + j(\bar{E}_0 \sinh \alpha x \sin \beta x - \bar{Z}_l \bar{I}_0 \cosh \alpha x \sin \beta x) = 0.$$

Therefore

$$\bar{I}_0 = \frac{\bar{E}_0(\cosh \alpha x \cos \beta x + j \sinh \alpha x \sin \beta x)}{\bar{Z}_l(\sinh \alpha x \cos \beta x + j \cosh \alpha x \sin \beta x)},$$

and

$$\bar{I}_x = \frac{\bar{Z}_l \bar{I}_0}{\bar{Z}_l \cosh \alpha x \cos \beta x + \bar{Z}_1 \sinh \alpha x \cos \beta x + j(\bar{Z}_l \sinh \alpha x \sin \beta x + \bar{Z}_1 \cosh \alpha x \sin \beta x)}$$

$$= \frac{\bar{E}_0}{Z_l(\sinh \alpha x \cos \beta x + j \cosh \alpha x \sin \beta x)}.$$

In case of an open circuit at distance x from the generator $\bar{I}$ becomes zero at that point, and

$$\bar{I}_0 \cosh \alpha x \cos \beta x - \frac{\bar{E}_0}{\bar{Z}_l} \sinh \alpha x \cos \beta x$$

$$+ j\left(\bar{I}_0 \sinh \alpha x \sin \beta x - \frac{\bar{E}_0}{\bar{Z}_l} \cosh \alpha x \sinh \beta x\right) = 0.$$

Therefore

$$\bar{I}_0 = \frac{\bar{E}_0(\sinh \alpha x \cos \beta x + j \cosh \alpha x \sin \beta x)}{Z_l(\cosh \alpha x \cos \beta x + j \sinh \alpha x \sin \beta x)},$$

and

$$\bar{E}_x = \frac{\bar{E}_0}{(\cosh \alpha x \cos \beta x + j \sinh \alpha x \sin \beta x)}.$$

In an infinitely long cable $\bar{I}_x = \bar{I}_0(\cosh Px - \sinh Px) = \bar{I}_0 \epsilon^{-Px}$, of which the real part is $I_0 \epsilon^{-\alpha x}$ and therefore $I_x = I_0 \epsilon^{-\alpha x}$.

As pointed out in an earlier paragraph, the space wave length measured in conductor miles is $\frac{2\pi}{\beta}$ when the resistance, inductance, insulation conductance, and capacity are given in terms of ohms, henrys, mhos, and farads per mile of conductor. For conductors of the ordinary sizes and spacings used in the long distance transmission of large amounts of power, and currents of the ordinary frequencies used for that purpose, a wave length is many hundreds of miles when measured along the line. For instance, an ungrounded 3-phase overhead line of No. 0 copper wires spaced ten feet apart has resistance of approximately .52 ohm per mile of conductor, self-inductance of approximately .0022 henry per mile of conductor, leakage conductance negligible at ordinary altitudes if the line voltage is below 125,000 volts, and capacity of $.0078 \times 10^{-6}$ farads per mile of conductor. This gives for the value of the wave length constant when $f = 25$ cycles per second or $\omega = 157$ radians per second,

$$\beta = .00077$$

and for the wave length

$$\lambda = \frac{2\pi}{\beta} = 8150 \text{ miles of conductor.}$$

The wave length in miles of line is one half as great as the length in miles of conductor, since the constants are all given on the assumption of an outgoing conductor and return. No overhead transmission line has yet approached this length, or even the length of 1925 miles, which are the miles of line required to give a complete wave length if the frequency were 60 cycles per second on the foregoing line.

Consequently, it is possible to neglect the formulas which recognize the effects of distributed resistance, self-inductance, leakage, and capacity when computing long distance power transmission lines of lengths within the limits of existing practice, and to make the computations on the assumption that the fall of potential is uniform along the conductors. Leakage being usually negligible, the effects of the electrostatic capacity may be approximated by assuming the capacity of the conductors concentrated at certain points instead of distributed. For short lines it is sufficient to consider the capacity all concentrated at the middle of the line; but for lines having lengths approaching the limits of the existing longest lines, it is sometimes essential to make a closer approximation to truth, and the capacity may be assumed to be concentrated in three divisions, one sixth of the whole capacity at each end of the line and the remaining two thirds at the middle of the line.

The transmission of large amounts of power over underground cables has not yet reached lengths of circuit approaching the lengths in individual overhead lines, and the voltage impressed on cables is usually less than 20,000 volts. The self-inductance of the conductors in the usual underground circuit is much less than the self-inductance of a corresponding overhead circuit, on account of the closer spacing of the conductors; but the electrostatic capacity of the cabled conductors is much greater than when the circuit is overhead, on account of the closer spacing of the conductors and the relatively high dielectric constant of the solid insulating material. An ungrounded 3-phase cable of three symmetrically spaced No. 0 wires with rubber insulation having $\frac{5}{16}$ inch thickness of wall around each conductor has a resistance of approximately .52 ohm per mile of conductor, a self-inductance of approximately .00065 henry per mile of conductor, leakage negligible, and a capacity of approximately $.0545 \times 10^{-6}$ farads per mile of

conductor. The wave length constant when the frequency is twenty-five cycles per second is therefore $\beta = .001645$ and the wave length in miles of conductor is

$$\lambda = \frac{2\pi}{\beta} = 3820.$$

This corresponds with 1910 miles in length of cable. This far exceeds the lengths of circuit ordinarily found in underground cables for heavy electric power transmission, but in the case of the longer underground cables used with voltages higher than a few thousand volts, it is desirable to utilize the exact formulas for determining the effects of capacity.

In the case of long distance telephone service the situation is different on account of the higher frequency of the current harmonics which it is desirable to retain without great attenuation, and also because it is desirable to adjust the relation of resistance, self-inductance, and capacity so that the attenuation and the velocity of propagation shall be substantially the same for current harmonics of a considerable range of frequencies (such as from 200 periods per second to 1800 periods per second), in order that the intelligibility of the irregular speech waves may be maintained. This requires what is called a **Distortionless circuit** and the exact formulas are needed in the computation of such circuits.

When leakage is negligible, the values of α and β heretofore given reduce to

$$\alpha = \sqrt{\frac{\omega C}{2}}(\sqrt{R^2 + \omega^2 L^2} - \omega L)^{\frac{1}{2}},$$

$$\beta = \sqrt{\frac{\omega C}{2}}(\sqrt{R^2 + \omega^2 L^2} + \omega L)^{\frac{1}{2}}.$$

If the inductive reactance ωL is at the same time negligibly small compared with the resistance, these reduce to

$$\alpha = \beta = \sqrt{\frac{\omega C R}{2}}.$$

When ωL is quite large compared with R, $\sqrt{R^2 + \omega^2 L^2} = \omega L + \frac{R^2}{2\omega L}$ approximately, since the square root of $a + b$ is very nearly

equal to $\sqrt{a}+\frac{b}{2\sqrt{a}}$ when a is large compared with b, and when leakage is negligible under these circumstances,

$$\alpha=\frac{R}{2}\sqrt{\frac{C}{L}}.$$

Also, leakage being negligible, if $2\omega L$ is very large compared with R^2,

$$\beta=\omega\sqrt{CL},$$

and

$$v=\frac{\omega}{\beta}=\frac{1}{\sqrt{LC}}.$$

Whence, by making ωL for the various harmonics of a periodic current very large compared with R, the attenuation and the velocity of wave propagation of the harmonics are both made independent of the frequency of the harmonics and dependent solely on R, L, and C.

The latter condition may also be accomplished when leakage is not negligible, provided $LG=RC$, except that attenuation and velocity of propagation are dependent upon R, L, G, and C instead of R, L, and C, only. In this case, the original expressions for α and β reduce to

$$\alpha=\frac{1}{\sqrt{2}}[\sqrt{(RG+\omega^2LC)^2+\omega^2(LG-RC)^2}+(RG-\omega^2LC)]^{\frac{1}{2}}$$

$$=\sqrt{RG},$$

$$\beta=\frac{1}{\sqrt{2}}[\sqrt{(RG+\omega^2LC)^2+\omega^2(LG-RC)^2}-(RG-\omega^2LC)]^{\frac{1}{2}}$$

$$=\omega\sqrt{LC}.$$

The importance of this sort of treatment in the case of a long distance telephone line may be recognized from the facts that an overhead line of No. 8 B. & S. gauge wire spaced twelve inches apart has resistance of 3.3 ohms per mile of conductor, self-inductance of 2×10^{-3} henrys per mile of conductor, capacity of $.004\times10^{-6}$ farads per mile of conductor, and the insulation resistance ought to be in the order of some megohms per mile of conductor; and that it is desirable to maintain the attenuation and velocity of propagation substantially the same for harmonics of the voice currents within the range of 200 to

1800 cycles per second. The foregoing constants give wave lengths which are but fractional compared with the length of many long distance telephone circuits.

In the case of telephone cables, the relations are even more impressive. With the usual lead-covered cables with loose dry paper wrapping for the conductors, the constants for No. 19 B. & S. gauge conductors are per mile of conductor approximately 45 ohms, 5×10^{-3} henrys, $.035 \times 10^{-6}$ farads, and insulation resistance of one half a megohm or more.

This gives a wave length of less than sixty miles of cable for the harmonic corresponding to an angular velocity, ω, of 5000 radians per second ($f = 800$ periods per second, approximately), which is about a mean value for the harmonics of voice currents that are necessary to retain an approximately unaltered relative magnitude during transmission in order that the transmitted speech may be fully articulated by the receiver. For the harmonic of frequency 1800 periods per second, the wave length on this cable is less than thirty-five miles. Since telephony through underground cables is now practiced over many tens of miles, the transmission computations must obviously be executed with the formulas which include all the features of the wave transmission.*

The foregoing formulas relate to uniform circuits. They may be applied to each uniform part of a non-uniform circuit, provided the voltage impressed on the part concerned and the vector impedance of the succeeding part are known; but changes of character, especially if they are abrupt, cause reflection points which greatly alter the distribution of voltage compared with the distribution of voltage over a uniform section of the same length.

217. Analogies of the Electric Circuit. — The deductions of Chapter VI have shown very clearly that concentrated self-inductance and concentrated capacity in a circuit may be made to neutralize each other when a sinusoidal voltage is applied to the circuit, and the self-inductance and capacity are constant. In this case the self-inductance and capacity act in opposition, so that at each instant energy is being restored or released in

* The simpler problems of long distance telephone transmission are dealt with admirably by Fleming in a book entitled *The Propagation of Electric Currents in Telephone and Telegraph Conductors.*

the magnetic field at exactly the same rate as energy is being released or stored in the charge of the condenser. The self-inductance and capacity may therefore be said to supply each other's demands, and the power delivered by a generator to the circuit may be wholly utilized in doing work on a non-reactive receiver such as incandescent lamps and in heating the wires of the circuit. The vector power which is transferred back and forth between the self-inductance and capacity may be many times as great as that given to the circuit by the generator, and volt-amperes at the terminals of the self-inductance and of the condenser must then be proportionally greater than the volt-amperes delivered by the generator.

This condition can fully exist only when the frequency of the impressed voltage produces the relation $2\pi fL = \frac{1}{2\pi fC}$,* or $LC = \frac{1}{4\pi^2 f^2}$, and the natural period of the circuit is $2\pi\sqrt{LC}$.† From the condition $2\pi fL = \frac{1}{2\pi fC}$ it is seen that $\frac{1}{f} = 2\pi\sqrt{LC}$. The natural period of discharge of the circuit is therefore equal to the period of the cycles of impressed voltage, or, as may be said, equal to the rate of the electrical vibrations impressed on the circuit by the generator. This relation between the vibrations of the line and of the generator is similar to that of a vibrating tuning fork or string and a sounding board when they are in resonance, and therefore the term **Electrical resonance** has, on account of the analogy, been applied to the electric circuit. *An electric circuit is said to be in resonance with an impressed voltage when the natural period of the circuit is equal to the period of the impressed voltage.* When this condition exists, the maximum current is caused to flow in the circuit by the application of a given impressed voltage, the value of the current in a resonant circuit from which no external work is supplied being

$$I = \frac{E}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}} = \frac{E}{R}, \text{ since } 2\pi fL = \frac{1}{2\pi fC}.$$

If the self-inductance, capacity, and resistance are in series in the circuit, it is evident that when the periods of the circuit

* Art. 69 (4). † Art. 58, Case (2).

and the impressed frequency are in resonance, the voltage between the terminals of the capacity $\left(=\frac{I}{2\pi fC}\right)$ is a maximum, since the circuit current is a maximum; and the same is true of the voltage between the terminals of the inductance. If either the frequency, the self-inductance, or the capacity is changed in value, the value of the current falls, and the condenser voltage falls, unless the other elements are changed in value in such a way as to continue the condition of resonance. A condenser in a resonant circuit may be used as a transformer of voltage by connecting non-reactive apparatus across its terminals, as has been suggested by Blakesley, Loppe et Bouquet, Pupin, and others.

If the self-inductance and capacity are in parallel in the circuit, the voltage at their terminals cannot be greater than that impressed upon the circuit minus the loss of voltage in the lead wires; but when the circuit is resonant, the current furnished to the circuit by the generator is at a minimum which is equal to the power consumed by the circuit divided by the impressed voltage, while the current transferred between the inductance and capacity is a maximum which may be a great many times as great as the value of the generator current.

Resonant circuits in the hands of experimenters such as Hertz, Lodge, and others have produced remarkable results, which have led to great advances in our knowledge of electricity, while mathematical analysis of such circuits has led to further discoveries. These results have caused some to expect remarkable effects to be gained from the use of resonant circuits (or **Tuned circuits**, as they are sometimes called) for the purposes of the electrical transmission of power.

Circuits which are installed for the transmission of power over considerable distances (whether the wires are overhead or underground) always contain capacity and self-inductance distributed along their lengths. It would be possible in such lines to adjust the capacity and self-inductance so as to give resonance, and the results to be gained from so doing may be examined through analogy.

A mechanical analogue of an electric circuit is shown in Fig. 523. This consists of a tube fitted with two plungers and filled with a perfectly elastic fluid. The properties of this fluid

may be used to represent electrical quantities according to the analogies; fluid velocity — electric current; fluid pressure — voltage; inertia — self-inductance; compressibility * — electrostatic capacity; frictional resistance — electrical resistance. Now suppose the fluid to be without inertia and perfectly incompressible; then if plunger A is moved toward D, a uniform current is instantly set up in the whole length of the tube, the velocity of which is equal (in proper units) to the pressure applied to the plunger divided by the frictional resistance. If plunger A is caused to move up and down harmonically, the plunger A'

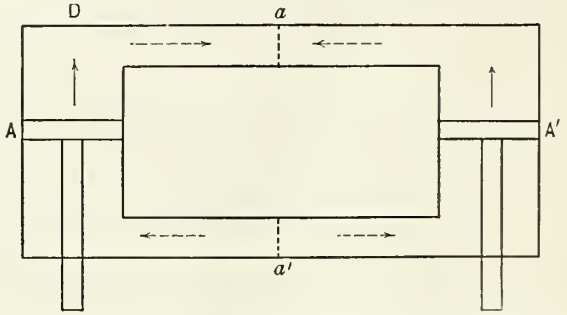


Fig. 523. — Illustration of a Mechanical Analogue of an Electric Circuit.

at the other end of the line will have an exactly equal synchronous harmonic motion. This is analogous to the state of an electric circuit without inductance or capacity. Figure 524 shows diagrammatically the state of the circuit, the distance of the broken line from the heavy line being equal to the current at each point. The light full line shows the gradual fall of pressure between A and A', caused by the resistance, and the sudden fall of pressure at A', caused by the external work done by plunger A'.

If the fluid is compressible but has no inertia, it is evident that the motion of the plunger at A' will be less than that at A, which is analogous to the decadence of current as it flows along a circuit having capacity, due to the quantity of electricity entering into the electrostatic charge on the conducting wires. The movements of the plungers are isochronous but not in synchronism.

* Compressibility of a fluid is the ratio of compression (change of volume) to the pressure producing it, and electrical capacity is the ratio of the charge (change of quantity) to the voltage producing it.

In this case the motion of the plunger A will exert its maximum pressure when the fluid is most compressed, or at the end of its stroke where its velocity is least. Hence the velocity of the fluid at the generator (*i.e.* the generator current), which is greatest at the middle of the stroke, leads the pressure by 90° of phase. The magnitudes of the current and voltage throughout the circuit are illustrated in Fig. 525, after the manner explained in connection with Fig. 524.

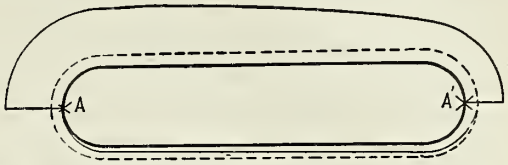


FIG. 524. — Illustration showing Conditions of Currents and Pressures when the Pistons of Fig. 523 move in an Incompressible Fluid which is without Inertia.

If the fluid has inertia but is incompressible, the velocities at A and A' are equal; that is, the current throughout the circuit is uniform, but the pressure exerted upon piston A is greatest when the acceleration is greatest, which is at the beginning of the stroke where the velocity is least. Consequently the current lags behind the pressure by 90°. This is analogous to the electric circuit with self-inductance but no capacity.

If the fluid has both inertia and compressibility, the column of fluid in the tube then takes upon itself the usual properties of material elastic bodies, and possesses a natural rate of vibration of its own which depends upon the dimensions of the column and the inertia and compressibility of the fluid. The period of this vibration, as is proved in elementary mechanics, is proportional to the square root of the density divided by the elasticity, or to the square root of the product of the inertia and compressibility. Hence $T = a\sqrt{MK}$ where a is a constant, M mass, K compressibility, and T time of vibration.

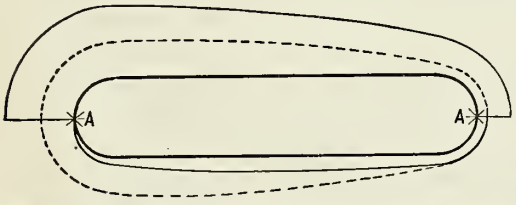


FIG. 525. — Illustration showing the Conditions of Currents and Pressures when the Pistons of Fig. 523 move in a Compressible Fluid having No Inertia.

In this case, if the plunger A (Fig. 523) is moved with a sinusoidal velocity having a period of T seconds, which is the

same as the natural period of vibration of the column of fluid, the fluid will be thrown into vibrations which require one complete traversal of the circuit to make a wave length. Hence, if there is no power taken from the circuit, there are nodes or points of no motion at a and a', and antinodes or points of maximum motion at the plungers. Since the directions of motion in the two halves of a wave are opposite, the two plungers move in opposite directions in the tube. As the velocity of the fluid varies from node to antinode as a sinusoidal function, the loss of power by friction is reduced to one half the value which it has for an equal plunger velocity in the inertialess, incompressible fluid. The velocity of propagation of the disturbance through the fluid from plunger A to A' is equal to $\frac{l}{a\sqrt{MK}}$ centimeters per second, where l is the length of column.

Since the velocity of movement of the fluid falls off from the plungers toward the nodes, the pressure upon the fluid exerted by the plungers must be proportionately multiplied at the nodes, in order that the same power may be transmitted through the fluid across the nodes as is applied at the prime plunger A. The condition of pressure and velocity is diagrammatically represented in Fig. 526. If power is transferred to an outside object by plunger A', it is impossible for the velocity at the nodal points to be zero, as the product of velocity and pressure must be sufficiently great to transfer the power through the nodal point. The relative motions of the plungers, under the conditions here cited, require that *the power shall be transferred from one plunger to the other wholly through the absorption and redelivery of the power by the fluid by means of the effects of inertia and elasticity.* The fluid must therefore have a sufficient mass so that even at the slow velocity at the nodal

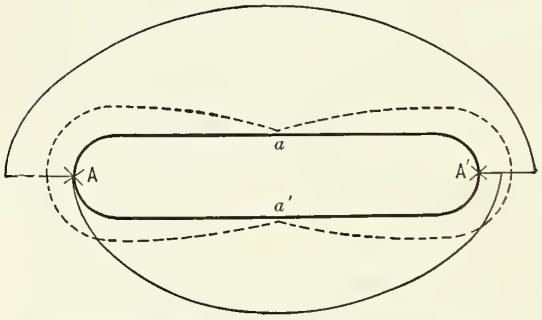


Fig. 526. — Illustration showing the Conditions of Currents and Pressures when the Pistons of Fig. 523 move in a Fluid having both Compressibility and Inertia.

points its kinetic energy shall be sufficient to carry the energy in the circuit across those points.

This analogue represents the conditions in the resonant electric circuit with distributed self-inductance and capacity. Carrying in mind the analogue and the diagrammatic representation of current and pressure in Fig. 526, it is easy to draw definite conclusions in regard to the effect of resonance on the operation of electric circuits for the transmission of power.

The advantages of a resonant circuit for electrical transmission are then: (1) a gain of upwards of one half of the I^2R loss that would be caused by the transmission of an equal amount of power at an equal receiving voltage over the same circuit when out of resonance; (2) more satisfactory regulation than would be found in a non-resonant but reactive line, since the difference in voltage between generator and receiver is equal to current times resistance instead of current times an impedance which is greater than the resistance.

The principal disadvantage of a resonant circuit for electrical transmission is: a very large excess of voltage on the line at certain points, or nodes of current, which excess decreases toward the antinodes. If satisfactory resonance is to be gained by adjusting the self-inductance and capacity of the circuit so that the voltage at the nodes of current is no greater than ten times that at the antinodes, the average voltage along the line must be caused to be seven times $\left(\frac{10}{\sqrt{2}}\right)$ that of the antinodes, using a sinusoidal function. In other words, if the voltage which is safe for use is limited by the insulation, we may say that the average strength of insulation on the line must be seven times as great as would be necessary at the generators. This enormous increase of insulation must be made to save fifty per cent of the I^2R loss caused by the transmission of a certain amount of power over a given line. A much more reasonable plan for heavy power transmission would be to reduce the self-inductance and capacity of the line to a minimum, avoiding resonance and raising the generator voltage to 1.4 its previous value. Now the same power could be transferred over the line with the same resistance as before, the I^2R loss being the same as when the line was resonant, but the average strain on the insulation would be only one fifth as great as in the resonant line.

The highest voltage which can be economically used on circuits for the electrical transmission of large amounts of power over long distances is generally conceded to be set at the limit which may be properly insulated. If this is true, the preceding paragraph shows that, with equal insulation, the generator voltage may be safely made $\frac{X}{\sqrt{2}}$ times greater on a non-resonant, long distance transmission line than that which is safe on a resonant line, where X is the ratio of the maximum voltage to the generator voltage on the resonant line. This shows that the non-resonant line would be by far the most economical for long distance heavy power transmission, even if it were commercially possible to maintain resonance on service circuits. For the distribution of power over short distances, the voltage is usually quite low, and the insulation limit is not approached, so that resonance might be introduced without adding to the insulation; but the reactions of transformers and motors on the line make it practically impossible to keep the line in resonance. Similar defects are seen in the propositions for using resonant lines for various other classes of electrical transmission except telephony.

These deductions in regard to resonance have been made upon the assumption of sinusoidal currents. In practice these are now seldom exactly realized, since iron-cored transformers and motors, and tooth-cored alternators, introduce distortions, and a circuit which is resonant for the fundamental wave is not resonant for its harmonics. As the question of resonance now rests, it does not enter into problems relating to ordinary electric power circuits in such a way as to modify practice except as it in certain instances causes undue stresses on the insulation through its accidental occurrence in connection with transient effects caused by switching, arcing between wires or from wires to ground, effects of lightning induction, and the like.

In telephony (which consists of the electric transmission of power in very minute quantities under certain special conditions), the voltages are relatively low and satisfactory insulation for resonant transmission may be readily accomplished. It is, therefore, good practice to load long distance telephone lines with self-inductance by inserting coils at frequent intervals, for the purpose of making the lines more nearly distortionless.

218. Corona. — One of the earliest observed manifestations of electricity was the brush discharge between the electrodes of an electrostatic machine or near a highly charged pointed body. It was early recognized that this is accompanied by convection effects whereby a current of electrically charged air is caused to flow away from the body concerned, with a tendency to discharge the body. This is illustrated by holding a lighted candle near a point protruding from the electrode of an electrostatic machine in operation, when it may be observed that the candle flame is blown aside by the air current caused by the particles of air receding from the electrode on account of repulsion after they have become charged by contact. The phenomenon is accompanied by a silent transfer of electricity from one electrode of the machine to the other by means of the air particles.

While the usual electric circuits of commerce were only of relatively low voltage, it was thought that the phenomena of the brush discharge and electrostatic convection might pertain only to the extraordinary voltages of the so-called electrostatic machines; but the development of high voltage power transmission has proved that those phenomena are associated with voltages of only a few tens of thousands of volts and that they occur in striking degree on aërial power lines when the voltage exceeds a hundred thousand or more volts. Such lines, when the voltage exceeds 125,000 volts, become luminous from the discharge, and seem, when viewed in the dark, to be covered with a brush discharge having the appearance of the corresponding discharge between the electrodes of an electrostatic machine.

The discharge is established at a voltage which is inversely related to the radius of curvature of the conductors on which it is seated, and, in the case of aërial lines for the transmission of power by alternating currents, it results in a serious loss of power when the voltage much exceeds 125,000 volts and the wires are of the usual transmission sizes such as from No. 6 to No. 0000 B. and S. gauge. The loss is greater when the barometric pressure of the atmosphere is low, and it is therefore more strikingly observed in connection with high voltage lines in mountainous regions. The phenomena thus denoted are called **Corona** effects, on account of the luminous corona which may be observed in a dark room around conductors on which the phenomena are active. The phenomena are accompanied

by the crackling or hissing sounds and the production of nitric oxide and ozone (which may be observed by the odor) that are recognized accompaniments of brush discharges at the terminals of an electrostatic machine.

The amount of power lost by convection between power transmission conductors of the usual diameters increases slightly with the voltage up to a certain critical point, at which the corona seems to be fully established, and the ratio of power lost to line voltage is much higher for voltages above the critical point. The critical voltage for ordinary transmission lines at the usual central altitudes of this country seems to be about 125,000 volts, and the loss would be prohibitive on long lines at higher voltages when conductors not exceeding one half inch in effective diameter are used. The loss also increases with the frequency of the cycles of the voltage.

These phenomena have been given much study, but no reliable laws have yet been formulated. The consensus of knowledge may be obtained fairly well from various papers in the *Transactions of the American Institute of Electrical Engineers* for the years 1910 to 1912.*

It has not yet been conclusively determined whether or not the effects of corona may occur in liquids or solids, but the best opinion leans to the hypothesis that corona is a result of ionization in a gas and requires a gaseous medium for its development. There are, however, certain recognized effects of excessive electrostatic stresses in solid dielectrics which indicate that some phenomena occur in solids which are at least similar to corona effects.

* Steinmetz, Ryan, Mershon, Whitehead, Peek, and others.

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